Hecke algebras, Intersection cohomology of Schubert varieties and Representation Theory

B.Sury

Stat-Math Unit
Indian Statistical Institute
8th Mile Mysore Road
Bangalore 560 059, India.

These notes were prepared by me in 1984 expanding the notes of a lecture-course by George Lusztig on 'Intersection cohomology of Schubert varieties' at the Tata Institute, Bombay during January-March 1984. Feedback from anybody who may care to give it will be appreciated.

It is a real pleasure to thank Ms. Ashalata for deciphering and typing my old set of 'yellowed' notes. Although we have corrected many typographical errors, there are likely to be some more as-yet-unspotted ones. Another problem may be that some of the cross-references in the written notes may occur with the wrong numbering here. This will hopefully be done soon.

Contents

1 Coxeter Groups

Definition 1.1 *A pair (W, S) is said to be a Coxeter system if*

1. *W is a group with the generating set S and*
2. *the relations are $m(s,s) = 1$ and $m(s,t) = m(t,s)$ where $m(s,t)$ denotes the order of st $\forall$ $s,t \in S$.*

Remark 1.1 1. *$m(s,t) = \infty$ is allowed and has the obvious meaning.*

2. *If (W, S) is a Coxeter system, we also call W a Coxeter group by abuse of language.*

Examples

1. Let s_n denote the symmetric group of degree $n \geq 2$. If s_i is the transposition $(i, i+1)$ for $1 \leq i \leq n-1$ and $S = \{s_1, \ldots, s_{n-1}\}$, then (S_n, S) is a Coxeter system.

2. Let $\tilde{S}_n^1 = \left\{ \varphi : \mathbb{Z} \overset{\text{Perm}}{\to} \mathbb{Z} : \varphi(a+n) = \varphi(a) + n \; \forall \; a \in \mathbb{Z} \right\}$.

   Then, we have a map

   $$\begin{aligned} \alpha \quad : \quad & \tilde{S}_n^1 \to \mathbb{Z} \\ & \varphi \mapsto \sum_{i \bmod n} (\varphi(i) - i). \end{aligned}$$

   If $\tilde{S}_n = \text{Ker } \alpha$ and S is the subset of n elements induced from the transpositions $(\bar{i}, \overline{i+1})$ on $\{1, 2, \ldots, n+1\}, 1 \leq i \leq n$ then $(\tilde{S}_n, S)$ is a Coxeter system.

   The group $\tilde{S}_n$ is infinite.

3. Suppose $\mathcal{J}$ is a complex semi-simple Lie algebra and h, a Cartan subalgebra of $\mathcal{J}$. Let Φ denote the system of roots of $\mathcal{J}$ with respect to h. Then, the Weyl group of Φ is a Coxeter group.

4. In a semi-simple algebraic group over a local field, the affine Weyl group corresponding to a system of roots is a Coxeter group. This is infinite.

Definition 1.2 *Suppose (W, S) is a Coxeter system and $w \in W$. We define the length of $\ell(w)$ to be the smallest integer $\ell(w)$ such that $w = s_1 s_2 \dots s_{\ell(w)}, s_i \in S$. An expression of w, of length $\ell(w)$ over S, is called a reduced expression of w.*

Proposition 1.1 *([B]) Suppose (W, S) is a Coxeter system. Then,*

(i) $\ell(w) = \ell(w^{-1}) \ \forall \ w \in W$.

(ii) $|\ell(w) - \ell(w')| \leq \ell(ww'^{-1}) \ \forall \ w, w' \in W$.

(iii) For $w, w' \in W, \ell(ww') \equiv \ell(w) + \ell(w') \mod 2$.

(iv) For $s \in S$ and $w \in W$ with a reduced expression $w = s_1 \dots s_q$, exactly one of the following holds: (a) $\ell(sw) = \ell(w)+1$ with $ss_1 \dots s_q$ a reduced expression of sw, (b) $\ell(sw) = \ell(w) - 1$ and $s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q$ is a reduced expression (for some $1 \leq i \leq q$) of sw (Exchange property).

(v) For $s, t \in S$ and $w \in W$, if $\ell(w) = \ell(swt)$ and $\ell(sw) = \ell(wt)$, then $sw = wt$.

(vi) Suppose M is a monoid and f a function of S into M.

For $s, t \in S$, let $m(s, t)$ denote the order of st; define

$$a(s,t) = \begin{cases} (f(s)f(t))^{\ell}, & \text{if } m(s,t) = 2\ell < \infty \\ (f(s)f(t))^{\ell} f(s), & \text{if } m(s,t) = 2\ell + 1 < \infty \\ 1, & \text{if } m(s,t) = \infty. \end{cases}$$

If we have $a(s, t)$ for $s, t \in S$, then there exists a function $g : W \to M$ such that $g(w) = f(s_1) \dots f(s_q)$ for each $w \in W$ and each reduced expression $s_1 \dots s_q$ of w.

Proof:

(i) If $w = s_1 \dots s_q$ is a reduced expression, then $w^{-1} = s_p \dots s_1$ and so $\ell(w^{-1}) \leq \ell(w)$. On the same count, $\ell(w) \leq \ell(w^{-1})$.

(ii) Taking reduced expressions for w, w', it is clear that $\ell(ww') \leq \ell(w) + \ell(w')$.

Replacing w by ww'^{-1}, we get $\ell(w) - \ell(w') \leq \ell(ww'^{-1})$. Interchanging w and w', $\ell(w') - \ell(w) \leq \ell(w'w^{-1}) = \ell(ww'^{-1})$.

(iii) The map $s \mapsto -1$ on S extends to the homomorphism $w \mapsto (-1)^{\ell(w)}$ of W into $\{-1, 1\}$.

(iv) If $\ell(sw) > \ell(w)$, then $\ell(sw) = \ell(w)+1$ as $\ell(w_1 w_2) \leq \ell(w_1)+\ell(w_2) \, \forall \, w_1, w_2 \in w$.

Let $\ell(sw) \leq \ell(w)$.

Since $\ell(sw) \equiv \ell(w) + 1 \ \ mod \ \ 2$ and $|\ell(w) - \ell(sw)| \leq \ell(s) = 1$, we have $\ell(sw) = \ell(w) - 1$.

We will prove the exchange property in three steps.

Step 1: Let $T = \{wsw^{-1} | w \in W, s \in S\}$. Given a string $\bar{s} = (s_1, \dots, s_q), s_i \in S$, let $\Phi(\bar{s}) = (t_1, \dots, t_q)$ where $t_i = s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1$. If $s_1 \dots s_q$ is a reduced expression, then t_i are all distinct.
For, if $t_i = t_j$ for some $i < j$, we have $s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1 = s_1 \dots s_{j-1} s_j s_{j-1} \dots s_1$ so that

$$1 = s_{i+1} \dots s_{j-1} s_j s_{j-1} \dots s_i.$$

If we multiply by $s_1 \dots s_{i-1}$ on the left and $s_i s_{i+1} - s_q$ on the right, we get $s_1 \dots s_q = s_1 \dots s_{i-1} s_{i+1} \dots s_{j-1} s_{j+1} \dots s_q$. This gives a contradiction because $s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_q$ has lesser length.

Step 2: For any string $\bar{s} = (s_1, \dots, s_q)$, define $n(\bar{s}, t) = \#\{i | t_i = t\}$. Then $(-1)^{n(\bar{s},t)}$ depends only on the expression $s_1 \dots s_q$.

To show this, we consider for $s \in S$, the map $U_s : \{\pm 1\} \times T \to \{\pm 1\} \times T$ defined by $U_s(\epsilon, t) = (\epsilon \cdot (-1)^{\delta_{s,t}}, sts^{-1})$.
The map $s \mapsto U_s$ extends to a homomorphism of $F(s)$, the free group on S into $\mathrm{Aut}(\{\pm 1\} \times T)$. We check that this goes down to W. For this, consider any string $\bar{s} = (s_1, \dots, s_q)$ and write $w = s_1 \dots s_q$ and $U_{\bar{s}} = U_{s_1} \dots U_{s_q}$. We show, by induction on q that $U_{\bar{s}}(\epsilon, t) = (\epsilon \cdot$

$(-1)^{n(\bar{s},t)}, wtw^{-1})$. This is clear for $q = 0$ or 1. For $q > 1$, write $\overline{s'} = (s_2, \ldots, s_q)$ and $w' = s_2 \ldots s_q$. We get

$$\begin{aligned} U_{\bar{s}}(\epsilon, t) &= U_{s_1}(\epsilon \cdot (-1)^{n(\overline{s'},t)}, w'tw'^{-1}) \\ &= (\epsilon \cdot (-1)^{n(\overline{s'},t)+\delta_{s_1,w'tw'1}}, wtw^{-1}). \end{aligned}$$

But $\Phi(\bar{s}) = (w'^{-1}s_1w', \Phi(\overline{s'}))$. So $n(\bar{s}, t) = n(\overline{s'}, t) + \delta_{w'^{-1}s_1w',t}$ and hence we have $U_{\bar{s}}(\epsilon, t) = (\epsilon \cdot (-1)^{n(\bar{s},t)}, wtw^{-1})$. So, to complete this step, it is enough to show that if $s, s' \in S$ and ss' has order r, then $(U_sU_{s'})^r =$ Identity. Consider the string $\bar{s} = (s_1, \ldots, s_{2r})$ where s_i is s for i odd and s' for i even. Then, $t_i = s_1 \ldots s_{i-1}s_is_{i-1} \ldots s_1 = (ss')^{i-1}s \ \forall\ 1 \leq i \leq 2r$. Thus $t_1, \ldots, t_r$ are distinct and $t_{r+i} = t_i$ for $1 \leq i \leq r$. So, $\forall\ t \in T, n(\bar{s}, t) = 0$ or 2. Thus $(U_sU_{s'})^r = U_{\bar{s}} =$ Identity.

Step 3: To complete the proof of the exchange property, let $w = s_1 \ldots s_q$ be a reduced expression. Since $\ell(sw) = q - 1$, let $sw = s'_1 \ldots s'_{q-1}$. Then $w = s_1 \ldots s_q = ss'_1 \ldots s'_{q-1}$ are reduced expressions for W. Now call the string $\bar{s} = (s_1, \ldots, s_q)$ and the string $\overline{s'} = (s, s'_1, \ldots, s'_q)$. Since $s \in \Phi(\overline{s'})$ and $n(\overline{s'}, s) = 1$, therefore $n(\bar{s}, s) =$ odd and hence $s \in \Phi(\bar{s})$ i.e. $s = s_1 \ldots s_{i-1}s_is_{i-1} \ldots s_1$ for some $1 \leq i \leq q$ i.e. $ss_1 \ldots s_{i-1} = s_1 \ldots s_i$.

(v) Either $\ell(swt) = \ell(w) < \ell(wt) = \ell(sw)$ or $\ell(swt) = \ell(w) > \ell(wt) = \ell(sw)$. Let us first assume $\ell(w) < \ell(wt)$.

If $w = s_1 \ldots s_q$ is a reduced expression, then $wt = s_1 \ldots s_q$ is a reduced expression. Since $\ell(swt) < \ell(wt)$, the exchange property implies that $ss_1 \ldots s_{i-1} = s_1 \ldots s_i$ for some $0 \leq i \leq q + 1$ where we denote $s_0 = s, s_{q+1} = t$.

Therefore if $i \leq q, ss_1 \ldots s_{i-1}s_i \ldots s_q = s_1 \ldots s_{i-1}s_{i+1} \ldots s_q$ i.e $sw = s_1 \ldots s_{i-1}s_{i+1} \ldots s_q$ i.e. $\ell(sw) \leq q - 1 < \ell(w)$, a contradiction.

Thus $i = q + 1$.

Therefore $ss_1 \ldots s_q = s_1 \ldots s_{q+1} \Rightarrow sw = wt$.

For the other case when $\ell(w) > \ell(wt)$, we use wt instead of w to get $s(wt) = (wt)t \Rightarrow sw = wt$.

(vi) For each $w \in W$, let D_w be the set of tuples $(s_1, \ldots, s_q)$ such that $w = s_1s_2 \ldots s_q$ is a reduced expression and let $F_w : D_w \rightarrow M$ be defined by $F_w(s_1, \ldots, s_q) = f(s_1) \ldots f(s_q)$.

We will show by induction on $\ell(w)$ that F_w is constant. The cases $\ell(w) = 0$ or 1 are trivial and we suppose that $\ell(w) = q \geq 2$ and assume the assertion proved for the elements with length $< q$. Let $s = (s_1, \ldots, s_q)$ and $s' = (s'_1, \ldots, s_{q'})$ be in D_w.

Firstly we prove $F_w(s) = F_w(s')$ in the two cases:

(1) $s_1 = s'_1$ or $s_q = s_{q'}$ and
(2) there exist $u, v \in S$ such that $s_j = s'_k = u$ and $s_k = s'_j = v$ for j odd and k even.
To prove the result in the case (1), we consider

$$F_w(s_1, \ldots, s_q) = f(s_1)F_{w''}(s_2, \ldots, s_q) = F_{w'}(s_1, \ldots, s_{q-1})$$

for $w' = s_1 s_2 \ldots s_{q-1}$ and $w'' = s_2 \ldots s_q$ and use the induction hypothesis to get $F_w(s) = F_w(s')$ if $s_1 = s'_1$ or $s_q = s_{q'}$.

In the case (2) when there are $u, v \in S$ such that $s_j = s'_k = u \neq v = s_k = s'_j$ for j odd and k even (the case $u = v$ is obvious). Let $s_1 \ldots s_q$ and $s'_1 \ldots s'_q$ be reduced expressions. Then, uv has finite order; in fact, if q is even, $s_1 \ldots s_q = (uv)^{q/2} = (vu)^{q/2} = s'_1 \ldots s'_q \Rightarrow (uv)^q = 1$ and if q is odd, $s_1 \ldots s_q = (uv)^{\frac{q-1}{2}} \cdot u = (vu)^{\frac{q-1}{2}} \cdot v = s'_1 \ldots s_{q'} \Rightarrow (uv)^q = 1$. Moreover, order of uv divides q and since $s_1 \ldots s_q$ is a reduced expression, therefore $q =$ order of uv. Then it is clear that $F_w(s) = a(u, v) = a(v, u) = F_w(s')$. Now, we will show, in general, that $F_w(s) = F_w(s')$ for any two strings $s = (s_1, \ldots, s_q)$ and $s' = (s'_1, \ldots, s'_q)$ in D_w. Let, if possible, $F_w(s) \neq F_w(s')$. Consider the string $t_1 = (s'_1, s_1, \ldots, s_{q-1})$. Now, $w = s'_1 \ldots s_{q'}$ is reduced $\Rightarrow s'_1 w < w = s_1 \ldots s_q \Rightarrow s'_1 w = s_1 \ldots s_{i-1} s_{i+1} \ldots s_q$ is a reduced expression for some $1 \leq i \leq q$. Thus $(s'_1, s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_q) \in D_w$ and its image under F_w is equal to $F_w(s')$ and is also equal to $F_w(s)$ unless $i = q$. Since $F_w(s) \neq F_w(s')$, we must have $i = q$ i.e. $(s'_1, s_1, \ldots, s_{q-1}) \in D_w$ i.e. $t_1 \in D_w$ and $F_w(t_1) = F_w(s') \neq F_w(s)$.

Recursively, again (taking s and t_1 in place of s' and s respectively), we get $t_2 = (s_1, s'_1, s_1, \ldots, s_{q-2}) \in D_w$ and $F_w(t_2) = F_w(s) \neq F_w(t_1)$ and so on. Finally, we get t_{q-1} and t_q to be of the form of case (2) (where we have proved the result) and such that $F_w(t_{q-1}) \neq F_w(t_w)$; this is a contradiction.

Hence, we have $F_w(s) = F_w(s') \; \forall \; s, s' \in D_w$.

2 Hecke Algebras

Definition 2.1 *Let (W, S) be a Coxeter system. We define the Hecke algebra H corresponding to (W, S) to be the free $\mathbb{Z}[q]$-module with a basis T_w, for each $w \in W$. The multiplication is defined by the rules*

$$\left.\begin{array}{rcl} T_w \cdot T_{w'} & = & T_{ww'} \text{ if } \ \ell(ww') = \ell(w) + \ell(w') \\ \text{and } \ T_s^2 & = & qT_e + (q-1)T_s \ \forall \ s \in S \end{array}\right\}. \tag{2.1}$$

Remark 2.1 *1. The fact that there is a unique associative algebra structure on H, satisfying (2.1), will be verified in Proposition 2.1.*

2. The same proposition will also show that the Hecke algebra can be defined purely in terms of generators $\{L_s\}_{s\in S}$ and the (slightly more general) set of relations ([I])

$$\left.\begin{array}{lcl} L_s^2 & = & q_s + q'_s L_s \ \forall \ s \in S \\ (L_sL_t)^{m_{st}} & = & (L_tL_s)^{m_{st}} \ \ if \ \ O(st) = 2m_{st} < \infty \\ (L_sL_t)^{m_{st}} \cdot L_s & = & (L_tL_s)^{m_{st}} \cdot L_t \ \ if \ \ O(st) = 2m_{st} + 1 < \infty \end{array}\right\} \tag{2.2}$$

where the constants q_s, q'_s satisfy $q_s = q_{s_0}$ and $q'_s = q'_{s_0}$ whenever $s_0 \in S$ is conjugate to s, in W; the earlier rules (2.1) arise as particular case on setting $q_s = q$ and $q'_s = q - 1$ for all s.

3. If W is the Weyl group of a Tits-system (G, B, N, S) with G finite, and if K is an algebraically closed field whose characteristic does not divide the orders of G and W, then it is a Theorem of Tits that $H_K \cong K[W]$. This is shown by proving that if A is an associative algebra over $K[t_1, \ldots, t_r]$ having finite rank as a free $K[t_1, \ldots, t_r]$-module and if the discriminant $\Delta(t_1, \ldots, t_r)$ of A relative to a basis of A is not zero, then for any two specializations $(\alpha_i), (\beta_i) \in K^r$ of $t_1, \ldots, t_r$ such that $\Delta(\alpha_i) \neq 0 \neq \Delta(\beta_i)$, one has $A(\alpha_i) \simeq A(\beta_i)$ as K-algebras.

Thus, if q is specialized to a prime power, we will have $H \bigotimes_{\mathbb{Z}[q]} \mathbb{C} \cong \mathbb{C}[W]$. But, the isomorphism itself may not be definable over $\mathbb{Z}[q]$, without introducting $q^{1/2}$. We will show in §4 that over $Q(q^{1/2})$, there is a canonical isomorphism (Theorem 4.1). That $q^{1/2}$ is necessary can be seen through examples after

Theorem 4.1. The representation theory of a Chevalley group G over $\mathbb{F}_q$ is connected with the study of irreducible representations of $H \bigotimes_{\mathbb{Z}[q]} \mathbb{C}$.

Concrete realization of H as an algebra of double cosets

The Hecke algebra can also be realized as an algebra of double cosets as follows. Let G be a group and B be a subgroup such that $[B : B \cap \sigma B \sigma^{-1}]$ is finite for all σ in G. Let $H = H(G, B)$ be the free $\mathbb{Z}$-module spanned by the double cosets $T_\sigma = B\sigma B (\sigma \in G)$. A multiplication is defined in H as $T_\sigma \cdot T_\tau = \sum_\mu m^\mu_{\sigma,\tau} \cdot T_\mu$ where $m^\mu_{\sigma,\tau}$ is the number of cosets Bx which are contained in $B\sigma^{-1}B\mu \cap B\tau B$.

