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On SU(1,D)/[U(1,D),U(1,D)] for a quaternion division algebra D

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Abstract. Let D be a quaternion division algebra with an involution of the second kind. We show that the quotient group SU(1,D)/[U(1,D),U(1,D)] is nontrivial in general. For global fields, we completely determine this group.

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Introduction. Let k be a field of characteristic $\neq 2$ and let K be a quadratic extension. Let the non-trivial Galois automorphism be denoted by σ. Let D be a quaternion division algebra with center K such that σ extends to an involution of D. It is known that there exist such D for global fields K, k but that an analogous triple does not exist with k a local field. If $\mathbf{N}$ denotes the reduced norm map from D to K, one has the group

$$SL(1, D) := \{d \in D : \mathbf{N}(d) = 1\}$$

which is the group of K-rational points of a simple, anisotropic, algebraic group of type ${}^1A_1$ defined over K. In the case of global fields, it is a well-known classical theorem of Wang that

$$SL(1, D) = [D^*, D^*]$$

where $D^* = D\backslash\{0\}$. Given σ as above, the *unitary group* $U(1, D)$ is defined as

$$U(1, D) := \{d \in D : dd^\sigma = 1\}.$$

On intersecting $U(1, D)$ with the group $SL(1, D)$, one gets the group $SU(1, D)$. This is the group of k-rational points of an anisotropic algebraic group of type ${}^2A_1$ defined over k. It is obvious that $[U(1, D), U(1, D)] \leq SU(1, D)$ and the first basic question here is the analogue of Wang's theorem (see [3, p. 536]) :

Question. Is $SU(1, D) = [U(1, D), U(1, D)]$?

Here, we first show that the answer to this question is in the negative for general quaternion division algebras and determine how far it is from being true. We shall prove some results for general k and later specialize to global fields to get more complete results applying a theorem of Margulis [2]. In contrast with the case of global fields, it was proved in [7] that, if K is a function field in one variable over a number field, and if D is an algebra with center K and with an involution of the second kind, the group $SU(1,D)/[U(1,D),U(1,D)]$ can be infinite in general; we gave an infinite class of such examples. *We believe that the results of this note would be known to experts but it has not been set down in print.*

1. Case of a general field. In this section, we let K be an arbitrary field of characteristic not equal to 2. We assume that there exists a division algebra D of degree 2 over K with an involution σ of the second kind on D. This means that σ does not fix K element-wise; let k be the field fixed under (the restriction to K of) σ. Then $SU(1,D)$ corresponds to the k-rational points of a group of type ${}^2A_1$ over k.

Lemma 1.1. *There is a quaternion division algebra A over k such that $D = A \otimes_k K$ and σ is induced by the canonical involution on A, and the non-trivial Galois automorphism σ on K.*

Proof. See [5, Theorem 11.2 (ii)]. □

Remark 1.2. Note that the canonical involution of D is induced by the canonical involution of A and the trivial automorphism of K.

We will denote by N_0, the reduced norm of A and by $\bar{x}$, the canonical involution of A i.e. $N_0(x) = x\bar{x}$ for all $x \in A$.

Proposition 1.3.

$$\frac{SU(1,D)}{[U(1,D),U(1,D)]} \cong \frac{SL(1,A)}{[G,G]}$$

where $G := \{x \in A^* : N_0(x) \in N_{K/k}K^*\}$, with $N_{K/k}$, the norm map for the field extension K/k.

Proof. Now, we may write $K = k + \alpha k$ with $\alpha^\sigma = -\alpha$. Let us write $\alpha^2 = -t \in k$ (so, $t = N_{K/k}\alpha$). Therefore, any element of D can be written in the form $z = (x \otimes 1) + (y \otimes \alpha)$ with $x, y \in A$. If $z \in U(1,D)$, then

$$1 = zz^\sigma = (x \otimes 1 + y \otimes \alpha)(\bar{x} \otimes 1 - \bar{y} \otimes \alpha) = (x\bar{x} + ty\bar{y}) \otimes 1 + (y\bar{x} - x\bar{y}) \otimes \alpha$$

Since $\{1 \otimes 1, 1 \otimes \alpha\}$ is a basis of D over A and since $1 \otimes 1 = 1_D$, we have

$$\begin{aligned} x\bar{x} + ty\bar{y} &= 1 \\ y\bar{x} - x\bar{y} &= 0 \end{aligned}$$

