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A Tropical Power Sum

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The so-called "tropical" sums $a \oplus b = \min(a, b)$ have been studied by David Speyer, Bernd Sturmfels and others; see [1, 2] for instance. However, the following identity connecting them does not seem to have been noticed earlier by anyone:

Theorem 1. *Let n, r be arbitrary positive integers. Then we have*

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_r=1}^{n}\min(i_1,\ldots,i_r) = \sum_{i=1}^{n} i^r.$$

The use of the word "tropical" in this context has an interesting history. According to Speyer and Sturmfels, it was coined by a group of French mathematicians in honor of their Brazilian colleague Imre Simon, who was one of the pioneers in this area. There is no deeper meaning in the adjective "tropical"; it seems simply to stand for the French view of Brazil!

The ideas used to prove Theorem 1 also yield other results and polynomial identities such as:

Theorem 2. *Let n, r be arbitrary positive integers. Then we have:*

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_r=1}^{n}\max(i_1,\ldots,i_r) = n^{r+1} - \sum_{i=1}^{n-1} i^r.$$

For the variables $x_1, \ldots, x_n$, define the polynomials

$$F_r(x_1,\ldots,x_n) = \sum_{i_1,\ldots,i_r=1}^{n} x_{\min(i_1,\ldots,i_r)}$$

and

$$G_r(x_1,\ldots,x_n) = \sum_{i_1,\ldots,i_r=1}^{n} x_{\max(i_1,\ldots,i_r)}.$$

Then, we have:

$$\begin{aligned} F_r(x_1,\ldots,x_n) &= G_r(x_n, x_{n-1},\ldots,x_1) \\ &= \sum_{i=1}^{n}\{(n-i+1)^r - (n-i)^r\}x_i. \end{aligned}$$

The inductive step

The proof of Theorem 1 will be demonstrated by induction on n, with the case $n = 1$ being obvious. To make it transparent, we use the notation

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$$S_r(n) := \sum_{i_1,\ldots,i_r=1}^{n} \min(i_1, \ldots, i_r).$$

We wish to prove that

$$S_r(n) = \sum_{i=1}^{n} i^r.$$

We start with the binomial theorem: For any n and any nonnegative integer r, we have

$$(n+1)^r = \sum_{k=0}^{r} \binom{r}{k} n^{r-k} = n^r + 1 + \sum_{k=1}^{r-1} \binom{r}{k} n^{r-k} \tag{1}$$

We claim that the following identity holds:

Lemma 3. *For $n, r \geq 1$, we have*

$$n^r - n = \sum_{k=1}^{r-1} \binom{r}{k} \sum_{i=1}^{n-1} i^{r-k}. \tag{2}$$

Proof. Both sides are zero if either $n = 1$ or $r = 1$. The case $n = 2$ for equation (2) reduces to equation (1) for $n = 1$. Assuming that equation (2) holds for a certain n, consider the statement of the lemma for $n + 1$ and subtract the statement for n from it. This is the assertion

$$(n+1)^r - n^r - 1 = \sum_{k=1}^{r-1} \binom{r}{k} n^{r-k}$$

which is just the identity (1). Thus, the lemma follows by induction. ■

PROOF OF THEOREM 1.

Proof. We are trying to prove

$$S_r(n) = \sum_{i_1,\ldots,i_r=1}^{n} \min(i_1, \ldots, i_r) = \sum_{i=1}^{n} i^r \tag{3}$$

As observed earlier, this is clear for $n = 1$. For each $k \leq n$, it is convenient to introduce the notation $S_r(n; k)$ for the sub-sum of $S_r(n)$ consisting of terms where exactly k of the indices i_j are equal to n. Clearly, $S_r(n; r) = n$ because each of the r indices equals n and there is only one term and that equals n. When we consider $S_r(n + 1; k)$ for $0 \leq k < r$, there are $r - k$ indices i_j that are less than $n + 1$; therefore,

$$S_r(n+1; k) = \binom{r}{k} S_{r-k}(n).$$

Then

$$S_r(n+1) = n + 1 + \sum_{k=0}^{r-1} \binom{r}{k} S_{r-k}(n).$$

Assuming that equation (3) holds for n, this is equivalent to

$$S_r(n+1) = n+1+\sum_{k=0}^{r-1}\binom{r}{k}\sum_{i=1}^{n} i^{r-k}$$
$$= n+1+\sum_{i=1}^{n} i^r + \sum_{k=1}^{r-1}\binom{r}{k}\sum_{i=1}^{n} i^{r-k}.$$

As we wish to show that the left hand side equals $\sum_{i=1}^{n+1} i^r$, the previous equality implies that we need to prove

$$(n+1)^r = (n+1)+\sum_{k=1}^{r-1}\binom{r}{k}\sum_{i=1}^{n} i^{r-k}.$$

This is just the identity (2) for $n+1$. Therefore, the identity (3) of the main Theorem 1 is proved by induction. ■

Remark. The expression for sums of rth powers of the first n positive integers is well known in terms of the Bernoulli polynomials $B_{r+1}(x)$:

$$S_r(n) = \sum_{i=1}^{n} i^r = \frac{B_{r+1}(n+1) - B_{r+1}(0)}{r+1},$$

where

$$\frac{te^{xt}}{e^t - 1} = \sum_{r\geq 0} B_r(x)\frac{t^r}{r!}.$$

Polynomial identities

Now, we go on to show how the above proof helps us make more results such as those stated in Theorem 2.

PROOF OF THEOREM 2.

Proof. In order to simultaneously discuss maximum functions, we change notation and write

$$m_r(n) = \sum_{i_1,\ldots,i_r=1}^{n} \min(i_1,\ldots,i_r)$$

and

$$M_r(n) = \sum_{i_1,\ldots,i_r=1}^{n} \max(i_1,\ldots,i_r).$$

We have already shown that $m_r(n) = \sum_{i=1}^{n} i^r$. We now prove that:

$$M_r(n) = n^{r+1} - \sum_{i=1}^{n-1} i^r.$$

Now,

$$M_r(n+1) = M_r(n) + \sum_I \max(i_1, \ldots, i_r)$$

where the sum is over the r-tuples of indices $I = (i_1, \ldots, i_r)$ where at least one of the indices i_j equals $n+1$. The maximum value equals $n+1$ in each of the terms of the sum over I. The number of terms is $(n+1)^r - n^r$. Therefore,

$$M_r(n+1) = M_r(n) + (n+1)\{(n+1)^r - n^r\}.$$

From this recursion

$$M_r(k) - M_r(k-1) = k^{r+1} - (k-1)^{r+1} - k^r,$$

it follows by adding from $k = 2$ to n that

$$M_r(n) = n^{r+1} - \sum_{i=1}^{n-1} i^r$$

since $M_r(1) = 1$.

The polynomial assertions are straightforward. Note also that

$$m_r(n) = F_r(1, 2, \ldots, n), \quad \text{and} \quad M_r(n) = G_r(1, 2, \ldots, n).$$

■

One further corollary that follows from the above observations is:

Corollary 4. *For any strictly increasing function $s(n)$ on the natural numbers, we have*

$$\sum_{i_1,\ldots,i_r=1}^{n} \min(s(i_1), \ldots, s(i_r)) = \sum_{i=1}^{n} \{(n-i+1)^r - (n-i)^r\} s(i)$$

and

$$\sum_{i_1,\ldots,i_r=1}^{n} max(s(i_1), \ldots, s(i_r)) = \sum_{i=1}^{n} \{(i^r - (i-1)^r\} s(i).$$

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Summary. We prove several combinatorial identities related to the so-called "tropical" sums $a \oplus b = \min(a, b)$.

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