Note that $B \cap x^{-1}Bx \backslash B \to B \backslash BxB$ is a bijection

$$(B \cap x^{-1}Bx)y \mapsto Bxy$$

and so $\#(B \cap x^{-1}Bx \backslash B) = \#(B \backslash BxB) < \infty \ \forall \ x \in G$.

Thus each double coset is a finite union of right cosets.

Now, it can be seen that $m^\mu_{\sigma,\tau}$ is independent of the choice of the representatives σ, τ, μ in the double coset and that, for a given σ, τ, the number of double cosets $B\mu B$ satisfying $m^\mu_{\sigma,\tau} \neq 0$ is finite. $H(G, B)$ becomes an associative algebra with the unit element over $\mathbb{Z}$. For example, let $\mathbb{F}_q$ be a finite field and consider $G = GL_n(\mathbb{F}_q)$, $B =$ upper triangular matrices inside G. Then, the set of double cosets is in bijection with S_n. (2.1) can be easily verified. For instance, for $s = (12)$, the set $B \cap sBs^{-1} = \{b \in B/b_{12} = 0\}$ and since $m^e_{s,s}$ is the index of B in $BsB =$ Index of $B \cap sBs^{-1}$ in B, we have $m^e_{s,s} = q$ as $B = \bigcup_{a=0}^{q-1} (B \cap sBs^{-1})u^a$, where u is the unipotent matrix whose only non-zero diagonal entry is $u_{12} = 1$.

Let us now show that the Hecke algebra can be defined by generators and relations as in (2.2).

Proposition 2.1 *Let $\mathcal{C}$ be the set of conjugacy classes of elements of S and let $\{u_c, v_c, c \in \mathcal{C}\}$ be indeterminates over $\mathbb{Z}$. We write u_s, v_s (for s in c)*

instead of u_c, v_c. Let V be the free $\mathbb{Z}[u_c, v_c]$-module spanned by W. Then, there exists an associative multiplication $*$ in V such that

$$s * w = \left\{ \begin{array}{ll} sw, & \text{if } \ell(sw) > \ell(w), \text{ and,} \\ u_s sw + v_s w, & \text{if } \ell(sw) < \ell(w) \end{array} \right\}. \tag{2.3}$$

Such a multiplication is unique. Also, then (2.3) are the defining relations for H over $\mathbb{Z}[u_c, v_c; c \in \mathcal{C}]$.

Proof: First note that the existence of a unique associative multiplication V satisfying (2.3) shows that (2.3) form defining relation for H, by virtue of Proposition 1 (vi).

Now, the uniqueness of the multiplication is clear from its associativity and from (2.3) let us show the existence.

Define $P_s \in$ End (V) for $s \in S$ by the right hand side of (2.3) and $Q_s \in$ End (V) similarly with s on the right instead of the left.

Now, Proposition 1(v) gives

$$P_s Q_t = Q_t P_s \ \forall \ s, t \in S. \tag{2.4}$$

Let $\mathcal{R}$ and $\mathcal{J}$ denote the sublagebras of End(V) generated by $\{P_s : s \in S\}$ and $\{Q_s; s \in S\}$ respectively. Then, the maps

$$\begin{array}{rlr} P : \mathcal{R} \to V & \text{and} & \psi : \mathcal{L} \to V \\ f \mapsto f(1) & & s \mapsto g(1) \end{array}$$

are bijective. In fact, for any reduced expression $w = s_1 \dots s_n$, we have $\varphi(P_{s_1} \dots P_{s_n}) = w$ from (2.3). Thus φ and ψ are surjective. Also, if $\varphi(f) = 0$, then $f(1) = 0 \Rightarrow g(f(1)) = 0 \ \forall \ g \in \mathcal{L}$

$$\begin{array}{ll} \Rightarrow & f(g(1)) = 0 \text{ from } f(1) = 0 \Rightarrow g(f(1)) = 0 \ \forall \ g \in \mathcal{L} \\ \Rightarrow & f = 0 \text{ from the surjectivity of } \psi. \end{array}$$

Thus, one can define the product $v * v' = \varphi(\varphi^{-1}(v)\varphi^{-1}(v^{-1}))$ for $v, v' \in V$.

Definition 2.2 *Let H be the Hecke algebra corresponding to a Coxeter system (W, S), and over $\mathbb{Z}[q, q^{-1}]$. We define a map: $H \to H$ by*

$$\overline{\sum_w p_w(q) T_w} = \sum_w p_w(q^{-1}) T_{w^{-1}}^{-1}$$

for $p_w(q) \in \mathbb{Z}[q, q^{-1}]$ and $\forall\ w \in W$.

Note that each T_w is invertible for H over $\mathbb{Z}[q, q^{-1}]$.

Lemma 2.2 *$- : H \to H$ is a ring involution.*

Proof: The only thing to check is that $\overline{T_s \cdot T_s} = \bar{T}_s \cdot \bar{T}_s$ for $s \in S$. But

$$\begin{aligned} \overline{T_s \cdot T_s} &= \overline{q + (q-1)T_s} = q^{-1} + (q^{-1} - 1)T_s^{-1} \\ &= (q^{-1}T_s + q^{-1} - 1)T_s^{-1} = T_s^{-1}T_s^{-1} = \bar{T}_s \cdot \bar{T}_s. \end{aligned}$$

Definition 2.3 *(Standard partial order on any Coxeter system)*

For $w, w' \in W$, we say $w \leq w'$ if w is obtained from w' by dropping some elements from a reduced expression for w'.

Theorem 2.3 *(Kazhdan-Lusztig) [KL1]*

For all $w \in W$, there exists a unique element $c'_w \in H$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ such that
(i) $\overline{c_{w'}} = c_{w'}$ and (ii) $c_{w'} = q^{-\frac{1}{2}\ell(w)} \sum\limits_{y \leq w} P_{y,w}(q) \cdot T_y$
where $P_{y,w}(q) \in \mathbb{Z}[q]$ is

$$\begin{cases} \text{of degree } \leq \frac{\ell(w) - \ell(y) - 1}{2} \text{ if } y < w \\ \equiv 1 \text{ if } y = w. \end{cases}$$

Before embarking upon the proof, let us see how this theorem is related to a problem concerning Verma modules - 'The Kazhdan-Lusztig Conjecture'. (Now solved independently by Brylinski-Kashiwara [BK] and Bernstein-Beilinson [BB].)

Let $\mathcal{J}$ be a complex reductive Lie algebra, $h \subseteq \mathcal{J}$ a fixed Cartan subalgebra, and $b = h + n$ a fixed Borel subalgebra containing h, with nil radical h. Let $\hat{\mathcal{J}}$ denote the set of equivalence classes of irreducible (possibly infinite dimensional) $\mathcal{J}$-modules L such that

(i) the center $Z(\mathcal{J})$ of the enveloping algebra of $\mathcal{J}$ acts on L as it does in the trivial representation; and

(ii) L has a highest weight vector i.e. a vector $v \neq 0$ killed by n.

Denoting $\rho = \rho(n) = \frac{1}{2} \sum_{\alpha \in s} \alpha$ and $W =$ Weyl group of h in $\mathcal{J}$, one knows from Harish-Chandra's theorem on $Z(\mathcal{J})$ that, for $L \in \hat{\mathcal{J}}$, the highest weight vector v is unique upto a scalar multiple and there is a $w \in W$ such that v has weight $-(w\rho + \rho)$ for h. This defines a bijection $\hat{\mathcal{J}} \leftrightarrow W$.

So, for $w \in W$, let L_w denote the unique element of $\hat{\mathcal{J}}$ of highest weight $-(w\rho + \rho)$ i.e. $M_w = U(\mathcal{J}) \otimes_b \mathbb{C}_{-(w\rho+\rho)}$. It is known that M_w has a finite composition series and all its irreducible composition factors lie in $\hat{\mathcal{J}}$. If we denote by $[L_w]$ and $[M_w]$, their formal characters, then we have the following theorem.

Theorem 2.4 $[M_w] = \sum_{y<w} m_{y,w}[L_y] + [L_w]$ *where $m_{y,w}$ are integers ≥ 0, the multiplicities of L_y in M_w. Further, these formulae can be inverted to give $[L_w] = \sum_{y<w} M_{y,w}[M_y] + [M_w]$ where $M_{y,w}$ are some integers.*

Also, M_{y,w_0} a $(-1)^{\ell(y)-\ell(w_0)}$, where w_0 is the largest element (this is Weyl character formulas). Kazhdan-Lusztig conjecture is an algorithm for computing $M_{y,w}$. It says that $M_{y,w} = (-1)^{\ell(y)-\ell(w)} \cdot P_{y,w}(1)$. (See Cor. 4.2.1 to deduce the Weyl character formula from Kazhdan-Lusztig conjecture.)

We will show in Corollary 2.10 that this is equivalent to saying that $m_{y,w} = P_{ww_0,yw_0}(1)$ where w_0 is the unique longest element of w. Later, we will prove a geometric version of the conjecture in relation to singularities of Schubert varieties (Theorem 3.4).

Before proving Theorem 2.3, we will define, for $y, w \in W, y \prec w$ if (a) $y < w$, (b) $\ell(y) \not\equiv \ell(w) \bmod 2$ and (c) $P_{y,w}$ (once it is known) has a non-zero coefficient of $q^{\frac{\ell(w)-\ell(y)-1}{2}}$. For $y \prec w$, we will write $\mu(y,w)$ for this coefficient.

We also define $y \mapsto w$ if either $y \prec w$ or $w \prec y$. $\mu(y,w)$ is also defined for $y \mapsto w$ if we write $\mu(y,w) = \mu(w,y)$ if $w \prec y$.

Proof of Theorem 2.3 We first note that if we put $\tilde{T}_w = q^{-\frac{\ell(w)}{2}} \cdot T_w \ \forall\, w \in W$, the statement of the theorem is equivalent to finding for any $w \in W$, a unique element c_w in H such that $\bar{c}_w = c_w$ and $c_w = \sum_{y \leq w} (-1)^{\ell(y)+\ell(w)} q^{\frac{\ell(w)-\ell(y)}{2}} \cdot \overline{P_{y,w}(q)} \cdot \tilde{T}_y$ where $P_{y,w}$ is a polynomial is q of degree $\leq \frac{\ell(w)-\ell(y)-1}{2}$ for $y < w$ and such that $P_{w,w} \equiv 1$. (See Corollary 2.6 for the relation between c_w and c'_w). We shall prove the theorem *in this formulation*.

Let us write $\bar{T}_y = \sum_x \overline{R^{(q)}_{x,y}} \cdot q^{-\ell(x)} \cdot T_x$, where $R_{x,y} \in \mathbb{Z}[q, q^{-1}]$. We can inductively compute $R_{x,y}$ as

$$R_{x,y} = \begin{cases} R_{sx,sy}, & \text{if } sx < x \text{ and } sy < y \\ R_{xs,ys}, & \text{if } xs < x \text{ and } ys < y \\ (q-1)R_{sx,y} + qR_{sx,sy}, & \text{if } sx > x \text{ and } sy < y. \end{cases}$$

It is clear that $R_{x,y} \neq 0 \Leftrightarrow x \leq y$ and that $R_{x,y} \in \mathbb{Z}[q]$ is of degree $\leq \ell(y) - \ell(x)$.

Uniqueness The equation $\bar{c}_w = c_w$ can be written as

$$\begin{aligned} & \sum_{x \leq w} (-1)^{\ell(x)+\ell(w)} \cdot q^{\frac{\ell(w)}{2}} \cdot q^{-\ell(x)} \overline{P_{x,w}} \cdot T_w \\ = & \sum_{y \leq w} (-1)^{\ell(y)+\ell(w)} \cdot q^{-\frac{\ell(w)}{2}} \cdot q^{\ell(y)} P_{y,w} \cdot \left(\sum_{x \leq y} q^{-\ell(x)} \cdot \overline{R_{x,y}} T_x \right). \end{aligned}$$

Equivalently, $\forall\, x \leq w$,

$$(-1)^{\ell(x)+\ell(w)} \cdot q^{\frac{\ell(w)}{2}} \cdot q^{-\ell(x)} \cdot \overline{P_{x,w}} =$$
$$\sum_{\substack{y \\ x \leq y \leq w}} (-1)^{\ell(y)+\ell(w)} \cdot q^{-\frac{\ell(w)}{2}} \cdot q^{\ell(y)-\ell(x)} \cdot \overline{R_{x,y}} \cdot P_{y,w}$$

i.e.,

$$\left(q^{\frac{\ell(w)-\ell(x)}{2}}\overline{P_{x,w}} - q^{\frac{\ell(x)-\ell(w)}{2}} \cdot P_{x,w}\right)(-1)^{\ell(x)+\ell(w)} =$$

$$\sum_{\substack{y \\ x<y\leq w}} (-1)^{\ell(x)+\ell(y)} \cdot q^{\ell(y)-\frac{\ell(w)}{2}-\frac{\ell(x)}{2}} \cdot \overline{R_{x,y}} P_{y,w} \ \forall\ x < w.$$

Thus, if $P_{y,w}$ for all y in $x < y \leq w$ are known ($x < w$ fixed), then the above equation cannot have more than one solution for $P_{x,w}$ since no cancellations can take place on the left-hand side due to the fact that $q^{\frac{\ell(w)-\ell(x)}{2}} \cdot \overline{P_{x,w}}$ is a polynomial in $q^{1/2}$ without constant terms and $q^{\frac{\ell(x)-\ell(w)}{2}} \cdot P_{x,w}$ is a polynomial in $q^{-1/2}$ without constant term. Hence, the uniqueness follows by induction on the length function.

Existence Clearly $C_e = T_e$.

Assume as induction hypothesis, that the existence of $c_{w'}$ has been already proved for each w' of length $< \ell(w)$.

Write $w = sv$ with $s \in S$ such that $\ell(w) = \ell(v) + 1$. Define

$$c_w = \left(q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}\right) c_v - \sum_{\substack{z \prec v \\ sz<z}} \mu(z,v) c_z. \tag{2.5}$$

Observe that $\overline{q^{-1/2}T_s - q^{1/2}} = q^{-1/2} \cdot T_s - q^{1/2}$ so that $\overline{c_w} = c_w$. So, we can write $c_w = \sum_{y\leq w} (-1)^{\ell(y)+\ell(w)} \cdot q^{\frac{\ell(w)}{2}-\ell(y)} \cdot \overline{P_{y,w}} T_y$, where

$$P_{y,w} = q^{1-c} \cdot P_{sy,v} + q^c \cdot P_{y,v} - \sum_{\substack{z \\ y\leq z\prec v \\ sz<z}} \mu(z,v) \cdot q^{\frac{\ell(v)-\ell(z)+1}{2}} \cdot P_{y,z} \text{ for } y \leq w, \tag{2.6}$$

and

$$c = \begin{cases} 1, & \text{if } sy < y \\ 0, & \text{if } sy > y. \end{cases}$$

We have also used the connection $P_{x,v} = 0$ for $x \not\leq v$.

By induction hypothesis, it is now clear that $P_{y,w}$ is a polynomial in q of degree $\leq \frac{\ell(v)-\ell(y)}{2}$ if $y < w$ and that $P_{w,w} \equiv 1$.

Thus, the proof of the theorem is complete. (Note that the proof gives an algorithm to compute $P_{y,w}$'s.)

Corollary (of Proof)

(i) If $v \in sv$, then $T_s c_v = qc_v + q^{1/2} c_{sv} + q^{1/2} \cdot \sum_{\substack{z \prec v \\ sz < z}} \mu(z,v) c_z$.

(ii) If $v > sv$, then $T_s c_v = -c_v$.

Proof: (i) is just the equation (2.5).

(ii) Replace v by sv in (i) to get

$$\begin{aligned} T_s c_{sv} &= qc_{sv} + q^{1/2} \cdot c_v + q^{1/2} \cdot \sum_{\substack{z \prec sv \\ sz < z}} \mu(z, sv) c_z \\ \Rightarrow q^{1/2} T_s c_v &= (q + (q-1)T_s) c_{sv} - qT_s c_{sv} - q^{1/2} \cdot \sum_{\substack{z \prec sv \\ sz < z}} \mu(z, sv) T_s c_z \\ &= -q^{1/2} c_v \text{ by induction.} \end{aligned}$$

Corollary 2.5 *(i) If $sv < v$, then $c'_s c'_v = (q^{1/2} + q^{-1/2}) c'_v$*

(ii) If $sv > v$, then $c'_s c'_v = c'_{sv} + \sum_{\substack{z \prec v \\ sz < z}} \mu(z,v) c'_z$.

Proof: Firstly, we note that the connection between c'_w is given as $c'_w = (-1)^{\ell(w)} \cdot j(c_w)$ where j is the algebra involution defined by

$$j(a_w(q) \cdot T_w) = (-1)^{\ell(w)} \cdot q^{-\ell(w)} \cdot \overline{a_w(q)} \cdot T_w.$$

(i)

$$\begin{aligned} c'_s c'_v &= (-1)^{\ell(v)+1} \cdot j(c_s) \cdot j(v) = (-1)^{\ell(v)+1} \cdot j c_s c_v \\ &= (-1)^{\ell(v)+1} \cdot j(-(q^{1/2} + q^{-1/2}) c_v) \\ &= q^{-1/2} T_s - q^{1/2} = (q^{1/2} + q^{-1/2}) c'_v \end{aligned}$$

using (ii) of the previous corollary.

(ii)

$$\begin{aligned}
C'_s C'_v &= (-1)^{\ell(w)+1} \cdot j \left(C_{sv} + \sum_{\substack{z \prec v \\ sz < z}} \mu(z,v) C_z \right) \\
&\quad \text{since } sv > v \text{ and from Cor.2.5(i)} \\
&= C'_{sv} + \sum_{\substack{z \prec v \\ sz < z}} (-1)^{\ell(v)-\ell(z)+1} \cdot \mu(z,v) \cdot C'_z \\
&= C'_{sv} + \sum_{\substack{z \prec v \\ sz < z}} \mu(z,v) C'_z \text{ since } z \prec v \Rightarrow \ell(v) \not\equiv \ell(z) \bmod 2.
\end{aligned}$$

We give a simpler proof of Theorem 2.3 by Gabber.

Proof (Gabber) As before, the problem is to find $\forall\ x \in W$, a unique element $C_x \in H$ such that $\bar{C}_x = C_x$ and $C_x = \sum_{y \le x} \pi_{y,x}^{(q)} \tilde{T}_y$ where $\pi_{y,x}(q) \in \mathbb{Z}[q^{1/2}]$ is without constant term and has degree (as a polynomial in $q^{1/2}$) $\le \frac{\ell(x)-\ell(y)}{2}$ for $y < x$ and $\pi_{x,x} \equiv 1$.

Let us write, for convenience $\overline{\tilde{T}_y} = \sum_{x \le y} r_{x,y} \cdot \tilde{T}_x$ i.e.

$$r_{x,y} = q^{\frac{\ell(y)-\ell(x)}{2}} \cdot \overline{R_{x,y}} \text{ for } x \le y.$$

We also know that $R_{x,y} \in \mathbb{Z}[q]$ is of degree $\ell(y) - \ell(x)$ for $x \le y$.