But, then $y\bar{x}$, being an element of A fixed by its canonical involution, has to be in its center k. So, we may write $y = ax$ with $a \in k$ if $x \neq 0$ and, similarly, we may write $x = by$ with $b \in k$ if $y \neq 0$. Thus,

$$z = x \otimes 1 + y \otimes \alpha = x \otimes (1 + a\alpha) \quad \text{if} \quad x \neq 0;$$

$$z = y \otimes (b + \alpha)) \quad \text{if} \quad y \neq 0.$$

In either case, $z = u \otimes s$ with $u \in A, s \in K$. Now, the condition $z \in U(1, D)$ gives $1 = zz^\sigma$; that is, $1 \otimes 1 = u\bar{u} \otimes ss^\sigma = N_0(u)N_{K/k}(s)(1 \otimes 1)$. Therefore,

$$U(1, D) = \{x \otimes s : x \in A^*, s \in K^*, N_0(x) = N_{K/k}(s)^{-1}\} \ldots (\spadesuit)$$

Now, $SU(1, D) = U(1, D) \cap SL(1, D) = \{g \in U(1, D) : gg^\theta = 1\}$, where θ denotes the canonical involution of D. Since $(x \otimes s)^\theta = \bar{x} \otimes s$, we have

$$x \otimes s \in U(1, D) \Leftrightarrow N_0(x)N_{K/k}(s) = 1$$

and

$$x \otimes s \in SL(1, D) \Leftrightarrow 1 = N_0(x)s^2.$$

Therefore,

$$x \otimes s \in SU(1, D) \Leftrightarrow s \in k\ ,\ xs \in SL(1, A).$$

This gives $SU(1, D) = \{x \otimes 1 : x \in SL(1, A)\}$.

Consider the subgroup $G := \{x \in A^* : N_0(x) \in N_{K/k}K^*\}$ of A^*.

Using the description of $U(1, D)$ in equation $(\spadesuit)$, we have the homomorphism

$$\pi : SU(1, D) \to \frac{SL(1, A)}{[G, G]}\ ;\ x \otimes 1 \mapsto x[G, G]$$

which is surjective. If $x \otimes 1$ is in Ker π, then we have

$$x = [x_1, y_1] \cdots [x_n, y_n]$$

for some $x_i, y_i \in G$. Writing $N_0(x_i) = N_{K/k}(s_i), N_0(y_i) = N_{K/k}(t_i)$, we have $u_i := x_i \otimes s_i^{-1}, v_i := y_i \otimes t_i^{-1} \in U(1, D)$ and

$$x \otimes 1 = [u_1, v_1] \cdots [u_n, v_n] \otimes 1$$

in $U(1, D)$. Thus $x \in [U(1, D), U(1, D)]$; so Ker $\pi \leq U(1, D), U(1, D)]$.

Conversely, for any $x \otimes s, y \otimes t \in U(1, D)$, the element

$$[x \otimes s, y \otimes t] = [x, y] \otimes 1$$

of $SU(1, D)$ maps under π to $[x, y][G, G] = [G, G]$ since $x, y \in G$. This proves the proposition. $\square$

2. Case of a global field. Let k, K as above be global fields and let the other notations D, A, σ, N_0 etc. be as before. In this section, we use a theorem of Margulis to compute the group $SU(1,D)/[U(1,D), U(1,D)]$. Let us write k_v for the completion of k with respect to a place v of k; let T denote the (finite) set of non-archimedean places of k where A is ramified i.e. such that $A \otimes_k k_v$ is a division algebra over k_v.

For $v \in T$, let A_v denote the division algebra $A \otimes_k k_v$, and let $\mathbf{P}_v$ denote the unique maximal two-sided ideal of the valuation ring of A_v.

