A Very Short Evaluation of an Integral

In [1], the author used partial fractions to evaluate the integral of an odd power of sec $\theta$. Here, we give a one-sentence evaluation of this integral:

$$\int \sec^{2n+1} \theta \, d\theta = \int \frac{\sec^{2n+1} \theta (\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} \, d\theta$$

$$= \int \left( t + \frac{1}{t} \right)^{2n} \frac{dt}{t} \quad (t := \sec \theta + \tan \theta)$$

$$= \frac{1}{2^{2n}} \sum_{r=0}^{2n} \binom{2n}{r} \int t^{2r-2n} \, dt$$

$$= \frac{1}{2^{2n}} \binom{2n}{n} \log t + \frac{1}{2^{2n}} \sum_{r=0}^{2n} \binom{2n}{r} \frac{t^{2r-2n}}{2r-2n} + C$$

$$= \frac{1}{2^{2n}} \binom{2n}{n} \log[\sec \theta + \tan \theta]$$

$$+ \sum_{r=0}^{n-1} \binom{2n}{r} \frac{(\sec \theta + \tan \theta)^{2n-2r} - (\sec \theta - \tan \theta)^{2n-2r}}{2^{2n+1}(n-r)} + C.$$

In the second step, we have used the fact that $1/t = \sec \theta - \tan \theta$, and in the last step, for each $r$ between 0 and $n - 1$ we have combined the terms of the sum for $r$ and $2n - r$. One can, of course, rewrite the expression in terms of $1 \pm \sin \theta$ easily because

$$(\sec \theta + \tan \theta)^2 = \frac{(1 + \sin \theta)^2}{1 - \sin^2 \theta} = \frac{1 + \sin \theta}{1 - \sin \theta}.$$

References


—Submitted by B. Sury, Indian Statistical Institute, Bangalore, India