Uniqueness (This is completely similar)

The condition $C_x = \overline{C_x}$ is equivalent to

$$\begin{aligned}
C_x &= \sum_{z \le x} \sum_{\substack{y \\ z \le y \le x}} \overline{\pi_{y,x}} r_{z,y} \tilde{T}_z \\
i.e. \sum_{z \le x} \pi_{z,x} \cdot \tilde{T}_z &= \sum_{\substack{z \\ z \le x}} \sum_{\substack{y \\ z \le y \le x}} \bar{\pi}_{y,x} r_{z,y} \tilde{T}_z \\
i.e.\ \pi_{z,x} &= \sum_{\substack{y \\ z \le y \le x}} \bar{\pi}_{y,x} r_{z,y}\ \forall\ z \le x \\
i.e.\ \pi_{z,x} - \bar{\pi}_{z,x} &= \sum_{\substack{y \\ z < y \le x}} \bar{\pi}_{y,x} r_{z,y}\ \forall\ z \le x. \qquad (2.7)
\end{aligned}$$

Now, if $\pi_{y,x}$ is known $\forall\ y$ such that $z < y \le x$ (for a fixed $z < x$), then (2.8) gives at the most one solution for $\pi_{z,x}$ since $\pi_{z,x}$ and $\bar{\pi}_{z,x}$ would be in $\mathbb{Z}[q^{1/2}]$ and $\mathbb{Z}[q^{-1/2}]$ respectively and are without constant terms so that no cancellations can take place on the left hand side of (2.8). Thus, by induction, the uniqueness follows $\forall\ \pi_{x,w},\ \forall\ y \le w$ in W.

Exsitence: Clearly $C_e = \tilde{T}_e$. We will use induction. Call the RHS of (2.8) to be $\sum\limits_{i \in \mathbb{Z}} C_i q^{i/2} = \varphi(q)$, say, which is assumed to be known. Then, if we show $\varphi(q) + \overline{\varphi(q)} = 0$, it is clear that from (2.8) we can define $\pi_{z,x} = \sum\limits_{i>0} C_i q^{i/2}$.

Therefore, let us show $\varphi + \bar{\varphi} = 0$.

Now $\varphi + \bar{\varphi} = \sum\limits_{\substack{y \\ z<y\le x}} \bar{\pi}_{y,x} r_{z,y} + \sum\limits_{\substack{v \\ z<v\le x}} \pi_{v,x} \overline{r_{z,v}}$. Writing for $\pi_{v,x}$ from (2.8) since $z < v \le x$, we have

$$\pi_{v,x} = \sum_{\substack{y \\ v\le y\le x}} \bar{\pi}_{y,x} \cdot r_{v,y}.$$

Therefore

$$\begin{aligned}
\varphi + \bar{\varphi} &= \sum_{\substack{y \\ z<y\le x}} \bar{\pi}_{y,x} r_{z,y} + \sum_{\substack{y,v \\ z<v\le y\le x}} \bar{p}i_{y,x} \bar{r}_{z,v} r_{v,y} \\
&= \sum_{\substack{y \\ z<y\le x}} \bar{\pi}_{y,x} r_{z,y} + \sum_{\substack{y \\ z<\le x}} \left\{ \bar{\pi}_{y,x} \cdot \sum_{\substack{v \\ z<v\le y}} \overline{r_{z,v}} r_{v,y} \right\} \\
&= \sum_{\substack{y \\ z<y\le x}} \left\{ \bar{\pi}_{y,x} \cdot \sum_{\substack{v \\ z\le v\le y}} \overline{r_{z,v}} \cdot r_{v,y} \right\} = \sum_{\substack{y \\ z<y\le x}} \bar{\pi}_{y,x} \cdot \delta_{z,y} = 0.
\end{aligned}$$

Also, since $R_{x',y'} \in \mathbb{Z}[q]$ is of degree $\ell(y') - \ell(x')$ for each $x' \le y'$ in W.

Therefore $r_{x',y'} = \sum\limits_{i=-a}^{a} C_i q^{i/2}$, where $a = \ell(y') - \ell(x')\ \forall\ x' \le y'$ in W. By induction hypothesis, if we assume $\pi_{y,x} = \sum\limits_{i=1}^{u} a_i q^{i/2}$, where $u = \ell(x) -$

$\ell(y)\ \forall\ y, z < y \leq x$, then from (2.8) we see that $\pi_{z,x}$ is a polynomial in $q^{1/2}$ with degree ≥ 1 and $\leq \ell(x) - \ell(z)$ (since $\sum\limits_{\substack{y \\ z<y\leq x}} \bar{\pi}_{y,x} r_{z,y} = \sum\limits_{i=-c}^{c} b_i q^{i/2}$ with $c = \ell(x) - \ell(z)$). Hence, the proof is complete.

We have the following which will be used in Theorem 2.9:-

Lemma 2.6 *(i)* $\overline{R_{x,y}} = (-1)^{\ell(x)+\ell(y)} \cdot q^{\ell(x)-\ell(y)} R_{x,y}$.

(ii) $\sum\limits_{x\leq t\leq y} (-1)^{\ell(t)+\ell(x)} \cdot R_{x,t} \cdot R_{t,y} = \delta_{x,y}\ \forall\ x \leq y$.

(iii) If W is finite and w_0, its longest element, then $R_{w_0y,w_0x} = R_{x,y}$.

Proof: Firstly recall that $R_{x,y}$ are defined by

$$\bar{T}_y = T_{y^{-1}}^{-1} = \sum_{x\leq y} \overline{R_{x,y}} \cdot q^{-\ell(x)} \cdot T_x$$

and we have the inductive procedure for their computations, as follows:

$$R_{x,y} = \left\{ \begin{array}{ll} R_{sx,sy} & \text{if} \quad sx < x \text{ and } sy < y \\ R_{xs,ys} & \text{if} \quad xs < x \text{ and } ys < y. \end{array} \right\} \tag{2.8}$$

and

$$R_{x,y} = (q-1)R_{sx,y} + qR_{sx,sy} \text{ if } sx > x \text{ and } sy < y. \tag{2.9}$$

(i) We assume the result to be time for all y' with $\ell(y') \leq \ell(y')$, and take $s \in S$ such that $sy > y$. Also, we assume that the result holds for all $R_{u,sy}$ with $\ell(x) < \ell(u)$ where $u \leq sy$.

We will prove the result for $R_{x,sy}$.

If $sx < x$, then the equations (2.9) give $R_{x,sy} = R_{sx,y}$ and so

$$\begin{aligned} \overline{R_{x,sy}} &= \overline{R_{sx,y}} = (-1)^{\ell(sx)+\ell(y)} \cdot q^{\ell(sx)-\ell(y)} \cdot R_{sx,y} \text{ by induction hypothesis} \\ &= -(-1)^{\ell(x)+\ell(y)} \cdot q^{\ell(x)-1-\ell(y)} \cdot R_{x,sy} \text{ since } \ell sx = \ell(x) - 1 \\ &= (-1)^{\ell(x)+\ell(sy)} \cdot q^{\ell(x)-\ell(sy)} \cdot R_{x,sy}. \end{aligned}$$

If $sx > x$, then (2.10) gives $R_{x,sy} = (q-1)R_{sx,sy} + qR_{sx,y}$ so that

$$\begin{aligned}
\overline{R_{s,xy}} &= (q^{-1}-1)(-1)^{\ell(sx)+\ell(sy)} \cdot q^{\ell(sx)} \cdot q^{-\ell(sy)} \cdot R_{sx,sy} \\
&\quad + q^{-1}(-1)^{\ell(sx)+\ell(y)} \cdot q^{\ell(sx)-\ell(y)} \cdot R_{sx,y} \\
&= (-1)^{\ell(sx)+\ell(y)} \cdot q^{\ell(x)-\ell(sy)} \cdot \{-q(q^{-1}-1)R_{sx,sy} + q^{-1} \cdot q^2 \cdot R_{sx,y}\} \\
&\quad \text{since } sx > x, sy > y \\
&= (-1)^{\ell(sx)+\ell(y)} \cdot q^{\ell(x)-\ell(sy)} \cdot R_{x,sy}.
\end{aligned}$$

(ii) Now

$$\begin{aligned}
\overline{T_y} &= \sum_{t \le y} \overline{R_{t,y}} q^{\ell(t)} \cdot T_t \\
&\Rightarrow\ T_y = \overline{\overline{T_y}} = \sum_{t \le y} R_{t,y} q^{+\ell(t)} \left(\sum_{x \le t} \overline{R_{x,t}} q^{-\ell(x)} T_x \right) \\
&= \sum_{x \le y} \sum_{\substack{t \\ x \le t \le y}} R_{t,y} q^{\ell(t)} \cdot \overline{R_{x,t}} q^{-\ell(x)} T_x.
\end{aligned}$$

Therefore, $\sum_{\substack{t \\ x \le t \le y}} R_{t,y} q^{\ell(t)} \overline{R_{x,t}} q^{-\ell(x)} = \delta_{x,y}\ \forall\ x \le y.$

Therefore, from (i), we will get

$$\sum_{\substack{t \\ x \le t \le y}} R_{t,y} \cdot (-1)^{\ell(x)+\ell(t)} \cdot R_{x,t} = \delta_{x,y}\ \forall\ x \le y.$$

(iii) Again, we apply induction. Note that $w_0 u < w_0 v \Leftrightarrow u > v$. Assume the result for all $R_{a,b}$ with $\ell(b) \le \ell(y)$ and also for all $R_{x',ys}$ with $\ell(x') > \ell(x)$ where $x' \le ys$.

We will prove the result for $R_{x,ys}$.

If $xs < x$, then

$$\begin{aligned}
R_{x,ys} &= R_{xs,y} \text{ from (2.9)} \\
&= R_{w_0 y, w_0 xs} \text{ by induction hypothesis} \\
&= R_{w_0 ys, w_0 x} \text{ by (2.9).}
\end{aligned}$$

If $xs > x$, then $R_{x,ys} = (q-1)R_{xs,ys} + qR_{xs,y}$ by (2.10) $= (q-1)R_{w_0ys,w_0xs} + qR_{w_0y,w_0xs} = R_{w_0ys,w_0x}$.

Theorem 2.7 *Assuming W to be finite and w_0 to be its longest element, we have* $\sum\limits_{\substack{z \\ x\le z\le y}} (-1)^{\ell(x)+\ell(z)} \cdot P_{x,z} \cdot P_{w_0y,w_0z} = \delta_{x,y}\ \forall\ x \le y$ *in W.*

Proof: Let $M_{x,y}$ be the left-hand side above.

As induction hypothesis, we assume that $x < y$ and that $M_{t,s} = 0$ for all $t < s$ in W such that $\ell(s) - \ell(t) < \ell(y) - \ell(x)$.

We start with the identity equation (2.5) in the Proof of Theorem 2.3):

$$P_{x,z} = \sum_{x\le t\le z} (-1)^{\ell(x)+\ell(t)} \cdot R_{x,t}\overline{P_{t,z}}q^{-\ell(t)+\ell(z)}\ \forall\ x \le z \text{ in } W.$$

Substituting this and for P_{w_0y,w_0z} in $M_{x,y}$, we get

$$M_{x,y} = \sum_{\substack{t,s \\ x\le t\le s\le y}} (-1)^{\ell(y)+\ell(s)} \cdot q^{-\ell(t)+\ell(s)} \cdot R_{x,t}R_{w_0y,w_0s} \cdot \overline{M_{t,s}}.$$

By induction hypothesis, the only t, s which contribute to the above sum are those such that either $t = s$ or $t = x$ and $s = y$. Thus

$$M_{x,y} = q^{-\ell(x)+\ell(y)} \cdot \overline{M_{x,y}} + \sum_{x\le t\le y} (-1)^{\ell(y)+\ell(t)} \cdot R_{x,t} \cdot R_{w_0y,w_0t}.$$

But

$$\begin{aligned} & \sum_{\substack{t \\ x\le t\le y}} (-1)^{\ell(y)+\ell(t)} \cdot R_{x,t} \cdot R_{w_0y,w_0t} \\ = \ & (-1)^{\ell(y)-\ell(x)} \sum_{\substack{t \\ x\le t\le y}} (-1)^{\ell(x)+\ell(t)} \cdot R_{x,t} \cdot R_{t,y} \text{ by Lemma 2.9(iii)} \\ = \ & (-1)^{\ell(y)-\ell(x)} \cdot \delta_{x,y} \text{ by Lemma 2.9(ii)} \\ = \ & 0. \end{aligned}$$

Therefore $x < y$. Thus $M_{x,y} = q^{-\ell(x)+\ell(y)} \cdot \overline{M_{x,y}}$.

Hence

$$q^{\frac{\ell(x)-\ell(y)}{2}} \cdot M_{x,y} = q^{-(\frac{\ell(x)-\ell(y)}{2})} \cdot \overline{M_{x,y}}. \tag{2.10}$$

But the bounds on the degree of the polynomials $P_{y,w}$ give that $q^{-(\frac{\ell(x)-\ell(y)}{2})} \cdot \overline{M_{x,y}}$ is a polynomial in $q^{1/2}$ without constant term since ${}_{x,z} \cdot P_{w_0 y, w_0 z} = P_{x,z}$: $P_{z,y}$ is a polynomial in q of degree $\leq \frac{\ell(y)-\ell(x)-2}{2}$.

Thus (2.11) is not possible unless it is $\equiv 0$. Hence, the theorem is proved.

Corollary 2.8 *For finite W, $\sum\limits_{\substack{z \\ x \leq z \leq y}} (-1)^{\ell(x)+\ell(z)} = \delta_{x,y} \; \forall \; x \leq y$ in W.*

Proof: Putting $q = 0$ in Theorem 2.9, we get

$$\sum_{\substack{z \\ x \leq z \leq y}} (-1)^{\ell(x)+\ell(z)} \cdot P^{(0)}_{x,z} P^{(0)}_{w_0 y, w_0 z} = \delta_{x,y} \; \forall \; x \leq y \text{ in } W.$$

But, $\forall \; x \leq y$ in any Coxeter group $W, P_{x,y}$ is a polynomial in q with constant term 1 as seen (by the equation (2.6) in the proof of Theorem 2.3) by induction.

3 Intersection Cohomology and Schubert Varieties

Definition 3.1 *Let X be a complex algebraic variety of dimension n. Then X admits a locally finite decomposition into disjoint connected nonsingular analytic subvarieties $\{X_\alpha\}$ of varying dimension called strata, which satisfies a homogeneity condition along the strata: for any two points p and q on a stratum X_α, there exists a homeomorphism of X to X, preserving all the strata and taking p to q. If c_α denotes the codimension of X_α, then the space Σ which is the union of all X_α such that $c_\alpha > 0$ contains all singularities of X.*

Example: If G is a connected semi-simple linear algebraic group $/\mathbb{C}$, and $\mathcal{B} = G/B$ is the variety of all Borel subgroups of G, then for w in the Weyl group W of G, we define the Schubert cell

$$\mathcal{B}_w = \{B^1 \in \mathcal{B} | B^1 = gBg^{-1}, \text{ for some } g \in BwB\}.$$

The Schubert variety $\overline{\mathcal{B}_w}$ is the closure of $\mathcal{B}_w$ in $\mathcal{B}$. It can be verified that $\{\mathcal{B}_x\}_{x \leq w}$ is a stratification of $\mathcal{B}_w$ with the given properties.

Remark 3.1 *We will attach to $\overline{\mathcal{B}}_w$, a collection $\{\mathcal{H}^i(\overline{\mathcal{B}}_w)\}$ of sheaves of vector-spaces on $\overline{\mathcal{B}}_w$. If $\overline{\mathcal{B}}_w$ is smooth at a point x, then the stalks at x satisfy $\mathcal{H}^i_x(\overline{\mathcal{B}}_w) = \begin{cases} 0, i \neq 0 \\ \mathbb{C}, i = 0. \end{cases}$*

If $\overline{\mathcal{B}}_w$ is singular at x, the stalks $\mathcal{H}^i_x(\overline{\mathcal{B}}_w)$ measure the failure of local Poincaré duality at x.

Definition 3.2 *Let X be an algebraic variety $/\mathbb{C}$.*

1. *A sheaf $\mathcal{F}$ of complex vector-spaces on X, is said to be constructible if $X = \bigcup_{\text{finite}} X_i$, where X_i are locally closed and $\mathcal{F}/X_i$ is locally constant.*
   (Recall that a sheaf S on X is locally constant if $\forall\ x \in X$, $\exists$ a neighbourhood U such that $S_x \leftarrow \Gamma(U, S) \rightarrow S_y$ are isomorphisms for each $y \in U$).

2. *If $\hat{\mathcal{F}}$ denotes a complex of sheaves (of degrees ≥ 0), then $\hat{\mathcal{F}}$ is said to be constructible iff the homology sheaves are all constructible and are zero for large degrees. Here the cohomology sheaf $H^p(\hat{\mathcal{F}})$ is defined to be the sheafification of the presheaf whose space of sections over an open set U is the p-th homology of the chain complex*

$$\cdots \to \Gamma(U, \mathcal{F}^{p-1}) \to \Gamma(U, \mathcal{F}^p) \to \cdots$$

3. *If $\hat{P}$ and $\hat{Q}$ are two complexes of sheaves, a quasi-isomorphism from $\hat{P}$ to $\hat{Q}$ is a diagram of complexes $\hat{P} \xleftarrow{\alpha} \hat{R} \xrightarrow{\beta} \hat{Q}$ so that α and β induce isomorphisms $H^i\hat{P} \xleftarrow{\sim} H^i\hat{R} \xrightarrow{\sim} H^i\hat{Q}\ \forall\ i$. We note that quasi-isomorphism is an equivalence relation and quasi-isomorphic sheaves are interchangeable for all calculations with cohomological functors.*

4. *Any complex $\hat{\mathcal{F}}$ of sheaves is quasi-isomorphic to a complex $\hat{I}$ with each sheaf injective i.e., we take the canonical injective resolution of each sheaf and construct the total complex corresponding to the double complex. The cohomology of the global section complex $0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \ldots$ is called the hypercohomology of X in $\hat{\mathcal{F}}$ and the hypercohomology groups are denoted by $\mathrm{I\!H^i(X, \hat{\mathcal{F}})}$. Similarly, taking sections with compact support, we can define the hypercohomology groups with compact support, which are denoted by $\mathrm{I\!H^i_c(X, \hat{\mathcal{F}})}$.*

5. *Let $\hat{\mathcal{F}}$ be a complex of sheaves over X. For $x \in X$, we choose any neighbourhood $\cup$ in X and define the local cohomology groups $\mathcal{H}^i(\hat{\mathcal{F}})_x = \mathrm{I\!H^i(U, \hat{\mathcal{F}}/U)}$.*

The definition is independent of the set U chosen. Similarly, we define $\mathcal{H}^i_{\mathrm{c}}(\hat{\mathcal{F}})_x = \mathrm{I\!H^i_c(U, \hat{\mathcal{F}}/U)}$.

Theorem 3.1 *Suppose X is an irreducible complex algebraic variety of complex dimension n. Then, there is a unique (upto quasi-isomorphism) constructible complex $\hat{\mathcal{F}}$ of sheaves (of degrees ≥ 0) such that*
(i) codim. $\{x \in X | \mathcal{H}^i(\hat{\mathcal{F}})_x \neq 0\} > i$ for $i = 1, 2, \ldots$
(ii) codim. $\{x \in X | \mathcal{H}^{2n-i}_c(\hat{\mathcal{F}})_x \neq 0\} > i$ for $i = 1, 2, \ldots$
(iii) $\hat{\mathcal{F}}|_{\text{open dense subset}} \cong \mathbb{C}$ (the constant sheaf).
(A single sheaf S is thought of as the complex $S \to 0 \to 0 \to \ldots$)

Definition 3.3 *$\hat{\mathcal{F}}$ is called the intersection cohomology sheaf of X in $\mathbb{C}$. It is denoted by $IC(X, \mathbb{C})$.*

Remark 3.2 *(i) For a non-singular variety, $IC(X, \mathbb{C})$ is quasi-isomorphic with $\mathbb{C}$.*
(ii) The intersection homology of a singular algebraic variety satisfies many of the special properties of the ordinary homology of a Kähler manifold like Poincaré duality, Künneth theorem over a field, Lefschetz hyperplane theorem and Hard Lefschetz theorem which are not satisfied by the usual homology of the singular variety.