Let $\pi : A \to \prod_{v \in T} A_v$ be the diagonal embedding. The groups $SL(1, A_v)$ are profinite, and the group $SL(1,A)$ gets a topology (the T-adic topology) via the map π. We use the convention that when T is empty (this can happen when k is a number field) the T-adic topology is indiscrete (that is, the only two open sets are $\emptyset$ and the whole of $SL(1,A)$). We recall an old theorem of Margulis :

Theorem 2.0 (Margulis [2]). *Any noncentral normal subgroup N of $SL(1,A)$ is T-adically open; that is, $N = \pi^{-1}(W)$ where W is the closure of $\pi(N)$ in $\prod_{v \in T} SL(1, A_v)$.*

In our situation, we apply Margulis's theorem to derive :

Lemma 2.1. *Let $K, D, \sigma, k, A, N_0, T$ be as above. Let $G = \{x \in A^* : N_0(x) \in N_{K/k} K^*\}$. For each $v \in T$, let G_v denote the closure of G in A_v^*. Then,*

$$\frac{SU(1,D)}{[U(1,D), U(1,D)]} \cong \prod_{v \in T} \frac{SL(1, A_v)}{[G_v, G_v]}.$$

Proof. Now, each G_v is closed as well as open in A_v^* and therefore, $\prod_{v \in T} [G_v, G_v]$ is open in $\prod_{v \in T} SL(1, A_v)$ since $[A_v^*, A_v^*] = SL(1, A_v)$ (see [3, p. 31]). On the other hand, By the weak approximation theorem, the image under the diagonal embedding π of $SL(1,A)$ in $\prod_{v \in T} SL(1, A_v)$ is dense. This implies that

$$\pi(SL(1,A)) \prod_{v \in T} [G_v, G_v] = \prod_{v \in T} SL(1, A_v).$$

By Margulis's Theorem 2.0 applied to $[G,G] \subseteq SL(1,A)$, we get

$$[G,G] = \pi^{-1}\left(\prod_{v \in T} [G_v, G_v] \right).$$

That is,

$$\pi([G,G]) = \pi(SL(1,A)) \cap \prod_{v \in T} [G_v, G_v].$$

Thus,

$$\begin{aligned} \frac{\pi(SL(1,A))}{\pi([G,G])} &\cong \frac{\pi(SL(1,A))}{\pi(SL(1,A)) \cap \prod_{v \in T} [G_v, G_v]} \\ &\cong \prod_{v \in T} \frac{SL(1, A_v)}{[G_v, G_v]}. \end{aligned}$$

As $\frac{SU(1,D)}{[U(1,D),U(1,D)]}$ is isomorphic to the above left hand side by proposition 1.3, the lemma follows. □

Remarks and notations 2.2. We need to compute for each $v \in T$, the closure G_v of G and its commutator subgroup $[G_v, G_v]$. Since the definition of G involves K, we need to look at $K \otimes_k k_v$ for $v \in T$; this is either a quadratic extension field of k_v or isomorphic to $k_v \oplus k_v$. Now, for $v \in T$, it is well known ([6, p. 194]; [1, p. 159]) any quadratic extension E_v of k_v can be embedded isomorphically in A_v and splits it; in particular, the norm map N_{E_v/k_v} coincides with the reduced norm map of A_v. By local class field theory, the quadratic extensions of k_v are characterized uniquely by their images under their norm maps to k_v. If $v \in T$ and $K \otimes_k k_v$ is a field, we denote by L_v a subfield of A_v whose norm subgroup is the same as that of $K \otimes_k k_v$; that is,

$$N_{L_v/k_v} L_v = N_{K \otimes_k k_v/k_v}(K \otimes_k k_v).$$

We shall recall and use some results of C. Riehm ([4]) on local division algebras which are also recalled in [3]. Let B be a quaternion division algebra over a local field l and let N_{red} denote its reduced norm. The valuation v of l extends to one on B as $\tilde{v}(x) = \frac{v(N_{\mathrm{red}}(x))}{2}$ (see [3, p. 28]). Let $\mathbf{O}_B$ denote the valuation ring of B and P_B denote its unique maximal two-sided ideal. Let U_B denote the group of units of $\mathbf{O}_B$. Then $U_B \supset SL(1,B) := \mathrm{Ker}(N_{\mathrm{red}})$. Also, B contains an isomorphic copy of each quadratic extension of l. Let W denote an unramified quadratic extension of l contained in B. One may choose a uniformizing parameter (generator) π for P_B such that $p := \pi^2$ is a uniformizer for W (as well as l) and so that the conjugation by π produces the nontrivial Galois automorphism $a \mapsto \bar{a}$ of W over l (see [4, p. 502]). Further, any uniformizing parameter for P_B is of the form $u\pi$ for some $u \in U_B$ (see [3, p. 29]).