Definition 3.4 *For any complex algebraic variety X, $D_c^b(X)$ denotes the bounded constructible derived category of the category of sheaves of Q-vector spaces on X i.e. it is the set of bounded complexes of sheaves of Q-vector spaces on X which are locally constant on the strata X_α for some stratification of X.*

We will talk if objects in the derived category $D_c^b(X)$ and complexes of Q-sheaves interchangeably. Therefore if $\hat{S} \in D_c^b(X)$, and $U \subseteq X$, then $\mathrm{I\!H^k(U, \hat{S})}$ denotes the hypercohomology of $\hat{S}/U$.

If $p \in X$, the 'open disk' $\mathcal{D}_p^0$ of points at distance (usual Euclidean distance with respect to a local analytic embedding of a neighbourhood of p in $\mathbb{C}^N$) less than ϵ from p is such that, for any $\hat{S} \in D_c^b(X)$, $\mathrm{I\!H^k(\mathcal{D}_p^0, \hat{S})}$ is independent of the local embedding, provided ϵ is small enough.

A *local system* on a space X is locally constant Q-sheaf on X. We have, more generally then Theorem 3.1, the following:

Let dim.$X = n$ and U be a non-singular Zariski open dense subvariety and $\mathcal{L}$ be a local system on U. Then, there is an object $IC(X, \mathcal{L})$ in $D_c^b(X)$ defined upto canonial isomorphism in $D_c^b(X)$ satisfying:

1. $IC(X, \mathcal{L})/_U = \mathcal{L}[-n]$
2. X can be stratified with strata $\{X_i\}$ with dim $X_i = i$ such that if $p \in X_i$, then $\mathrm{I\!H^i(\mathcal{D}_p^0, IC(X, \mathcal{L})) \neq 0 \Rightarrow j > n+i}$ and $\mathrm{I\!H_c^j(\mathcal{D}_p^0, IC(X, \mathcal{L})) \neq 0 \Rightarrow j < n-i}$.

Also, $IC(X, \mathcal{L})$ is independent of U.

Definition 3.5 *Let X, Y be complex algebraic varieties. Let $f : X \to Y$. Let $\hat{P}$ and $\hat{Q}$ be complexes of sheaves on X and Y respectively. We define the pushforward complex $f_*(\hat{P})$ on Y by $\Gamma(U, f_*(\hat{P})) = \{\gamma \in \Gamma(f^{-1}(U), \hat{P}), \gamma$ has compact support$\}$. Thus, the stalk cohomology*

$$\mathcal{H}^i(f_*(\hat{P}))_y \cong \mathrm{I\!H^i(f^{-1}(y), \hat{P})}.$$

Also the pullback complex $f^*(\hat{Q})$ on X can be defined by

$$\mathcal{H}^i(f^*(\hat{Q}))_x = \mathcal{H}^i(\hat{Q})_{f(x)}.$$

Theorem 3.2 (BBD) *I. If $f : X \to Y$ is a locally trivial fibration with non-singular fibres, then*

$$f^*(IC(Y, \mathbb{C})) = IC(X, \mathbb{C}).$$

II. (The decomposition theorem :) If $f : X \to Y$ is a proper projective map of complex algebraic varieties, then

$$Rf_*(IC(X, \mathbb{C})) \cong IC(Y, \mathbb{C}) \oplus \bigoplus_{Y_i \not\subseteq Y} IC(Y_i, \mathcal{L}_i)[-N_i]$$

where Y_i are proper closed subvarieties of Y and $\mathcal{L}_i$ are locally constant sheaves, and N_i are some integers.

(In the above, for a complex of sheaves $\{A^i\}$, $A[N]$ denotes the complex of sheaves $\{B^i\}$ where $B^i = A^{i+N}$.)

Corollaries:

1. Poincaré duality [GM 2]

   Denoting by Loc (Y, n) the direct sum of the $\mathcal{L}_i$ for those i such that $Y = Y_i$ and $n = N_i$, Poincaré directly simply says that there is an isomorphism Loc $(Y, n) \to$ Hom $(\mathrm{Loc}(Y, -n), Q)$.

2. Hard Lefschetz theorem [BBA]

   $\exists$ a map $\wedge$: Loc $(Y, n) \to \mathrm{Loc}(Y, n+2)$ for all n, such that for $n > 0$

   $$\wedge^n : \mathrm{Loc}(Y, -n) \xrightarrow{\cong} \mathrm{Loc}(Y, n) \text{ is an isomorphism.}$$

3. If $f : X \to Y$ is a resolution of singularities of Y, then since $IC(X) = Q_X$, therefore $H^*(IC(Y))$ is contained in $H^*(X)$.

4. [GM 3] Let X be non-singular and $f : X \to Y$ be a proper projective algebraic map. For any point $p \in Y$, the closed disc $\mathcal{D}_p \subseteq Y$ is the set of points at a distance less than ϵ from p (where distance is the usual Euclidean one with respect to some local analytic chart at p; note that cohomologies of $\mathcal{D}_p$ with 'nice' complexes of sheaves are independent of the choice of the local embedding provided ϵ is sufficiently small).

   Call $M = f^{-1}(\mathcal{D}_p)$ and $B = f^{-1}(S)$, where S is the boundary of $\mathcal{D}_p$.

   For small enough ϵ, M is a compact manifold with boundary B.

   Let $K = \mathrm{Ker}\ (H_*(B) \xrightarrow{i_*} H_*(M))$.

   Now, since B is the boundary of M, K is a maximal (totally) isotropic subspace for the intersection form on $H_*(B)$. In particular dim $K = \frac{1}{2} dim\ H_*(B)$. We will show the stronger :

   Theorem 3.3 *If Y is an n-dimensional variety with an isolated singular point at p, and $f : X \to Y$ is a resolution of singularities (so $\bar{f}$ is a homeomorphism), then*

   $$K = H_n(B) \oplus H_{n+1}(B) \oplus \ldots \oplus H_{2n-1}(B).$$

   Proof: We have only to see which cycles in $H_*(S)$ are boundaries in $H_*(IC(\mathcal{D}_p))$ because, in $\mathcal{D}_p$ the decomposition looks like $IC(Y)$ and terms concentrated at p, of which, only $IC(Y)$ gives contribution to K.

   But now since $\mathcal{D}_p$ is topologically a cone with base S and vertex p, any cycle in S is the boundary of a cone with vertex p. So, a cycle in S will be a boundary in $IC(\mathcal{D}_p)$ provided the cone of the cycle to p is a chain in $IC(B)$ which is so, by definition, if the dimension of the cycle is $\geq n$. Thus $H_n(S) \oplus \ldots \oplus H_{2n-1}(S) \subseteq K$. But, the left-handed side being a maximal isotropic subspace of $H_*(S)$, has to equal K. Hence, the theorem is proved.

5. **Generalized invariant cycle theorem**

Before stating the theorem, let us define some things.

Let, once again, $\bar{f} : B \to S$ be as before.

We will construct a complement to K in $H_*(B)$.

Let $\{U_\varphi\}$ be a stratification of S with strata of odd dimension, such that $\bar{f}$ is a topological fibration onto each U_φ. Triangulate S such that each U_φ is a union of interiors of simplices. Define R to be the union of all simplices Δ of the barycentric subdivision of the above triangulation which satisfy

$$dim(\Delta \cap U_\varphi) < \frac{1}{2} dim\ U_\varphi\ \forall\ \varphi.$$

Define $J \subseteq H_*(B)$ to be $Im(H_*(\bar{f}^{-1}(R)) \to H_*(B))$.

For example, if $\bar{f}$ is a topological fibration and S is a manifold of dimension $2m-1$, then $J = F_{m-1}H_*(B)$ where F_s denotes the filtration of $H_*(B)$ of the Leray spectral sequence of $\bar{f}$.

Then, we have :

Theorem 3.4 (GM 3) *(Generalized invariant cycle theorem :)*
J is independent of the choices (U_φ) and triangulation of S, and is a maximal isotropic subspace of $H_*(B)$. K is a vector space complement to J in $H_*(B)$.

Proof: The decomposition theorem gives the decomposition

$$H_*(B) = \bigoplus_i H_*(IC(Y_i, \mathcal{L}_i)).$$

As in the proof the last theorem, we have

$$K = \bigoplus_i H_{a_i}(IC(S \cap Y_i, \mathcal{L}_i) \oplus \ldots \oplus H_{2a_i-1}(IC(S \cap Y_i, \mathcal{L}_i))$$

where $a_i = dim_{\mathbb{C}} Y_i$.

We *claim* that

$$J = \bigoplus_i H_0(IC(S \cap Y_i, \mathcal{L}_i)) \oplus \ldots \oplus H_{a_i - 1}(IC(S \cap Y_i, \mathcal{L}_i)).$$

This will clearly prove the theorem.

To prove the claim, taking an open regular neighbourhood R^0 of R, we have

$$J = \bigoplus_i Im[H_*(IC(R^0 \cap Y_i, \mathcal{L}_i)) \to H_*(IC(S \cap Y_i, \mathcal{L}_i))].$$

We will show that $\forall\ i$, the term above, inside the direct sum

$$= H_0(IC(S \cap Y_i, \mathcal{L}_i)) \oplus \ldots \oplus H_{a_i - 1}(IC(S \cap Y_i, \mathcal{L}_i)).$$

Firstly, since we can find a homeomorphism (using stratified general position) $h : Y_i \to Y_i$ isotopic to the identity such that $h(R \cap Y_i) \cap R \cap Y_i$ is empty, it follows that J is self-annihilating under the intersection pairing. Thus the containment $\supseteq$ follows.

To show the inclusion $\subseteq$, we note that Y_i is a union of strata U_φ and the fact ([GM2], §3.4) that we have some basic subsets R_j in $Y_i (1 \leq j \leq 2a_i)$ such that $H_j(IC(Y_i, \mathcal{L}_i)) \cong Im(H_j(R_j) \to H_j(R_{j+1}))$. In fact, in [GM 2], they analogously construct such a family and show Poincaré duality and the independence of $H(IC(X))$ of the stratification. By the construction in [GM 2], §3.4, it is also clear that $R_j \subseteq R^0 \cap Y_i$ for $1 \leq j \leq a_i - 1$ (recalling that $\Delta \in R \Leftrightarrow dim(\Delta \cap U_\varphi) < \frac{1}{2} dim\ U_\varphi$).

Thus, for

$$\begin{aligned} 1 \leq j \leq a_i - 1, H_j(IC(S \cap Y_i, \mathcal{L}_i)) &= Im(H_j(R_j, \mathcal{L}_i) \to H_j(R_{j+1}, \mathcal{L}_i)) \\ &\subseteq Im(H_j(IC(R^0 \cap Y_i, \mathcal{L}_i)) \to H_j(IC(S \cap Y_i, \mathcal{L}_i))). \end{aligned}$$

Hence the claim is proved.

Example: If Y is a curve, the theorem is just invariant cycle theorem, which says that the composed map $H_{i+1}(M, B) \xrightarrow{\psi} H_{i+1}(B) \xrightarrow{\varphi^*} H_i(F)$ is a surjection onto the Kernel of $1 - \mu$, where $\varphi : F \hookrightarrow B$ is the inclusion of a fiber and $\mu : H_*(F) \to H_*(F)$ is the monodromy map.

To see this, we show that the Wang sequence for the fibration $\bar{f} : B \to S^1$ is exact ie. $H_{i+1}(F) \xrightarrow{\varphi_*} H_{i+1}(B) \xrightarrow{\varphi^*} H_i(F) \xrightarrow{1-\mu} H_i(F)$ is exact. Once we have shown this, then the invariant cycle theorem is clear from the theorem since, by definition, $Im\ \varphi_* = J \cap H_{i+1}(B)$ and so, $Im\ \varphi^* = Im\ \varphi^*\psi$ because for any $b \in H_{i+1}(B)$, writing $b = j + k$ as in the theorem we have $k \in \mathrm{Ker}(H_{i+1}(B) \to H_{i+1}(M)) = Im\ \psi$ and $b - k = j \in Im\ \varphi_* = Ker\ \varphi^*$.

Now, we will show the exactness of the sequence,

$$H_{i+1}(F) \xrightarrow{\varphi_*} H_{i+1}(B) \xrightarrow{\varphi^*} H_i(F) \xrightarrow{1-\mu} H_i(F).$$

More generally, we show

Lemma 3.5 *If $p : B \to S^n$ is a fibration with fiber F. Then $\ldots \to H_q(F) \to H_q(B) \to H_{q-n}(F) \to H_{q-1}(F) \to \ldots$ is exact.*

Proof: Consider the exact sequence

$$\ldots \to H_q(F) \to H_q(B) \to H_{q-n}(F) \to H_{q-1}(F) \to \ldots$$

We are thinking of S^n as the suspension $S(S^{n-1})$ of S^{n-1} and the fiber $F = p^{-1}(y_0)$ for some $y_0 \in S^{n-1}$.

Now, denote the upper and lower cones of S^{n-1} as E_n^+ and E_n^- respectively. Since E_n^+, E_n^- are contractible, we have $p\mid_{p^{-1}(E_n^+)}$ and $p\mid_{p^{-1}(E_n^-)}$ to be fiber homotopically equivalent to the respective trivial fibrations i.e $\exists\ f_- : E_n^- \times F \to p^{-1}(E_n^-)$ and $g_+ : p^{-1}(E_n^+) \to E_n^+ \times F$ such that $f_-\mid_{y_0 \times F}$ is homotopic to the map $(y_0, z) \mapsto z$ and $g_+\mid_F$ is homotopic to the map $z \mapsto (y_0, z)$ where f_- and g_+ are homotopy equivalences preserving the fibers.

So, we have a commutative diagram

$$\begin{array}{ccccccc}
H_q(B,F) & \xrightarrow{\cong} & H_q(B,p^{-1}(E_n^+)) & \xleftarrow{\cong} & H_q(p^{-1}(E_n^-),p^{-1}(S^{n-1})) & \overset{(f_-)_*}{\underset{\cong}{\leftarrow}} & H_q((E_n^-,S^{n-1}) \times F) \\
\partial \swarrow & & \partial \swarrow & & \partial \swarrow & & \partial \swarrow \\
H_{q-1}(F) & \overset{j_*}{\underset{\cong}{\rightarrow}} & H_{q-1}(p^{-1}(E_n^+)) & \xleftarrow{i_*} & H_{q-1}(p^{-1}(S^{n-1})) & \overset{(f_-)_*}{\underset{\cong}{\leftarrow}} & H_{q-1}(S^{n-1} \times F).
\end{array}$$

But $(j_*)^{-1}$ is the composite isomorphism

$$H_{q-1}(p^{-1}(E_n^+)) \overset{(g_+)_*}{\cong} H_{q-1}(E_n^+ \times F) \overset{(proj)_*}{\cong} H_{q-1}(F)$$

as $g_+|_F$ is homotopic to the map $z \mapsto (y_0, z)$.

Let $\bar{\mu}_* = j_*^{-1} i_*(f_-)_*$; i.e., $\bar{\mu} : S^{n-1} \times F \to F$ is defined by $g_+ f_-(y,z) = (y, \bar{\mu}(y,z))$; it is called a clutching function for p.

We have the commutative diagram

$$\begin{array}{ccc} H_q(B,F) & \overset{\cong}{\to} & H_{q-n}(F) \\ \partial \downarrow & & \partial \downarrow \\ & \overset{\bar{\mu}_*}{\leftarrow} & \\ H_{q-1}(F) & \cong & H_{q-1}(S^{n-1} \times F). \end{array}$$

Hence the lemma is proved.

For our case $n = 1$, we can choose f_-, g_+ such that $\bar{\mu}(y,-1) =$ Identity and $\bar{\mu}(y,1) = \mu(y)$ and so we have $\bar{\mu}_* \partial =$ Identity $-\mu_*$.

Definition 3.6 *Let G be a connected reductive complex algebraic group. Let B_0 be a (fixed) Borel subgroup.*
Let $\mathcal{B}$ = set of Borel subgroups; this can be identified with G/B_0. The B_0-orbits in $\mathcal{B}$ are parametrized by $w \in$ Weyl group. For $w \in W$, the B_0-orbit of wB_0w^{-1} is called a Bruhat cell and is denoted by $\mathcal{B}_w$. These are smooth, simply connected algebraic varieties; indeed $\mathcal{B}_w$ is an affine space of dimension $\ell(w)$. One can also write, $\mathcal{B}_w = \{B | B_0 \overset{w}{\to} B\}$ where $B \overset{w}{\to} B'$ denotes $gBg^{-1} = B_0 = w^{-1}gB'g^{-1}w$ for some $g \in G$.
The closures $\overline{\mathcal{B}}_w$ are called Schubert varieties (in general, they are singular). We have

$$\overline{\mathcal{B}}_w = \bigcup_{y \leq w} \mathcal{B}_y.$$

Since B_0 acts on $\overline{\mathcal{B}}_w$, we consider B_0-equivalent complexes of sheaves on $\overline{\mathcal{B}}_w$.

If $\hat{A}_w$ is a B_0-equivariant constructible complex of sheaves on $\overline{\mathcal{B}}_w$ for some $w \in W$ and if $s \in S$ such that $ws > w$, we define a new B_0-equivariant constructible complex of sheaves on $\overline{\mathcal{B}}_{ws}$ as $\hat{A}_w \circ \hat{A}_s = pr_{2_*}(pr_1^*(\hat{A}_w))$ where

$$pr_1 = \{(B,B') | B_0 \overset{\leq w}{\to} B \overset{\leq s}{\to} B'\} \to \overline{\mathcal{B}}_w = \bigcup_{y \leq w} \mathcal{B}_y$$

and

$$pr_2 = \{(B,B')|B_0 \overset{\leq w}{\to} B \overset{\leq s}{\to} B'\} \to \overline{\mathcal{B}}_{ws} = \bigcup_{y \leq ws} \mathcal{B}_y.$$

We note that $\mathcal{H}^i(\hat{A}_w)_{\mathcal{B}_y} = \mathcal{H}^i(\hat{A}_w)_x\ \forall\ x \in \mathcal{B}_y$ *and for all* $y \leq w$*, since* $\hat{A}_w$ *is a* B_0*-equivariant complex.*
Now, we define a map of B_0*-equivariant complexes from* $\overline{\mathcal{B}}_w$ *to the Hecke algebra* H *by writing*

$$h(\hat{A}_w) = \sum_{y \leq w} \left\{ \sum_{i \geq 0} q^{i/2} dim\ \mathcal{H}^i(\hat{A}_w)_{\mathcal{B}_y} \right\} T_y.$$

Lemma 3.6 *Let* $w \in W, s \in S$ *such that* $ws > w$*. Let* $\hat{A}_w$ *be a* B_0*-equivariant constructible complex of sheaves on* $\overline{\mathcal{B}}_w$ *and* $\hat{A}_w \circ \hat{A}_s$ *be the corresponding complex induced on* $\overline{\mathcal{B}}_{ws}$*. Assume that* $\forall$ *odd* i*,* $\mathcal{H}^i(\hat{A}_w)_B = 0$ *for every point* B *in* $\overline{\mathcal{B}}_w$*. Then the same is true for* $\hat{A}_w \circ \hat{A}_s$ *and we have*

$$h(\hat{A}_w \circ \hat{A}_s) = h(\hat{A}_w) \cdot (T_s + 1).$$

Proof: Let $\Omega = \{(B,B')|B_0 \overset{\leq w}{\to} B \overset{\leq s}{\to} B'\}$. Let p_1 and p_2 denote the projections $\Omega \to \overline{\mathcal{B}}_w$, $(B,B') \mapsto B$ and $\Omega \to \overline{\mathcal{B}}_{ws}, (B,B') \mapsto B^1$ respectively.

Let $B' \in \mathcal{B}_y, y \leq ws$. We want to show that $\mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B'} = 0$ for all odd i. Now, by the definition of $\hat{A}_w \circ \hat{A}_s, \mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B'} = \mathrm{IH}^i_C(p_1 p_2^{-1}(B'), \hat{A}_w)$. But $p_1 p_2^{-1}(B') = \{B|B_0 \overset{\leq w}{\to} B \overset{\leq s}{\to} B'\} \subseteq \{B|B \overset{\leq s}{\to} B'\} \simeq \mathbb{P}^1$. Thus $p_1 p_2^{-1}(B')$ is either a point or $\mathbb{P}^1$.

If it is a point, then by hypothesis we get $\mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B'} = 0$ for odd i. If $p_1 p_2^{-1}(B')$ is the whole of $\{B|B \overset{\leq s}{\to} B'\} \simeq \mathbb{P}^1$, we write it as $\{\text{point}\} \cup \mathbb{A}^1$.

If $ys > y$, then the point will be $\{B'\}$ and $\mathcal{B}_{ys}$ will be $\mathbb{A}^1$ (note that $\overline{\mathcal{B}}_{ys} \supseteq \mathcal{B}_y$). Applying long exact sequence for IH^i_C associated to a partition of a space into an open and a closed subspace, we have a short exact sequence

$$0 \to \mathcal{H}^i(\hat{A}_w)_{B^1} \to \mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B^1} \to \mathrm{IH}^i(\mathbb{A}^1, \hat{A}_w) \to 0.$$

Therefore $\dim \mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B'} = \dim\ \mathcal{H}^i(\hat{A}_w)_{B'} + \dim\ \mathrm{IH}^i(\mathbb{A}^1, \hat{A}_w)$. Projecting $\mathbb{A}^1$ to a point we will get $\mathrm{IH}^i(\mathbb{A}^1, \hat{A}_w) = \mathcal{H}^{i-2}(\hat{A}_w)_B$, where $B \in \mathcal{B}_{ys}$. Thus, we finally get the formulae:

$$dim\mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B'} = \begin{cases} dim\mathcal{H}^i(\hat{A}_w)_{B'} + dim\mathcal{H}^{i-2}(\hat{A}_w)_B, & \text{if } ys > y \\ dim\mathcal{H}^{i-2}(\hat{A}_w)_{B'} + dim\mathcal{H}^i(\hat{A}_w)_B, & \text{if } ys < y. \end{cases}$$

In any case, by hypothesis it follows that $\mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_{B'} = 0$ for *all odd* i. The above formula is just equivalent to saying that

$$h(\hat{A}_w \circ \hat{A}_s) = h(\hat{A}_w)(T_s + 1).$$

Theorem 3.7 (KL 2) *(a)* $\mathcal{H}^i(IC(\overline{\mathcal{B}}_w, \mathbb{C}))_B = 0 \forall$ *odd* $i, B \in \overline{\mathcal{B}}_w$.

(b) $h(IC(\overline{\mathcal{B}}_w, \mathbb{C})) = \sum_{y \leq w} P_{y,w}(q) \cdot T_y$ *where* $P_{y,w}$ *are the Kazhdan-Lusztig polynomials as in Theorem 2.3.*

Proof: We apply induction on $\ell(w)$.
(a) and (b) are clear for $w = e$. So we assume the results for w and prove them for $ws > w$.

(a) Let us denote $IC(\overline{\mathcal{B}}_w, \mathbb{C})$ by $\hat{A}_w$. Denoting by $\hat{A}_w \circ \hat{A}_s$ the corresponding complex on $\overline{\mathcal{B}}_{ws}$, we have $\mathcal{H}^i(\hat{A}_w \circ \hat{A}_s)_B = 0$ *for all odd* i by the lemma 3.3.

Now, by the decomposition theorem 3.2 (ii),

$$\hat{A}_w \circ \hat{A}_s = \hat{A}_{ws} \oplus \bigoplus_{\substack{y<ws \\ ys<y}} \oplus_i n_{y,i} \hat{A}_y[i]. \tag{3.1}$$

Since the odd cohomologies at a point vanish for the left-hand side, they continue to do so for the right-hand side and hence $\mathcal{H}^i(\hat{A}_{ws})_B$ being a direct summand, is zero for all odd i.

Thus, by induction, $\mathcal{H}^i(IC(\overline{\mathcal{B}}_w, \mathbb{C}))_B = 0$ *for all odd* i. Hence, we note that $h(IC(\overline{\mathcal{B}}_w, \mathbb{C})) \in \sum_{y \leq w} \mathbb{Z}[q] \cdot T_y$.

Corollary 3.8 $P_{y,w}(q)$ *has all coefficients* $\geq 0 \ \forall \ y, w \in W$.

Corollary 3.9 *The equality* $\overline{C'_w} = C'_w$ *of Theorem 2.3 simply expresses the fact that local Poincaré duality is satisfied by the Intersection cohomology sheaves of the Schubert variety* $\overline{\mathcal{B}}_w$.

Conjecture 3.1 *For any Coxeter group* W, $P_{y,w}(q)$ *has all coefficients* $\geq 0 \ \forall \ y, w \in W$.
(Goresky has verified for the finite Coxeter group H_3*, on a computer.)*

4 Representation of Hecke Algebras

Definition 4.1 *Let (W, S) be a Coxeter system. For $x \in W$, let* $\mathcal{L}(x) = \{s \in S | sx < x\}, \mathcal{R}(x) = \{s \in S | xs < x\}$.

(a) For $x, x^1 \in W$,
$x \underset{L}{\leq} x^1 \Leftrightarrow \exists$ *a sequence* $x = x_0, x_1, \ldots, x_n = x^1$ *in* W *such that (i)* $\mathcal{L}(x_i) \not\subseteq \mathcal{L}(x_{i+1})$ *and (ii) either* $x_i \prec x_{i+1}$ *or* $x_{i+1} \prec x_i, i = 0, \ldots, n-1$.

(b) $x \underset{R}{\leq} x^1$ *is similarly defined. Note that* $x \underset{R}{\leq} x^1 \Leftrightarrow x^{-1} \underset{L}{\leq} x'^{-1}$.

(c) $x \underset{LR}{\leq} x^1 \Leftrightarrow \exists x = x_0, x_1, \ldots, x_n = x^1$ *in* W *such that for each* $i = 0, 1, \ldots, n-1$, *(i) either* $\mathcal{L}(x_i) \not\subseteq \mathcal{L}(x_{i+1})$ *or* $\mathcal{R}(x_i) \not\subseteq \mathcal{R}(x_{i+1})$ *and (ii) either* $x_i \prec x_{i+1}$ *or* $x_{i+1} \prec x_i$.

(d) $x \underset{L}{\sim} x^1 \Leftrightarrow x \underset{L}{\leq} x^1 \underset{L}{\leq} x$. *The equivalence classes are called left cells.*

We have seen that the Hecke algebra corresponding to the Weyl group of a Chevalley group G over a finite field, plays an important role in the representation theory of G. So, it is desirable to construct irreducible representations of the Hecke algebra with a special basis. The left cells of any Coxeter group W, which are defined combinatorially can be used to construct representations of the Hecke algebra H over $\mathbb{Z}[q^{1/2}]$ via W-graphs.

In general, a W-graph is defined to be a set of vertices X, with a set Y of edges such that, for each vertex $x \in X$, we are given a subset I_x of S and, for each ordered pair of vertices y, x such that $\{y, x\} \in Y$, we are given an integer $\mu(y, x) \neq 0$. In the case of a left cell, X consists of elements $W, I_x = \mathcal{L}(x)$ and $\{y, x\} \in Y$ iff $y \mapsto x$ (recall that this means $y < x$ or $x < y$ and $\mu(x, y) \neq 0$. Here we are using the polynomials $P_{x,y}$).

We define (given a W-graph) a representation of H by taking E to be the free $\mathbb{Z}[q^{1/2}]$-module with basis X and defining $\forall\ s \in S$,

$$T_s(x) = \left\{ \begin{array}{ll} -x, & \text{if } s \in I_x \\ qx + q^{1/2} \cdot \sum\limits_{\substack{y \in X \\ \{y,x\} \in Y \\ s \in I_y}} \mu(y, x) y, & \text{if } s \notin I_x. \end{array} \right\}$$

Usually, for convenience (when, say, the function $\mu \equiv 1$) a W-graph will be represented by vertices as circles, inside which are described the corresponding subset I_x of S.

These will be used now to show a theorem on the structure of the Hecke algebra of a Chevalley group over a finite field.

Iwahori conjectured that the Hecke algebra over $\mathbb{Q}$ of a Chevalley group G over a finite $\mathbb{F}_q$ with respect to a Borel subgroup G isomorphic to the group algebra over $\mathbb{Q}$ of the Weyl group. As remarked in §2, the analogous statement with $\mathbb{Q}$ replaced by $\overline{\mathbb{Q}}$ was proved by Tits. Then, Benson and Curtis proved by case-by-case analysis that Iwahori's conjecture is true over $\mathbb{Q}$ for G simple of type $\neq E_7, E_8$ and false over $\mathbb{Q}$ for G of type E_7, E_8.

Using left cells etc., we prove the following theorem for a finite Weyl group W. Let W be a *(finite) Weyl* group and consider the Hecke algebra of W *over* $\mathbb{Q}[q^{1/2}]$. Let E be the free $\mathbb{Q}[q^{1/2}]$-module with basis $(\ell_w)_{w\in W}$.

We define a left H-module structure on E by

$$T_s\ell_w = \begin{cases} -\ell_w, & \text{if } sw < w \\ q\ell_w + q^{1/2}\ell_{sw} + q^{1/2}\sum\limits_{\substack{y\prec w\\ sy<y}} \mu(y,w)\ell_y, & \text{if } sw > w \end{cases}$$

and a left W-module structure on E by

$$s * \ell_w = \begin{cases} -\ell_w, & \text{if } sw < w \\ \ell_w + \ell_{sw} + \sum\limits_{\substack{y\prec w\\ sy<y}} \mu(y,w)\ell_y, & \text{if } sw > w. \end{cases}$$

We can similarly define right H-module and right W-module structures.

Theorem 4.1 *(a) There is a unique homomorphism of* $\mathbb{Q}[q^{1/2}]$*-algebras* $\varphi : H \to \mathbb{Q}[q^{1/2}][W]$ *such that,* $\forall\ h \in H$ *and* $w \in W, h\ell_w - \varphi(h) * \ell_w$ *is a linear combination of* $\ell_y, y \underset{LR}{<} w$.

(b) Given any homomorphism χ *of* $\mathbb{Q}[q^{1/2}]$ *into a field* K*, the* K*-homomorphism* $\varphi_\chi : H \otimes K \to K[W]$ *is such that its Kernel consists of nilpotents. In particular,* φ_χ *is an isomorphism if* $H \otimes K$ *is semi-simple.*

Let us mention some *combinatorial* conjectures now.

Conjecture 4.1 *An affine Weyl group has only finitely many left (right) cells (see §4.8). (Recently R. Bédard has shown that ∃ an infinite Coxeter group with 3 generators which is not an affine Weyl group and has infinitely many left (right) cells.)*

Conjecture 4.2 *For an affine Weyl group, the # of two-sided cells = # of unipotent conjugacy classes. (For example, this is 4 for $\tilde{B}_2$ ad 5 for $\tilde{G}_2$.)*

Conjecture 4.3 *Let (W, S) be any Coxeter system. ∃ a constant 'a' depending on W such that*

$$q^{a/2} \cdot \tilde{T}_w \tilde{T}_{w'} \in \sum_x \mathbb{Z}[q^{1/2}] C_x \ \forall \ w, w' \in W.$$

(For example, in case of finite or affine Weyl groups, by §4.4.4, this can be taken to be the # of the roots of the underlying finite Weyl group.)

Conjecture 4.4 *For any x, y, z in an affine Weyl group W, we have $C_{x,y,z} = C_{y,z,x} = C_{z,x,y}$ (see 4.4.2 for definition).*
Note that this conjecture ⇒ conjecture 4.1 (see 4.8 for a proof of this).

¿From now on, in this section we will be proceeding towards a proof of the Main theorem 4.1.

Lemma 4.2 *$y \prec y^1, \mathcal{L}(y^1) \not\subseteq \mathcal{L}(y) \Rightarrow y = sy^1$ and $\mu(y, y^1) = 1$ where $s \in \mathcal{L}(y^1) - \mathcal{L}(y)$. Similarly, $y \prec y^1, \mathcal{R}(y^1) \not\subseteq \mathcal{R}(y) \Rightarrow y = y^1 s$ and $\mu(y, y^1) = 1$ where $s \in \mathcal{R}(y^1) - \mathcal{R}(y)$.*

Proof: Assume that $s \in \mathcal{L}(y^1) \backslash \mathcal{L}(y)$. Now, since $sy^1 < y^1$, we have by Corollary 2.5 that $T_s C_{y^1} = -C_{y^1}$. Comparing the coefficients of T_{sy} on both sides, we get

$$(-1)^{\ell(sy)+\ell(y^1)}(q-1)q^{\ell(y^1)/2}q^{-\ell(sy)}\overline{P_{sy,y^1}} + (-1)^{\ell(y)+\ell(y^1)} \cdot q^{\frac{\ell(y^1)}{2}-\ell(y)} \cdot \overline{P_{y,y'}}$$

$$= (-1)^{\ell(sy)+\ell(y)+1} \cdot q^{\frac{\ell(y')}{2}-\ell(sy)} \cdot \overline{P_{sy,y'}}$$

i.e., $P_{y,y'} = P_{sy,y'}$.

Therefore if $sy \neq y^1$, then $P_{y,y^1} = P_{sy,y^1}$ has degree $\leq \frac{\ell(y^1)-\ell(sy)-1}{2}$ and $< \frac{\ell(y^1)-\ell(y)-1}{2}$, which is a contradiction of the fact that $\mu(y, y^1) \neq 0$.

Thus $sy = y^1$ and so $P_{y,y^1} \equiv 1$.

Corollary 4.3 *W finite $\Rightarrow P_{y,w_0} \equiv \ \forall\ y \in W$.*

Proposition 4.4 *$x \underset{L}{\leq} x' \Rightarrow \mathcal{R}(x) \supseteq \mathcal{R}(x')$. Hence, $x \underset{L}{\sim} x' \Rightarrow \mathcal{R}(x) = \mathcal{R}(x')$.*

Proof: We can assume without loss of generality that $\mathcal{L}(x) \not\subseteq \mathcal{L}(x')$ and either $x \prec x'$ or $x' \prec x$.

In the case $x' \prec x$, the lemma 4.2 gives $s \in \mathcal{L}(x) \backslash \mathcal{L}(x')$ such that $x' = sx$ and $\mu(x, x') = 1$. So, if $t \in \mathcal{R}(x')$, then $x't < x'$. Therefore $\ell(sxt) < \ell(sx)$. But $sx < x \Rightarrow \ell(sx) = \ell(x) - 1$. Therefore $\ell(sxt) \leq \ell(x) - 2$. But $\ell(sxt) = \ell(xt) \pm 1 = (\ell(x) \pm 1) \pm 1$. Therefore $\ell(xt) = \ell(x) - 1$ i.e. $t \in \mathcal{R}(x)$. Thus $\mathcal{R}(x') \subseteq \mathcal{R}(x)$.

In the other case $x \prec x'$, if we do have $\mathcal{R}(x') \not\subseteq \mathcal{R}(x)$, then again by the same argument we see that $\mathcal{L}(x) \subseteq \mathcal{L}(x')$, a contradiction of our assumption. Thus, in any case $\mathcal{R}(x') \subseteq \mathcal{R}(x)$.

Before proceeding further to the proof of Theorem 4.1, we will describe algorithmically the cells in S_n. In the proof of Theorem 4.1 we will see that any left cell (for any Coxeter group) gives rise to a representation of the Hecke algebra. In the case of S_n, the representation so obtained are irreducible and all the irreducible representations of the Hecke algebra of S_n, occur in this way. The left cells in S_n, arise from partitions of n, by the Robinson-Schensted algorithm. This algorithm gives a one-to-one correspondence between the elements of S_n and pairs (τ, τ') of standard Young tableaux of the same shape, of size n. A left cell in S_n corresponds to a subset (τ, τ') with τ' fixed and τ of the same shape as τ'. A two-sided cell corresponds to a set of pairs (τ, τ') which are of fixed shape. Thus S_n is decomposed into two sided cells (one for each partition of n) and each 2-sided cell is a disjoint union of a number of left cells equal to the dimension of the corresponding irreducible representation of S_n. This is the prototype of what applies to any Coxeter group.

We will describe the cells of the symmetric group S_n in a number of ways, some of which are combinatorial thereby enabling us to compute them more easily than would be possible by our definition.

(i) Robinson-Schensted algorithm [Kn]

They show (by actual construction), a one-to-one correspondence between permutations τ of $\{1, 2, \ldots, n\}$ and ordered pairs $(P(\tau), Q(\tau))$ of Young tableaux formed from $\{1, 2, \ldots, n\}$ where P, Q have the same shape. This correspondence is such that the inverse permutation goes to $(Q(\tau), P(\tau))$ so that involutions are parametrized by tableaux formed from $\{1, 2, \ldots, n\}$. (Note that, in S_n, each left cell (by our definition) contains a unique involution.) We will describe this in another way as follows:

Let $s, t \in S$ such that $O(st) = 3$. Let $\mathcal{D}_L(s,t) = \{w \in W | \mathcal{L}(w) \cap \{s,t\}$ has exactly one element$\}$. If $w \in \mathcal{D}_L(s,t)$, then exactly one of sw, tw is in $\mathcal{D}_L(s,t)$; this is denoted by ${}_{s,t}W$.

We *claim* that $w \underset{L}{\sim} w' =_{s,t} W$. We can assume $w' = sw$.

If $w < w'$, then $s \in \mathcal{L}(w') \backslash \mathcal{L}(w)$ so that $w' \underset{L}{\leq} w$. Also $w' \in \mathcal{D}_L(s,t)$ and $s \in \mathcal{L}(w') \Rightarrow t \notin \mathcal{L}(w')$, and $w \in \mathcal{D}_L(s,t)$ and $s \notin \mathcal{L}(w) \Rightarrow t \in \mathcal{L}(w)$ so that $\mathcal{L}(w) \nsubseteq \mathcal{L}(w')$ and so $w \underset{L}{\leq} w'$ Thus $w \underset{L}{\sim} w'$.

If $w > w'$, we can interchange the roles of w and w' since $w = sw'$.

(ii) *Joseph* defined $w \underset{L}{\leq} w' \Leftrightarrow ann\ L_w \supseteq ann\ L_{w'}$, where $ann\ L_w$ is the left annihilator of L_w in $U(\mathcal{J})$. This is also equivalent to our definition. (This is basically just Harish-Chandra's theorem that if $\lambda, \mu \in H^*$, then $\chi_\lambda = \chi_\mu \Rightarrow \lambda + \rho = (\mu + \rho)^w$ for some $w \in W$).