One may write each element of $\mathbf{O}_B = \mathbf{O}_W + \mathbf{O}_W \pi$, where $\mathbf{O}_W$ is the valuation ring of W. Let $F = \mathbf{O}_W / p\mathbf{O}_W$ and $f = \mathbf{O}_l / p\mathbf{O}_l$ denote the residue fields of W and l respectively. Then, we have (see [3, p. 32–34]) :

(a) $[B, B] = SL(1, B)$.

(b) The map $\rho_0 : \alpha_0 + \alpha_1 \pi \mapsto \alpha_0 \bmod p\mathbf{O}_W$ from U_B to F induces an isomorphism from $U_B/(1 + P_B)$ onto F^*.

(c) $[SL(1, B), SL(1, B)] = SL(1, B) \cap (1 + P_B)$.

(d) $[SL(1, B), SL(1, B) \cap (1 + P_B)] = SL(1, B) \cap (1 + P_B)$.

(e) ρ_0 gives an isomorphism of $SL(1, B)/[SL(1, B), SL(1, B)]$ onto

$$F^{(1)} := \{x \in F^* : N_{F/f}(x) = 1\}.$$

Hence $[U_B, U_B] \subseteq SL(1, B) \cap (1 + P_B) = [SL(1, B), SL(1, B)]$ since $U_B/(1 + P_B)$ is abelian by above and since clearly $[U_B, U_B] \subseteq \mathrm{Ker}(N_{\mathrm{red}}) = SL(1, B)$.

In particular,

$$[U_B, U_B] = [SL(1, B), SL(1, B)] \cdots\cdots\cdots\cdots\cdots \diamondsuit$$

For each $v \in T$, we shall be applying these results to the division algebra $B = A_v$ over $l = k_v$. The notations W, L, F, f etc. will be replaced by W_v, L_v, F_v, f_v. We shall also use $x \mapsto \bar{x}$ for the canonical involution on A_v.

Lemma 2.3. *For any $v \in T$, denote the closure of G in A_v^* as G. If $K \otimes_k k_v$ is a field for some $v \in T$, denote by L_v a subfield of A_v with $N_{L_v/k_v} L_v = N_{K \otimes_k k_v/k_v}$ $(K \otimes_k k_v)$. Then*

(i) $G_v = A_v^*$ if $K \otimes_k k_v \cong k_v \oplus k_v$;

(ii) $G_v = L_v^* SL(1, A_v)$ if $K \otimes_k k_v$ is a field.

Proof. Recall that $G = \{x \in A^* : N_0(x) \in N_{K/k} K^*\}$. Thus, for $v \in T$ as in case (i), G is dense in A_v^*. Suppose $v \in T$ and that we are in case (ii). Denote by N_v, the reduced norm on A_v. Then $x \in G_v$ if, and only if, $N_v(x) \in N_{L_v/k_v} L_v^*$. As L_v is a maximal subfield of A_v, we have $N_{L_v/k_v} = N_v$ on L_v. Thus $x \in G_v$ if and only if $N_v(y^{-1}x) = 1$ for some $y \in L_v^*$. That is, $y^{-1}x \in SL(1, A_v)$ which means $x \in L_v^* SL(1, A_v)$. The proof is complete. □

Finally, we introduce some notations to be used in the next theorem. Denote by q_v the order of the residue field of k_v for any nonarchimedean place v of k and, let n_v be the natural number defined as follows :

$n_v = q_v + 1$ if $K \otimes_k k_v$ is an unramified field extension of k_v;

$n_v = 2$ if $K \otimes_k k_v$ is a ramified quadratic extension and the residue characteristic of k_v is not 2;

$n_v = 1$ if $K \otimes_k k_v$ is a ramified quadratic extension and the residue characteristic of k_v is 2;

$n_v = 1$ if $K \otimes_k k_v$ is not a field.

We start with a description of G_v's before computing their commutators :

Theorem 2.4. *Let $v \in T$. Then $\frac{SL(1,A_v)}{[G_v,G_v]}$ can be identified with the following group:*

(i) $\{1\}$ if $K \otimes_k k_v \cong k_v \oplus k_v$;

(ii) $F_v^{(1)}$ if $K \otimes_k k_v$ is an unramified quadratic extension;

(iii) $F_v^{(1)}/(F_v^{(1)})^2$ if $K \otimes_k k_v$ is a ramified quadratic extension.

In particular, for $v \in T$, the order of $\frac{SL(1,A_v)}{[G_v,G_v]}$ equals n_v.

Proof. Let $P = P_{A_v}, F = F_v = \mathbf{O}_{A_v}/P, U = U_{A_v}$, and $S = SL(1, A_v)$. For $u \in U$, let $\tilde{u} = \rho_0(u) \in F$, the image of u in the residue field of the valuation on A_v. Similarly, for any $T \subseteq U$, let $\tilde{T} = \rho_0(T) \subseteq F$. Let f be the residue field of k_v, and let $\phi : F \to F$ be the nontrivial f-automorphism of F; so $\phi^2 = id_F$. As F is a field, every uniformizer π of A_v satisfies the property

$$\widetilde{\pi u \pi^{-1}} = \phi(\tilde{u}) \ \forall \ u \in U.$$

Fix a uniformizer π. Take arbitrary non-zero elements of A_v, which we can write as $a\pi^i, b\pi^j$ with $a, b \in U; i, j \in \mathbf{Z}$. Then observe :

$$[a\pi^i, b\pi^j] = a(\pi^i b\pi^{-i})(\pi^j a^{-1}\pi^{-j})b^{-1} \in U.$$

Hence, for the image in F, as F is commutative, one has

$$\widetilde{[a\pi^i, b\pi^j]} = \tilde{a}(\widetilde{\pi^i b\pi^{-i}})(\widetilde{\pi^j a^{-1}\pi^{-j}})\widetilde{b^{-1}} = (\tilde{a}\phi^j(\tilde{a})^{-1})(\phi^i(\tilde{b})\tilde{b}^{-1}) \tag{2.1}$$

This makes it clear that $\tilde{S} = \widetilde{[A_v^*, A_v^*]} = \{\phi(c)c^{-1} : c \in F^*\} = F^{(1)}$. Now, using (c) and (a) of Remark 2.2 we have,

$$(1 + P) \cap S = [S, S] \subseteq [G_v, G_v] \subseteq [A_v^*, A_v^*] = S.$$

Hence,

$$S/[G_v, G_v] \quad \cong \quad \tilde{S}/\widetilde{[G_v, G_v]} \tag{2.2}$$

Therefore, case (i) of the theorem follows from case (i) of Lemma 2.3 and (a) of Remark 2.2. To prove case (ii), we assume that $K \otimes_k k_v$ is an unramified quadratic extension of k_v; so L_v is unramified over k_v. So, as $S \subseteq U$ and every element of L_v^* has the form $a\pi^{2i}$ with $a \in U$, every element of $G_v = L_v^* S$ also has this form. Hence, equation (2.1) shows that $\widetilde{[G_v, G_v]} = \{1\}$, so formula (2.2) shows $S/[G_v, G_v] \cong \tilde{S} = F^{(1)}$ which proves case (ii). In case (iii), L_v is ramified over k_v, so the residue field of $\mathbf{O}_{L_v}$ is f, and we can assume that $\pi \in L_v$. So elements of L_v^* have the form $u\pi^i$ with $u \in U, \tilde{u}$ ranging over f, and i ranging over $\mathbf{Z}$. Since $S \subseteq U$ and $\tilde{S} = F^{(1)}$, elements of $G_v = L_v^* S$ have the form $u\pi^i$ with $u \in U$ where $\tilde{u}$ ranges over $fF^{(1)}$, and i ranges over $\mathbf{Z}$. Hence, as $\phi(c) = c^{-1}$ for $c \in F^{(1)}$, formula (2.1) shows that $\widetilde{[G_v, G_v]} = (F^{(1)})^2$; so (2.2) yields $S/[G_v, G_v] \cong F^{(1)}/(F^{(1)})^2$. This completes the proof. □

Corollary 2.5.

$$\left|SU(1, D)/[U(1, D), U(1, D)]\right| = \prod_{v \in T} n_v.$$

In particular, if $A \otimes_k k_v \cong M(2, k_v)$ for all finite places where $K \otimes_k k_v$ is a field, then

$$SU(1, D) = [U(1, D), U(1, D)].$$

Proof. By lemma 2.1, we have

$$\frac{SU(1, D)}{[U(1, D), U(1, D)]} \cong \prod_{v \in T} \frac{SL(1, A_v)}{[G_v, G_v]}.$$

By theorem 2.4, for $v \in T$, the cardinality $|\frac{SL(1,A_v)}{[G_v,G_v]}| = n_v$ as in the statement. □

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