(iii) *Vogan-Jantzen* defined $w \underset{L}{\sim} w' \Leftrightarrow$ (1) $\mathcal{R}(w) = \mathcal{R}(w')$, (2) $\mathcal{R}(w_{s,t;s',t'\ldots}) = \mathcal{R}(w'_{s,t;s',t'} \ldots)$ if $w_{s,t;s',t'\ldots}$ etc. make sense.

Therefore, clearly by Proposition 4.3, $w \overset{L\sim'}{w}$ in the sense of Vogan $\Rightarrow w \overset{L\sim'}{w}$ in the sense of our definition.

We will show that the converse implication also holds.

Let $y, w \in \mathcal{D}_L(s,t)$ and $y \underset{R}{\sim} w$. Then, it is enough to show that ${}_{s,t}y \underset{R_{s,t}}{\sim} w$.

Let $y = y_0, \ldots, y_n = w$ where $y_i \leftrightarrow y_{i+1}$ and $\mathcal{R}(y_i) \not\subseteq \mathcal{R}(Y_{i+1})$ and $\mathcal{R}(y_{i+1}) \not\subseteq \mathcal{R}(y_i)$.

Clearly $y_i \underset{R}{\sim} y\ \forall\ i$, so that by Proposition 4.3, $\mathcal{L}(y_i) = \mathcal{L}(y)\ \forall\ i$. Since $y \in \mathcal{D}_L(s,t)$. Therefore $y_i \in \mathcal{D}_L(s,t)$ since $\mathcal{L}(y_i) = \mathcal{L}(y) \forall\ i$. Thus ${}_{s,t}y_i$ are well-defined. We can also show that since $y_i \mapsto y_{i+1}$, we have ${}_{s,t}y_i \leftrightarrow_{s,t} y_{i+1}\ \forall\ i$.

Also

$$\begin{array}{rcl} \mathcal{R}({}_{s,t}y_i) & = & \mathcal{R}(y_i) \text{ by } (i) \\ & \not\subseteq & \mathcal{R}(y_{i+1}) = \mathcal{R}({}_{s,t}y_{i+1}) \text{ and vice versa.} \end{array}$$

Thus ${}_{s,t}y \underset{R}{\sim} {}_{s,t}w$. Therefore all the four definitions of cells are equivalent.

¿From now on, we assume *that W is a finite or affine Weyl group.*

Definition 4.2 *Let $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}], \mathcal{A}^+ = \mathbb{Z}[q^{1/2}]$.*

(i) For $z \in W, a(z)$ is defined to be the smallest natural number such that

$$q^{\frac{a(z)}{2}} C_x C_y \in \mathcal{A}^+ \cdot C_z + \sum_{z' \neq z} \mathcal{A} \cdot C_{z'}\ \forall\ x, y \in W.$$

We will show in Corollary 4.4.4 that $a(z)$ exists for affine Weyl groups also.

(ii) Let $Q_{x,y}$ be defined as $\sum\limits_{\substack{y \\ x \leq y \leq z}} (-1)^{\ell(x)+\ell(y)} P_{x,y}(q) Q_{y,z}(q) = \delta_{x,z}$. Then, we have $Q_{y,y} \equiv 1$ and deg $Q_{y,z} \leq \frac{\ell(z)-\ell(y)-1}{2}$ for $y < z$.

Define

$$\begin{aligned} D_x &= q^{-\frac{\ell(x)}{2}} \sum_{\substack{y \\ y \geq x}} Q_{y,x}(q) \cdot T_y \\ &\in \tilde{T}_x + q^{1/2} \sum_{y>x} \mathcal{A}^+ \cdot \tilde{T}_y. \end{aligned}$$

Therefore $\tilde{T}_x \in D_x + q^{-\frac{1}{2}} \cdot \sum_{y>x} \mathcal{A}^+ \cdot D_y$.

In the case of affine Weyl groups, the elements D_x *can be regarded as belonging to the set* $\hat{H}$ *of formal (possibly infinite)* $\mathcal{A}$*-linear combinations of the elements* $\tilde{T}_w$.

(iii) Let $K = \mathbb{Q}(q^{1/2})$ *and* H_K *denote* $H \bigotimes_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} K$. *Define* $\tau : H_K \to K$ *by* $\tau(\Sigma a_x \tilde{T}_x) = a_e$. *Then, we see that*

$$\begin{aligned} \tau(\tilde{T}_x \cdot \tilde{T}_{x'}) &= \tau(C_x \cdot D_{x'}) = \tau(\tilde{T}_{x'} \cdot \tilde{T}_x) = \tau(C_{x'} \cdot D_x) \\ &= \begin{cases} 0, & \text{if } x' \neq x^{-1} \\ 1, & \text{if } x' = x^{-1}. \end{cases} \end{aligned}$$

(iv) For $x, y, z \in W$, *define* $C_{x,y,z} \in \mathbb{Z}$ *by*

$$C_x \cdot C_y - (C_{x,y,z} q^{-\frac{a(z)}{2}} + \textit{Higher powers of } q) C_{z^{-1}} \in \sum_{u \neq z^{-1}} \mathcal{A} \cdot C_u.$$

This definition makes sense provided we show that $a(z) = a(z^{-1})\ \forall\ z \in W$. To show this, we first note that $\tilde{T}_w \mapsto \tilde{T}_{w^{-1}}$ defines an antiautomorphism of H.

Therefore, if we choose $x, y \in W$ such that

$$q^{\frac{a(z_0)-1}{2}} \tilde{T}_x \tilde{T}_y = \sum_z \alpha_{x,y,z} C_z \text{ with } \alpha_{x,y,z_0} \notin \mathcal{A}^+,$$

then we get

$$q^{\frac{a(z_0)-1}{2}} \tilde{T}_{y^{-1}} \tilde{T}_{x^{-1}} = \sum_z \alpha_{x,y,z} C_{z^{-1}} = \sum_z \alpha_{x,y,z^{-1}} C_z.$$

But, the left-hand side is also $= \sum_z \alpha_{y^{-1},x^{-1},z} \cdot C_z$. Thus $\alpha_{x,y,z^{-1}} = \alpha_{y^{-1},x^{-1},z}\ \forall\ z \in W$. Therefore,

$$\alpha_{y^{-1},x^{-1},z_0^{-1}} \notin \mathcal{A}.$$

Thus, by the definition of $a(z_0^{-1})$, we have $a(z_0^{-1}) > a(z_0) - 1$. Therefore $a(z_0^{-1}) \geq a(z_0)$. Similarly $a(z_0) \geq a(z_0^{-1})$ and hence,

$$a(z_0) = a(z_0^{-1})\ \forall\ z_0 \in W.$$

Theorem 4.5 *Let (W, S) be an irreducible affine Weyl group. Let ν be the number of positive roots. Then, for any $x, y, z \in W$ we have*

$$\tilde{T}_x \tilde{T}_y = \sum_z m_{x,y,z} \tilde{T}_{z^{-1}}$$

where $m_{x,y,z}$ is a polynomial in $\xi = (q^{1/2} - q^{-1/2})$ with integral coefficients ≥ 0 and of degree $\leq \nu$.

Corollary 4.6 *For any $z \in W$, we have $a(z) \in \gamma$.*

Proof of Corollary: This is clear since $\tilde{T}_w \in C_w + q^{1/2} \sum_{y<w} \mathcal{A}^+ \cdot C_y$.

Proof of Theorem 4.5 Before starting the proof, we will fix some notation for the affine Weyl group. Let E be an affine Euclidean space of finite dimension $\ell \geq 1$ with a given set F of hyperplanes (In the case of simply connected almost simple algebraic groups over an algebraically closed field of characteristic $p > 0, E = X(T) \otimes \mathbb{R}$ and the set F consists of the hyperplanes $H_{\check{\alpha},n} = \{\lambda \in E | \check{\alpha}(\lambda + \rho) = np\}$ where $\check{\alpha}$ is a coroot and $n \in \mathbb{Z}$).

Now each $H \in F$ defines an orthogonal reflection $\varrho \to \varrho\sigma_H$ in E with fixed point set H. We will take Ω, the group of affine transformations generated by $\{\sigma_H | H \in F\}$, to be acting on the right on E. We will also regard Ω as an infinite discrete subgroup of the group of affine motions of E, acting irreducibly on the space of translations of E, and leaving F stable. Now Ω acts simply transitively on the set X of connected components (= alcoves) of $E - \bigcup_{H \in F} H$.

If S denotes the set of Ω-orbits in the set of faces of codimension one of the alcoves, then S *consists of* $\ell+1$ *elements* which can be represented as the $\ell+1$ faces of any given alcove. For $s \in S$ and A, as alcove, we denote by sA the alcove ($\neq A$) which has with A a common face of type s.

The maps $A \to sA$ generate a group W of permutations of X which is a Coxeter group and we will regard our given affine Weyl group to be (W, S) as above. A special point v is a 0-dimensional facet of an alcove such that there are exactly ν hyperplanes (the maximum number possible) in F passing through v (Note that any hyperplane has exactly ν directions).

A quarter with vertex v is a connected component of $E - \bigcup\limits_{\substack{H \in F \\ v \in H}} H$.

It has ℓ walls. For a 0-dimensional facet v of an alcove, we define W_v to be the stabilizer in W of the set of alcoves A containing v in their closure; W_v is a standard parabolic subgroup of W and its longest element w_v has length $= \nu$. Note that W_ν is generated by ℓ elements of S. For each special point v, we choose quarter C_v^+ with vertex v such that for any two special points, v, v', the quarters $C_v^+, C_{v'}^+$ are translates of each other. Let A_v^+ denote the unique alcove contained in C_v^+ and having v in its closure, and put $A_v^- = w_v A_v^+$.

For any alcove A, we define a subset $\mathcal{L}(A)$ of S as follows:

For $s \in S$, if P is the hyperplane supporting the common face of type s of A and sA. Then $s \in \mathcal{L}(A)$ iff A is in the half-space determined by P which meets C_v^+ for every special point v of E. We will now define a 'length function' on the set of alcoves. For any $H \in F$, denote by E_H^+ and E_H^- the two connected components of $E - H$, where E_H^+ so that half-space which meets C_v^+ for every special point v. Given two alcoves A and B, there are only finitely many hyperplanes H separating them and we define
$d(A, B) = \sum_H (\pm 1)$, where we count $+1$ if $A \subset E_H^-, B \subset E_H^+$ and -1 if $A \subset E_H^+, B \subset E_H^-$.

We have $d(A, A) = 0$ and $d(A, B) + d(B, C) + d(C, A) = 0$. To see this last equation, we see that the hyperplanes which contribute to all the three terms, contribute zero to the sum clearly and if H is a hyperplane which contributes to $d(A, B)$ but not to $d(B, C)$, then it means that A and B are in different sides of H whereas B and C are on the same side so that A and C will be on different sides of H and clearly the contribution of H to $d(C,A) = -$ its contribution to $d(A,B)$. Thus, we have

$$d(A, sA) = \begin{cases} 1, \text{ if } s \notin \mathcal{L}(A) \\ -1, \text{ if } s \in \mathcal{L}(A). \end{cases}$$

and $d(A,B) = d(Bw_v, Aw_v)$ for any special point v.

A length function $\delta : X \to \mathbb{Z}$ is defined by $\delta(A) - \delta(B) = d(B,A)$. The existence and uniqueness (up to a constant) of δ are clear from the properties of d. Finally, let us define a partial order $\leq$ on X. We write $A \leq B$ if $\exists$ a sequence of alcoves $A = A_0, \ldots, A_n = B$ such that $\delta(A_i) = \delta(A_{i-1}) + 1$ and $A_i = A_{i-1} \cdot \sigma_{H_i}$ for some $H_i \in F$, $\forall\ i = 1, \ldots, n$.

Note that $A < B \Rightarrow d(A,B) > 0$.

Now, consider the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module $\mathcal{M}$ with basis corresponding to the alcoves. This can be regarded as a left H-module by defining

$$T_s A = \begin{cases} sA & ,\text{if } s \notin \mathcal{L}(A) \\ q(sA) + (q-1)A & ,\text{if } s \in \mathcal{L}(A). \end{cases}$$

Since

$$\delta(sA) = \begin{cases} \delta(A) - 1 \text{ if } s \in \mathcal{L}(A) \\ \delta(A) + 1 \text{ if } s \notin \mathcal{L}(A), \end{cases}$$

if we set $\tilde{A} = q^{-\frac{\delta(A)}{2}} A$, then

$$\tilde{T}_s \tilde{A} = \begin{cases} \widetilde{sA} & , \text{ if } s \notin \mathcal{L}(A) \\ \widetilde{sA} + (q^{1/2} - q^{-1/2})\tilde{A} & , \text{ if } s \in \mathcal{L}(A). \end{cases}$$

We have fixed all the necessary notations etc. To complete the proof of the theorem, we need the following three lemmata.

Lemma 1: For any x, y, z in W, $m_{x,y,z}$ is a polynomial in $\xi = (q^{1/2} - q^{-1/2})$ with integral nonnegative coefficients and of degree $\leq \min(\ell(x), \ell(y), \ell(z))$.

Proof: We will show first, by induction on $\ell(x)$, that $m_{x,y,z}$ is a polynomial in $\xi = q^{1/2} - q^{-1/2}$ with integral non-negative coefficients, of degree $\leq \ell(x)$.

Now, $\tilde{T_s}\tilde{T_y} = \tilde{T_{sy}}$ or $(q^{1/2} - q^{-1/2})\tilde{T_y} + tildeT_{sy}$ according as $sy > y$ or $sy < y$. Assume the induction hypothesis for some $x \in W$. Let $sx > x$. Now

$$\widetilde{T_{sx}}\tilde{T_y} = \tilde{T_s}\sum_z m_{x,y,z^{-1}}\tilde{T_z}$$

$$= \sum_{sz>z} m_{x,y,z^{-1}}\tilde{T_{sz}} + \sum_{sz<z}((q^{1/2} - q^{-1/2})m_{x,y,z^{-1}}\tilde{T_z} + m_{x,y,z^{-1}}\tilde{T_{sz}}.$$

Therefore, by induction, our claim follows easily.
Similarly, we also have that $m_{x,y,z}$ is a polynomial of degree $\leq \ell(y)$. Now, we have

$$m_{x,y,z} = \tau(\tilde{T_x}\tilde{T_y}\tilde{T_z}) = \tau(\tilde{T_y}\tilde{T_z}\tilde{T_x}) = m_{y,z,x}.$$

Hence deg $m_{x,y,z} =$ deg $m_{y,z,x} \leq \ell(z)$.

Lemma 2: If A is an alcove and $w \in W$, then $\tilde{T_w}\tilde{A} = \sum_B M_{w,A,B}\tilde{B}$ (finite sum), where $M_{w,A,B}$ are polynomials in $\xi = q^{1/2} - q^{-1/2}$ with non-negative integral coefficients, and of degree $\leq \nu$.

Proof: Again by induction on $\ell(w)$, it follows that $M_{w,A,B}$ are polynomials with non-negative integral coefficients.

Choose a special point v in the closure of A. Write uniquely $w = w' \cdot w_1$ where $w_1 \in W_v, w'$ has minimal length in $w'W_v$ and $\ell(w) = \ell(w') + \ell(w_1)$.

We write $A = w_2(A_v^-)$, where $w_2 \in W_v$ since A contains v in its closure and W_v is the stabilizer in W of the set of alcoves containing v in their closure. So $\tilde{A} = \tilde{T}_{w_2}\tilde{A_v^-}$ and hence

$$\begin{aligned}\tilde{T}_{w_1}\tilde{A} = \tilde{T_{w_1}}\tilde{T_{w_2}}\widetilde{A_v^-} &= \sum_{w_3 \in W_v} m_{w_1,w_2,w_3^{-1}}\tilde{T}_{w_3}\widetilde{A_v^-} \\ &= \sum_{w_3 \in W_v} m_{w_1,w_2,w_3^{-1}}\widetilde{w_3(A_v^-)},\end{aligned}$$

where, by Lemma 1, $m_{w_1,w_2,w_3^{-1}}$ has degree at the most $\ell(w_3)$ in $(q^{1/2} - q^{-1/2})$.

For a fixed $w \in W_v$, write $c = w_3(A_v^-)$. Let $w' = s_k \dots s_1$ be a reduced expression.

Then, it can easily be verified by induction on k, that

$$\tilde{T}_{w_1} \cdot \tilde{C} = \tilde{T}_{s_k} \ldots \tilde{T}_{s_1}(\tilde{C}) = \sum_I (q^{1/2} - q^{-1/2})^{p_I} \cdot s_k \ldots \hat{s}_{i_{p_I}} \ldots \widetilde{\hat{s}_{i_2} \ldots \hat{s}_{i_1} \ldots s_1(C)},$$

where the subsets $I = \{i_1 < i_2 < \ldots < i_{p_I}\}$ of $\{1, 2, \ldots, k\}$ are such that

$$s_{i_t} \ldots \hat{s}_{i_{t-1}} \ldots \hat{s}_{i_2} \ldots \hat{s}_{i_1} \ldots s_1(C) < \hat{s}_{i_t} \ldots \hat{s}_{i_{t-1}} \ldots \hat{s}_{i_2} \ldots \hat{s}_{i_1} \ldots s_1(C)\ \forall\, t = 1, 2, \ldots, p_I.$$

Therefore if $\mathcal{C}$ is the quarter with vertex v and containing C and if $\mathcal{J}(\mathcal{C})$ is the set of all directions i such that, for some hyperplane H with direction i, we have $\mathcal{C} \subset E_H^-$, then it can be proved by induction on $|\mathcal{J}(\mathcal{C})|$ that ([L7], §4.3) $p_I \leq |\mathcal{J}(\mathcal{C})|$. But for our C, we see that $|\mathcal{J}(\mathcal{C})| = \nu - \ell(w_3)$ because $d_i(A_v^-, C) = \ell(w_3)$ for those directions i such that $\mathcal{C} \subset E_H^+$ for each hyperplane with direction i. Therefore,

$$\tilde{T}_w \tilde{A} = \sum_{w_3, I} m_{w_1, w_2, w_3^{-1}} \xi^{p_I} \cdot s_k \ldots \hat{s}_{i_{p_I}} \widetilde{\ldots \hat{s}_{i_1} \ldots s_1(C)}.$$

Therefore $M_{w,A,B}$ has degree $\leq \nu$ since $\deg(m_{w_1, w_2, w_3^{-1}} \xi^{p_I}) \leq \nu$.

Lemma 4.7 *For $y \in W$, $\exists$ an alcove A such that $\tilde{T}_y \tilde{A} = \widetilde{yA}$.*

Proof: Let v be a special point in E and write $y = y' \cdot y_1$ with $y_1 \in W_v$ and y' of minimal length in $y'W_v$. Define A to be $(y_1^{-1} w_v) A_v^- = y_1^{-1} A_v^+$. Therefore $A_v^+ = y_1 A$. Now, there are $\ell(y_1)$ hyperplanes separating A and A_v^+ and each such H gives a contribution $+1$ to $d(A, A_v^+)$ since $A_v^+ \subseteq E_H^+$. Thus $d(A, A_v^+) = \ell(y_1)$ i.e. $\delta(y_1 A) = \delta(A) + \ell(y_1)$.

Similarly $d(y' A_v^+, A_v^+) = -\ell(y'^{-1}) = -\ell(y')$ i.e. $d(A_v^+, yA) = \ell(y')$. Therefore $d(A, yA) = \ell(y_1) + \ell(y') = \ell(y)$. Thus $\delta(yA) = \delta(A) + \ell(y)$. Thus $\tilde{T}_y \tilde{A} = \widetilde{yA}$.

Completion of proof of Theorem 4.5 : Let $x, y \in W$ be given. Select A corresponding to y as in the above lemma. Therefore

$$\begin{aligned} \tilde{T}_x \tilde{T}_y \tilde{A} &= \tilde{T}_x \widetilde{yA} = \sum_{B \in X} M_{x, yA, B} \tilde{B} = \sum_{z \in W} m_{x, y, z^{-1}} \cdot \tilde{T}_z \tilde{A} \\ &= \sum_{z, B} m_{x, y, z^{-1}} \cdot M_{z, A, B} \cdot \tilde{B}. \end{aligned}$$

Thus $M_{x,yA,B} = \sum_z m_{x,y,z^{-1}} \cdot M_{z,A,B} \; \forall \; B \in X$. By Lemma 2, $M_{x,yA,B} \in \mathbb{Z}[\xi]$ has ≥ 0 coefficients and has degree $\leq \nu$. Therefore $\forall \; z, B, m_{x,y,z^{-1}} \cdot M_{z,A,B} \in \mathbb{Z}[\xi]$ has degree $\leq \nu$.

Take $B = zA$; then $M_{z,A,B} \neq 0$. Therefore $M_{z,A,zA}(\xi = 0) = 1$. Thus $m_{x,y,z^{-1}}$ has degree $\leq \nu \; \forall \; z \in W$. Thus, the proof of the Theorem is complete.

Theorem 4.8 *(a)* $C_s C_x = \begin{cases} -(q^{1/2} + q^{-1/2})C_x, & \text{if } \; sx < x \text{ if } s \in \mathcal{L}(x) \\ \sum\limits_{\substack{y \leftrightarrow x \\ sy < y}} \mu(y,x) C_y, & \text{if } \; sx \not< x \text{ i.e. if } s \notin \mathcal{L}(x). \end{cases}$

(b) Writing $C_{w'}, C_w = \sum_y \alpha_{w',w,y}(q) C_y$ *and* $\alpha_{w',w'y}(q) = \sum_{i \in \mathbb{Z}} C_i q^{1/2}$, *we have* $(-1)^i C_i \geq 0$.

(c) $T(C_x C_y D_z) = q^{-\frac{a(z)}{2}} \cdot C_{x,y,z} +$ *Higher powers of* q.

(d) $C_{x,y,z} \neq 0 \Rightarrow z \underset{L}{\leq} x^{-1}, z^{-1} \underset{L}{\leq} y$.

Proof: Proof of (b) involves (i) intersection Cohomology (i.e. Theorem 3.4) (ii) interpreting the multiplication in H via sheaves and (iii) induction, and we will not give it here.

(a) If $sx < x$, then clearly the Corollary 2.5 (of proof) to the main theorem 2.3 gives $T_s C_x = -C_x$ and so $C_s C_x = (q^{-1/2} T_s - q^{1/2}) C_x = -(q^{-1/2} + q^{1/2}) C_x$.

If $sx > x$, the same Corollary gives

$$C_s C_x = C_{sx} + \sum_{\substack{y \prec x \\ sy < y}} \mu(y,x) C_y.$$

Now, if $x \prec y$ and $s \in \mathcal{L}(y) \backslash \mathcal{L}(x)$, then the lemma 4.2 gives $y = sx$ and $\mu(x,y) = 1$.

Thus, for $sx > x$, we can write

$$C_s C_x = \sum_{\substack{y \leftrightarrow x \\ s \in \mathcal{L}(y)}} \mu(y,x) C_y.$$

(c) Writing $C_x C_y = \sum_r \alpha_r(q) \cdot C_r$ we have $\tau(C_x C_y D_z) = \alpha^{z^{-1}}(q)$ since

$$\tau(C_r D_z) = \begin{cases} 0, r \neq z^{-1} \\ 1, r = z^{-1}. \end{cases}$$

But, by definition of $C_{x,y,z}$,

$$\alpha_{z^{-1}}(q) = q^{-\frac{a(z)}{2}} \cdot C_{x,y,z} + \text{ Higher powers of } q.$$

(d) Now $C_{x,y,z} \neq 0 \Rightarrow \tau(C_x C_y D_z) \neq 0$ by (c). Therefore $C_y D_z \neq 0$ and similarly $D_z C_x \neq 0$ because $\tau(C_r D_s) = \tau(D_r C_r)$. Write $C_y \cdot D_z = \sum_t \alpha_t \cdot D_t, \alpha_{z_0} \neq 0$ for some z_0. Then $\tau(C_{z_0^{-1}} \cdot C_y \cdot D_z) = \alpha_{z_0} \neq 0$. Now write $C_{z_0^{-1}} C_y = \sum_u \beta_u \cdot C_u$. Then we have $\tau(C_{z_0^{-1}} \cdot C_y \cdot D_z) = \beta_{z^{-1}}$.

$$\beta_{z^{-1}} = \alpha_{z_0} \neq 0.$$

Recall that $h \cdot C_x \in \sum_{u \underset{L}{\leq} x} \mathcal{A} \cdot C_u \; \forall \; h \in H$, from Corollary 2.5. (The basis $\{C_u\}$ is defined precisely for this reason.)

Thus, taking $h = C_{z_0^{-1}}$, we have $z^{-1} \underset{L}{\leq} y$. Similarly, we will have $z \underset{L}{\leq} x^{-1}$ from the fact that $D_z \cdot C_x \neq 0$.

Proposition 4.9 *Let $x, y, z, z' \in W$ such that $z' \leftrightarrow z$. Let $s \in \mathcal{R}(z') - \mathcal{R}(z)$.*
(1) If $q^{i/2} \cdot \tau(C_x C_y D_z)$ has a non-zero constant term, then $\exists\ x'$ such that $q^{i/2} \cdot \tau(C_{x'} C_y D_{z'})$ has a non-zero constant term.

(2) $a(z') \geq a(z)$.

(3) $C_{x,y,z} \neq 0 \Rightarrow \mathcal{R}(y) = \mathcal{L}(z), \mathcal{L}(x) = \mathcal{R}(z)$.

Proof: (1) Now $\tau(C_x C_y D_z) \neq 0$ i.e $\tau(C_y D_z C_x) \neq 0$ i.e. $D_z C_x \neq 0$. Therefore $z \underset{L}{\leq} x^{-1}$ from the proof of (d) in theorem 4.5.

Thus, $\mathcal{R}(z) \supseteq \mathcal{R}(z^{-1})$ by Proposition 4.4. Now, by hypothesis, $s \notin \mathcal{R}(z)$ and so $s \notin \mathcal{R}(x^{-1})$. Therefore $sx > x$.

Writing $C_x C_y = \sum\limits_w \alpha_w C_w$, we have

$$\begin{aligned} C_s C_x C_y &= \alpha_{z^{-1}} C_s C_{z^{-1}} + \sum_{w \neq z^{-1}} \alpha_w C_s C_w \\ &\in (\alpha_{z^{-1}} \cdot \mu(z'^{-1}, z^{-1}) + \delta) C_{z'^{-1}} + \sum_{w' \neq z'^{-1}} \mathcal{A} \cdot C_{w'} \end{aligned}$$

by (a) of Theorem 4.5 where δ denotes the coefficient of $C_{z'^{-1}}$ in $\sum\limits_{w \neq z^{-1}} \alpha_w C_s C_w$ and is equal to $-\alpha_{z'^{-1}}(q^{1/2}+q^{-1/2}) + \sum\limits_{\substack{w \neq z^{-1}, z'^{-1} \\ sw > w}} \alpha_w \cdot \mu(z'^{-1}, w)$. Writing $\beta_z'^{-1}$ for $\alpha_{z^{-1}} \cdot \mu(z'^{-1}, z^{-1}) + \delta$ and a_i, b_i, d_i for the coefficients of $q^{-i/2}$ in $\alpha_{z^{-1}}, \beta_{z'^{-1}}, \delta$ respectively, we have $b_i = a_i \cdot \mu(z'^{-1}, z^{-1}) + d_i$. But $(-1)^i a_i, (-1)^i d_i \geq 0$ by (b) of Theorem 4.5. By hypothesis $a_i \neq 0$ and $\mu(z'^{-1}, z^{-1}) \neq 0$. So $(-1)^i b_i > 0$. Thus $b_i \neq 0$.

Also, $(C_s C_x) C_y = \sum\limits_{\substack{x' \leftrightarrow x \\ sx' < x'}} \mu(x', x) C_{x'} C_y$ by (a) of Theorem 4.5 so that

$$\sum_{\substack{x' \leftrightarrow x \\ sx' < x'}} \mu(x', x) \cdot \tau(C_{x'} C_y D_{z'}) = \beta_{z'^{-1}} \tag{4.1}$$

since $C_s C_x C_y \in \beta_{z'^{-1}} \cdot C_{z'^{-1}} + \sum\limits_{w' \neq z'^{-1}} \mathcal{A} \cdot C_{w'}$.

Comparing coefficients of $q^{-i/2}$ on both sides of (4.2) we have some x' such that the constant term of $q^{i/2} \cdot \tau(C_{x'} C_y D_{z'}) \neq 0$.

(2) For $i = a(z) - 1$, $\exists$ some x, y in W such that $q^{i/2} \cdot \tau(C_x C_y D_z) \notin \mathcal{A}^+$, by the definition of $a(z)$. Therefore $\exists\, j > 0$ such that $q^{\frac{i+j}{2}} \cdot \tau(C_x C_y D_z)$ has non-zero constant term. Therefore, by (1) above, $\exists\, x'$ such that $q^{\frac{i+j}{2}} \cdot \tau(C_{x'} C_y D_{z'})$ has non-zero constant term. Thus $i = a(z) - 1 < a(z')$.

(3) $C_{x,y,z} \neq 0 \Rightarrow \mathcal{R}(z) \supseteq \mathcal{L}(x), \mathcal{L}(z) \supseteq \mathcal{R}(y)$ by (d) of Theorem 4.5 and Proposition 4.3. Assume that

$$t \in \mathcal{L}(z) - \mathcal{R}(y). \tag{4.2}$$

Writing $C_x C_y = \sum\limits_w \alpha_w C_w$, we have

$$\begin{aligned} C_x C_y C_t &= \alpha_{z^{-1}} C_{z^{-1}} C_t + \sum_{w \neq z^{-1}} \alpha_w \cdot C_w C_t \\ &\in \beta_{z^{-1}} \cdot C_{z^{-1}} + \sum_{w' \neq z^{-1}} \mathcal{A} \cdot C_{w'} \end{aligned}$$

where $\beta_{z^{-1}} = -\alpha_{z^{-1}}(q^{1/2} + q^{-1/2}) + \delta$, from the equation (4.B), δ being the coefficient of $C_{z^{-1}}$ from the sum $\sum\limits_{w \neq z^{-1}} \alpha_w C_w C_t$.

Denoting by m_i, n_i, p_i the coefficients of $q^{-i/2}$ in $\beta_{z^{-1}}, \alpha_{z^{-1}}, \delta$ resp., we get $m_i = -n_{i-1} - n_{i+1} + p_i$. Taking $i = a(z) + 1$, we have $m_{a(z)+1} \neq 0$ since $n_{a(z)+2} = 0$ and since $(-1)^{a(z)} \cdot n_{a(z)} > 0, (-1)^{a(z)+1} \cdot p_{a(z)+1} \geq 0$. Thus, the coefficient of $q^{\frac{-a(z)-1}{2}}$ in $\tau(C_x C_y D_z)$ is $\neq 0$ and so $q^{\frac{+a(z)}{2}} \cdot \tau(C_x C_y D_z) \notin \mathcal{A}^+$, which is a contradiction of the definition of $a(z)$.

Thus $\mathcal{L}(z) = \mathcal{R}(y)$. Similarly $\mathcal{R}(z) = \mathcal{L}(x)$.

Corollary 4.10 *(i) If $z' \underset{L}{\leq} z$, then $a(z') \geq a(z)$. Thus, a is a constant function on two-sided cells.*

(ii) If $z' \leftrightarrow z, \mathcal{L}(z') \not\subseteq \mathcal{L}(z), \mathcal{R}(z') \not\subseteq \mathcal{R}(z)$, then $a(z') > a(z)$.

In particular, by (i), z and z' are in different two-sided cells.

Proof: (i) It is clear that it is enough to assume $z' \leftrightarrow z$ and either $\mathcal{L}(z') \not\subseteq \mathcal{L}(z)$ or $\mathcal{R}(z') \not\subseteq \mathcal{R}(z)$. In either case, (2) of the Proposition 4.6 gives $a(z') \geq a(z)$.

(ii) We know by (i) that $a(z') \geq a(z)$. Assume $a(z') = a(z)$. Now, $\exists\, x, y$ such that $C_{x,y,z} \neq 0 \Rightarrow q^{\frac{-a(z)}{2}} \cdot \tau(C_x C_y D_z)$ has non-zero constant term. Therefore by (1) of Proposition 4.9, $\exists\, x'$ such that $q^{-\frac{a(z)}{2}} \cdot \tau(C_{x'} C_y D_{z'})$ has non-zero constant term.
Since $a(z) = a(z')$, the above statement gives $C_{x',y,z'} \neq 0$. By (3) of Proposition 4.9, we have $\mathcal{R}(y) = \mathcal{L}(z) = \mathcal{L}(z')$. This is a contradiction of the assumption that $\mathcal{L}(z') \not\subseteq \mathcal{L}(z)$. Therefore, $a(z') > a(z)$.

Remark 4.1 *The Corollary carries over to affine Weyl groups if we consider the elements D_y to be in the 'completion' $\hat{H}$ of formal (possibly infinite) $\mathcal{A}$-linear combinations of the $\tilde{T}_w$. No problem will arise since we do not have products like $D_y D_z$ etc.*

Proof of Theorem 4.1 Recall that we defined E to be the free $Q[q^{1/2}]$-module with a basis $(e_w)_{w\in W}$. E is a left H-module under the action

$$T_s e_w = \begin{cases} -e_w, \text{ if } s \in \mathcal{L}(w) \\ qe_w + q^{1/2} e_{s_w} + q^{1/2} \cdot \sum\limits_{\substack{y \prec w \\ sy < y}} \mu(y,w) e_y, \text{ if } s \notin \mathcal{L}(w) \end{cases}$$

i.e.

$$T_s e_w = \begin{cases} -e_w, \text{ if } s \in \mathcal{L}(w) \\ qe_w + q^{1/2} \cdot \sum\limits_{\substack{y \leftrightarrow w \\ s \in \mathcal{L}(y)}} \mu(y,w) e_y, \text{ if } s \notin \mathcal{L}(w) \end{cases}$$

by Lemma 4.2.

Similarly, we have a right H-module structure on E. E is also a left W-module under the action

$$s * e_w = \begin{cases} -e_w, \text{ if } s \in \mathcal{L}(w) \\ e_w + \sum\limits_{\substack{y \leftrightarrow w \\ s \in \mathcal{L}(y)}} \mu(y,w) e_y, \text{ if } s \notin \mathcal{L}(w). \end{cases}$$

The proof of the theorem would be much simpler if the left H-action and the right W-action commute on E but this does not happen. So, given any two-sided cell $X \subset W$, consider the $Q[q^{1/2}]$-submodule E_X of E spanned by $\{e_w | w \underset{LR}{\leq} X\}$.
Define E'_X to be the submodule spanned by $\{e_w | w \underset{LR}{\leq} X\}$. Then E_X/E'_X is a H-module.
Let $gr(E) = \bigoplus\limits_{\substack{X \\ 2-sided\ cell}} E_X/E'_X$. Hence $gr(E)$ is naturally a left H-module structure and a right W-module.

We, firstly, claim that the left H-module structure and the right W-module structure commute on $gr(E)$. (This is true for affine Weyl groups also as the proof uses only Corollary 4.10).

Note that $gr(E)$ has dimension $= dim.E$; infact it has a canonical basis $\{\overline{e_w}\}$, the images of $\{e_w\}$.
Let $s, t \in S$ and $w \in W$. Then

$$(T_s e_w) * t - T_s(e_w * t) = \begin{cases} 0, \text{ unless } s \notin \mathcal{L}(w), t \notin \mathcal{L}(w) \\ (q^{1/2}-1)^2 \sum\limits_{\substack{y \leftrightarrow w \\ sy<y \\ yt<y}} \mu(y,w)e_y, \text{ if } s \notin \mathcal{L}(w), t \notin \mathcal{L}(w). \end{cases}$$

But the terms in the summand satisfy the conditions of Corollary 4.10 (ii) and so satisfy $y \underset{LR}{<} w$.
Thus $(T_s\overline{e_w}) * t - T_s(\overline{e_w} * t) = 0$ in $gr(E)$.

Now, let $End_W gr(E)$ be the algebra of $Qq[1/2]$-endomorphisms of $gr(E)$ which commute with the right W-action. By what we have proved above, we have canonical homomorphisms $\alpha : H \to End_W gr(E)$ and $\beta : Q[q^{1/2}][W] \to End_W gr(E)$.

We *claim* that β is an isomorphism.

It is enough to show that for *any* homomorphism χ of $Q[q^{1/2}]$ into any field K, the homomorphism $\bar{\beta} : K[w] \to End_W gr(E) \otimes K$ is an isomorphism. Now $\bar{\beta}$ is the composite $K[W] \xrightarrow{\beta'} End_W \otimes K \xrightarrow{\beta''} End_W gr(E) \otimes K$ where $End_W E \otimes K$ denotes those endomorphisms of $E \otimes K$ which commute with the right W-action. Now β' is an isomorphism since the (W, w)-bimodule $E \otimes K$ is the two-sided regular representation of W (see Corollary 2.5). But β'' is an isomorphism since all $K[W]$-modules are semi-simple. Thus β is an isomorphism and (a) follows on taking $\varphi = \beta^{-1}\alpha$.

For (b), consider a homomorphism χ of $Q[q^{1/2}]$ into a field K. Let $h \in Ker\varphi_\chi$. Define $\hat{h}$ to be the endomorphism of $E \otimes K$ which is multiplication by h. Since $h \in Ker\varphi_\chi$, therefore $\hat{h}$ is the zero endomorphism of $gr(E \otimes K)$ and so $\hat{h}$ is a nilpotent endomorphism of $E \otimes K$.

Case (i): If $\chi(q) \neq 0$, then $E \otimes K$ is the left regular representation of H (see Corollary 2.5) and hence $h \mapsto \hat{h}$ from $H \otimes K \to End(E \otimes K)$ is an isomorphism (and thus injective) and so h is nilpotent.

Case (ii): If $\chi(q) = 0$, we consider the filtration of $H \otimes K$ by the two-sided ideals $J_i = \sum\limits_{\ell(w) \geq i} KT_w$. Then, clearly (as $\chi(q) = 0$) the associated graded

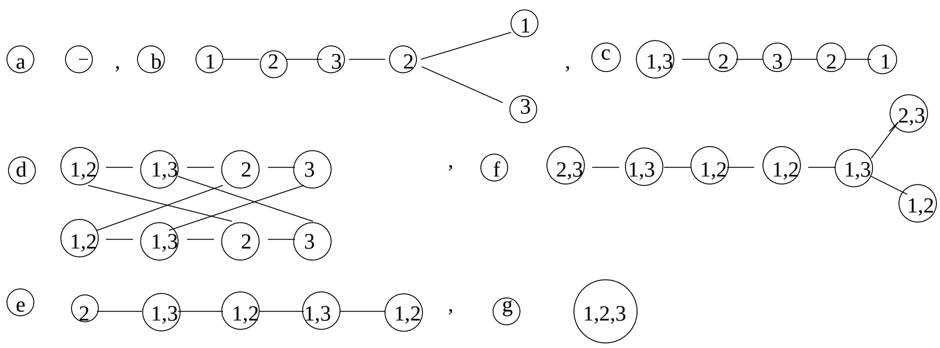
$\bigoplus_i J_i/J_{i+1}$ is isomorphic as a left H-module to $gr(E \otimes K)$. Thus $\hat{h}$ is zero on $\oplus J_i/J_{i+1}$ i.e $hJ_i \subseteq J_{i+1}$ and so h is nilpotent.

Remark 4.2 *(i) An elementary proof of Corollary 4.10 to arbitrary Coxeter groups, is desired.*

(ii) We give below an example which shows clearly that $q^{1/2}$ is necessary for the Theorem 4.1 to hold i.e we show that for the Coxeter group H_3, there are irreducible representations of the Hecke algebra which have characters involving $q^{1/2}$.

H_3 has generators s_1, s_2, s_3 and relations $s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^5 = (s_1 s_3)^5 = 1$. Consider the Hecke algebra H of H_3 over the field $Q(q^{1/2})$.
H has 10 irreducible representations which can be obtained from left cells as follows:

The left cells of H_3 give rise to the following W-graphs (each vertex x is represented by a circle and the set $I_x \subseteq S$ is given inside the circle; here the μ-function is $\equiv 1$).



As defined in the beginning of §4, these W-graphs give rise to seven representations of H of dimensions 1, 6, 5, 8, 5, 6, 1 respectively. The representations (a), (c), (e) and (g) are irreducible. The representations (b) and (f) split over $Q(q^{1/2}, \sqrt{5})$ into two three-dimensional irreducible non-equivalent representations. For example, in (b), the representation spaces has basis elements

$$\begin{array}{ccccc} & & & & x_1' \\ x_1 & x_2 & x_3 & x_2' & \\ & & & & x_3' \end{array}$$

corresponding to the vertices of (b) and the two subspaces $V_\alpha = \langle x_1 + \alpha x_1', x_2 + \alpha x_2', \alpha x_3 + x_3' \rangle$ where $\alpha^2 = \alpha + 1$, are H-stable. We will discuss (d) in details as it is the most interesting case. The representation space has basis elements

$$\begin{array}{cccc} x_{12} & x_{13} & x_2 & x_3 \\ x_{12}' & x_{13}' & x_2' & x_3' \end{array}$$

corresponding to the vertices of the W-graph (d).

We *claim* that the two subspaces $V_\epsilon = \langle x_{1_2} + \epsilon x_{1_2}', x_{1_3} + \epsilon x_{1_3}', x_2 + \epsilon X_2', X_3 + \epsilon X_3' \rangle, \epsilon = \pm 1$ are H-stable irreducible 4-dimensional representations and that the element T_{w_0} acts on V_ϵ as multiplication by $\pm q^{15/2}$ where w_0 is the longest element of H_3.

In fact, w_0 turns out to be $(s_3 s_2 s_1)^5$.

Also, $\langle x_{1_2} + x_{1_2}', x_{1_3} + x_{1_3}', x_2 + x_2', x_3 + x_3' \rangle$ is H-stable and, infact, the matrices $T_{s_1}, T_{s_2}, T_{s_3}$ with respect to this ordered basis are $\begin{pmatrix} -1 & 0 & q^{1/2} & 0 \\ 0 & -1 & q^{1/2} & q^{1/2} \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$,

$\begin{pmatrix} -1 & q^{1/2} & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & q^{1/2} & -1 & q^{1/2} \\ 0 & 0 & 0 & q \end{pmatrix}$ and $\begin{pmatrix} q & 0 & 0 & 0 \\ q^{1/2} & -1 & q^{1/2} & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & q^{1/2} & -1 \end{pmatrix}$ respectively.

Then $T_{s_3 s_2 s_1} = \begin{bmatrix} q & -q^{3/2} & q^2 - q^{3/2} & q^2 \\ q^{1/2} & -q & -q & q^2 + q^{3/2} \\ 0 & -q^{3/2} & 0 & q^2 + q^{5/2} \\ 0 & -q & 0 & q^{3/2} \end{bmatrix}$,

$$T_{(s_3 s_2 s_1)^2} = \begin{bmatrix} 0 & -q^{7/2} & q^3 & -q^{7/2} + q^{9/2} \\ 0 & -q^3 & q^{5/2} & -q^3 \\ -q^2 & q^{5/2} - q^3 - q^{7/2} & q^{5/2} & -q^3 + q^4 \\ -q^{3/2} & q^2 - q^{5/2} & q^2 & -q^{5/2} \end{bmatrix},$$

$$T_{(s_3s_2s_1)^4} = \begin{bmatrix} -q^6 & -q^{13/2}-q^7 & -q^6+q^{13/2} & q^{13/2} \\ 0 & 0 & -q^{11/2} & q^6+q^{13/2} \\ -q^{11/2} & q^6 & -q^{11/2} & q^6 \\ 0 & 0 & -q^5 & q^{11/2} \end{bmatrix} \text{ and } T_{w_0} = -q^{15/2}Id.$$

Thus, the character of T_{w_0} genuinely involves $q^{1/2}$.

Let us *consider the conjecture of finiteness of the number of left (right) cells* in the case of a *finite set S of generators* of the Coxeter system (W, S).

Let $\hat{W}$ denote the set of equivalence classes of irreducible representations of W. Then, there is a map (not necessarily one-one) from $\hat{W}$ to the set of two-sided cells.

Let V be an irreducible $\mathbb{Q}[W]$-module and $\mathcal{C}$ be the corresponding two-sided cell. Let $K = \mathbb{Q}(q^{1/2}), E_{\mathcal{C}} = \bigoplus\limits_{w\in\mathcal{C}} Ke_w$. Then $E_{\mathcal{C}}$ is a left H_K-module under the action $T_s \cdot e_w = \begin{cases} -e_w, \text{ if } sw < w \\ qe_w + \sqrt{q} \sum\limits_{\substack{y\leftrightarrow w \\ sy<y \\ y\in\mathcal{C}}} \mu(y,w)e_y, \text{ if } sw > w. \end{cases}$

$E_{\mathcal{C}}$ is also a right W-module under the action

$$e_w * s = \begin{cases} -e_w, \text{ if } ws < w \\ e_w + \sum\limits_{\substack{y\leftrightarrow w \\ ys<Y \\ y\in\mathcal{C}}} \mu(y,w)e_y, \text{ if } ws > w. \end{cases}$$

Consider $V(q) = (E_{\mathcal{C}} \otimes_{\mathbb{Q}} V)^W$ which is a left H-module by virtue of the action of H on $E_{\mathcal{C}}$, and is irreducible.

Thus $V \to V(q)$ defines a bijection between irreducible representations of W and those of H.

Now, each irreducible representation of W has a canonical direct sum decomposition as follows:

Let $\mathcal{C} = \bigcup\limits_i \Gamma_i$ where Γ_i are right cells. Then, corresponding $E_{\mathcal{C}} = \bigoplus\limits_i E_{\Gamma_i}$.

If each E_{Γ_i} is itself a right W-module, then the following conjecture will be true since $(E_{\Gamma_i} \bigotimes\limits_{\mathbb{Q}} V)^W$ is meaningful.

Conjecture: $V(q) = \bigotimes\limits_i (E_{\Gamma_i} \bigotimes\limits_{\mathbb{Q}} V)^W$.

In most of the cases $(E_{\Gamma_i} \bigotimes\limits_{\mathbb{Q}} V)^W$ will turn out to be one-dimensional spaces. So, to show that each E_{Γ_i} is a right W-module, we must show :

Conjectured lemma 4.8.1

$$y \underset{R}{\leq} y' \text{ and } y \underset{LR}{\sim} y' \Rightarrow y \underset{R}{\sim} y'.$$

More generally, it is enough to show the stronger :

Conjectured lemma 4.8.2

$$y \underset{R}{\leq} y' \text{ and } a(y) = a(y') \Rightarrow y \underset{R}{\sim} y'.$$

Assuming this lemma, we can write $E_a = \bigoplus\limits_{\substack{w \\ a(w)=a}} Ke_w = \bigoplus\limits_i E_{\Gamma_i}$. Then E_a would be a left as well as a right W-module. In the case of an affine Weyl group W, E_a can be regarded as a module over the translations in the affine Weyl group and will be a subquotient of the group ring of this translation part, and hence will be finitely generated. Thus, the conjecture of finiteness of the # of right (left) cells would be true.

We now prove the conjectured lemma in the case of *finite* W. (Of course, the 'finiteness of cells' itself is a trivial problem here.) For this we first prove the

Lemma 4.11 $C_{x,y,z} = C_{y,z,x} = C_{z,x,y}$.

Proof: If $c = C_{x,y,z} \neq 0$, then $q^{\frac{a(z)}{2}} \cdot \tau(\tilde{T}_y D_z D_x)$ has constant term $= C$. Also, $c \neq 0 \Rightarrow z \underset{L}{\leq} x^{-1} \Rightarrow a(x) = a(x^{-1}) \leq a(z)$ by Proposition 4.9 (2).
Now $q^{\frac{a(z)}{2}} \cdot \tau(\tilde{T}_y \cdot D_z \cdot D_x) \in q^{\frac{a(z)}{2}} \cdot \tau(\tilde{T}_y \tilde{T}_z \cdot D_x) + q^{\frac{a(z)+1}{2}} \cdot \sum\limits_{z' \neq z} \mathcal{A}^+ \cdot \tau(\tilde{T}_y \tilde{T}_{z'} \cdot D_x)$
writing for D_x in terms of the basis $\{\tilde{T}_w\}$. But the left-hand side and the first term on the right-hand side have constant term $= C$ and so the last term has constant term $= 0$. But then, by the definition of $a(x), a(z) + 1 < a(x)$ i.e $a(z) \leq a(x)$. Thus $a(x) = a(z)$.

Therefore $C_{x,y,z} = C_{y,z,x}$. Cyclically hence $C_{x,y,z} = C_{y,z,x} = C_{z,x,y}$.

Corollary 4.12 $C_{x,y,z} \neq 0 \Rightarrow x \underset{L}{\sim} y^{-1}, y \underset{L}{\sim} z^{-1}, z \underset{L}{\sim} x^{-1}$.

Proof of conjectural lemma 4.8.2 for finite W

We can assume without loss of generality that $z \leftrightarrow z'$ and $\mathcal{R}(z') \not\subseteq \mathcal{R}(z)$. Now, $\exists\ x, y$ such that $C_{x,y,z} \neq 0$ i.e. such that $q^{\frac{a(z)}{2}} \cdot \tau(C_x C_y D_z)$ has non-zero constant term. Therefore by (1) of Proposition 4.9, $\exists\ x'$ such that $q^{\frac{a(z)}{2}} \cdot \tau(C_{x'} C_y D_{z'})$ has a non-zero constant term. Thus $C_{x',y,z'} \neq 0$. Therefore by Corollary 4.12, $y^{-1} \underset{R}{\sim} z'$ and $y^{-1} \underset{R}{\sim} z$. Thus $z \underset{R}{\sim} z'$.

Remark 4.3 *Note that Lemma 4.8.2 would be true for affine Weyl groups if conjecture 4.4 is true.*

5 Representation Theory and Intersection Cohomology

Intersection cohomology theory plays an important role in representation theory. We have already seen its role in the first aspect mentioned below. We will discuss briefly the following aspects:

1. Multiplicities of Verma modules ($\Leftrightarrow$ problem about singularities of Schubert variety).

2. Character formula (conjectural) for finite-dimensional irreducible rational representations of a semi-simple algebraic group$/k = \bar{k}$ of char.p.

3. Multiplicities of weights in finite-dimensional irreducible representations of a semi-simple algebraic group$/\mathbb{C}$.

4. Representations of Weyl groups.

5. Representations of real Lie groups.

6. Representations of Chevalley groups over finite fields.

7. Representations of p-adic groups (Conjecture).

5.1 Conjectural character formula in positive characteristic

Let G be a simply-connected, almost simple algebraic group defined over $\overline{\mathbb{F}_{\mathrm{p}}}$. We wish to state a modular analogue of the conjecture of §2 stated for Verma modules in char. 0.

Let T be a maximal torus and B a Borel subgroup containing it. Let W denote the Weyl group. Let $X(T)$ be the character group of T and Q, the subgroup generated by the roots. Let W_{aff} be the group of affine transformation of $X(T)$-generated by W and by translations by elements in pQ.

Then W_{aff} is an infinite Coxeter group: its standard set of generators consists of those of W, together with the reflection in the hyperplane $\{\varphi \in X(T) | \check{\alpha}_0(\varphi) = p\}$, where $\check{\alpha}_0$ is the highest coroot.

If ρ is the element of $X(T)$ defined by the condition that it takes the value 1 on each simple coroot, then an element w of W_{aff} is said to be dominant if $-w\rho - \rho$ is dominant.

Equivalently, $w \in W_{aff}$ is dominant $\Leftrightarrow w = w'w_0$ with $\ell(w) = \ell(w') + \ell(w_0)$ and $w_0 =$ longest element of W. For each dominant $w \in W_{aff}$, let L_w denote the irreducible representation of G, of finite dimension over $\overline{\mathbb{F}}_p$, with highest weight $-w\rho-\rho$. Let V_w be the Weyl representation of G over $\overline{\mathbb{F}}_p$ obtained by reducing modulo p the irreducible representation with highest weight $-w\rho-\rho$ of the corresponding *complex* group. V_w is well-defined in the Grothendieck group.

In fact, one has Weyl character formula for character of V_w as:
$ch\ V_w = \sum\limits_{x \in W} (-1)^{\ell(x)} \cdot e^{-xw(\rho)-\rho} \cdot \prod\limits_{\alpha > 0} (1 - e^{-\alpha})^{-1}$ where $\{e^{\lambda}\}$ is a base of $\mathbb{Z}[X(T)]$ corresponding to $\{\lambda\}$ in $X(T)$.

Then, *as stated in* [L2]:

Conjecture 5.1 *Let $w \in W_{aff}$ be dominant and satisfy the Jantzen condi-*

tion $\check{\alpha}_0(-w\rho) \leq p(p-h+2)$, *where* h *is the Coxeter number. Then*

$$ch\ L_w = \sum_{\substack{y \in W_{aff} \\ y\ dominant \\ y \leq w}} (-1)^{\ell(w)-\ell(y)} \cdot P_{y,w}(1) \cdot ch\ V_y. \tag{5.1}$$

Kato [K1] shows that this conjecture is consistent with the Steinberg tensor product theorem and that, on using results of Jantzen [J], it follows that this modular conjecture contains as a special case the conjecture in §2 of Kazhdan and Lusztig on Verma modules in char. 0. Also, from (5.1) one can deduce the character formula for any irreducible finite dimensional representation of G over $\overline{\mathbb{F}}_p$, by making use of results of Jantzen and Steinberg. The evidence for this character formula is very strong. *Lusztig* has verified it in the cases where G is of type A_2, B_2 or G_2. (Jantzen computed $ch\ L_w$ in these cases).

5.2 Weight multiplicities for complex semisimple groups

Let $\mathcal{J}$ be a complex simple Lie algebra, $b \subseteq \mathcal{J}$ a Borel subalgebra, $h \subseteq b$ a Cartan subalgebra, W the Weyl group and $Q \subseteq h^*$ the subgroup generated by the roots. If $P \subseteq h^*$ is the subgroup consisting of those elements which take integral values on any coroot, then Q has finite index in P. $\tilde{W}_{aff} \subseteq$ set of affine transformations of h^*, is the semidirect product of W and P.

W_{aff} is the affine Weyl group which is generated by W and Q. For $\lambda \in P$, we denote by p_λ the same element regarded in $\tilde{W}_{aff}$ Thus $p_{\lambda+\lambda'} = p_\lambda \cdot p_{\lambda'}$ for $\lambda, \lambda' \in P$.

Though $\tilde{W}_{aff}$ is not a Coxeter group, it has a well-defined partial order and a well-defined length function induced from those of W_{aff}. In fact, $\tilde{W}_{aff} = \Omega \ltimes W_{aff}$ where $\Omega =$ Normaliser of S_{aff} in $\tilde{W}_{aff}$, S_{aff} being the set of simple reflections of W_{aff}.

So, we define the length function on $\tilde{W}_{aff}$ as

$$\ell(\delta w) = \ell(w\delta) = \ell(w)\ \forall\ w \in W_{aff}, \delta \in \Omega.$$

The partial order on W_{aff} is extended to one on $\tilde{W}_{aff}$ by defining

$$\delta w \leq \delta' w' \Leftrightarrow \delta \leq \delta'\ ,\ w \leq w',$$

for $\delta, \delta' \in \Omega$ and $w, w' \in W_{aff}$. By virtue of these definitions, the statement and proof of Theorem 2.3 carry over to $\tilde{W}_{aff}$.

If P^{++} denotes the set of dominant weights in P, then P^{++} parametrizes the double cosets $W \backslash \tilde{W}_{aff}/W$ as $\lambda \leftrightarrow W p_\lambda W$.

We note that for $\lambda, \lambda' \in P^{++}, \lambda \leq \lambda' \Leftrightarrow m_\lambda \leq m_{\lambda'}$ in $\tilde{W}_{aff}$, where m_λ is the longest element in $W p_\lambda W$. Let us denote, as usual, for $\lambda \in h^*$, by M_λ the Verma module for $\mathcal{J}$ with highest weight λ and L_λ its unique irreducible quotient. Then, if $\lambda \in P^{++}$, the $\mathcal{J}$-module L_λ is finite-dimensional. With respect to the action of h, it decomposes into a direct sum $L_\lambda = \bigoplus_\mu + L_{\lambda,\mu}$ of weight spaces parametrized by $\mu \in P$.

If $d_{\mu,\lambda} = dim\ L_{\lambda,\mu}$, then it depends only on the μ-orbit of W and thus it is enough to find $d_{\mu,\lambda}$ for $\mu \in P^{++}$. It is known that $d_{\mu,\lambda} = 0$ unless $\mu \leq \lambda$.

It is proved in [L1] that

$$d_{\mu,\lambda} = P_{m_\mu, m_\lambda}(1). \tag{5.2}$$

Here, of course, P_{m_μ, m_λ} is defined in terms of the Hecke algebra $\tilde{H}$ of $\tilde{W}_{aff}$. In fact $P_{\delta y, \delta w} = P_{y,w}\ \forall\ y, w \in W_{aff}$ and $\delta \in \Omega$. For type A, (5.2) is proved in [L3] where it is also shown that $P_{m_\mu, m_\lambda}(q)$ are the Green-Foulkes polynomials.

In fact, in [L1] a much stronger version of (5.2) is proved where $P_{m_\mu, m_\lambda}(q)$ is interpreted as a q-analogue of the multiplicity $d_{\mu,\lambda}$. These will be written in the following fashion.

Consider the elements

$$k_\lambda = \frac{1}{|w|} \cdot \sum_{w \in W p_\lambda W} w,$$

for $\lambda \in P^{++}$ and

$$j_\lambda = \left(\sum_{w \in W} (-1)^{\ell(w)} \cdot w^{-1} \right) \cdot p_\lambda \cdot \left(\sum_{w \in W} w \right),$$

for $\lambda \in P^{++} + \rho$ of the group algebra $\mathbb{Q}[\tilde{W}_{aff}]$. Then $k_\lambda (\lambda \in P^{++})$ form a $\mathbb{Z}$-basis for the subgroup

$$K^1 = \{x \in \frac{1}{|W|} \cdot \mathbb{Z}[\tilde{W}_{aff}] : \left(\sum_{w \in W} w \right) x = x \left(\sum_{w \in W} w \right) = |W| \cdot x\} \subseteq \mathbb{Q}[\tilde{W}_{aff}]$$

and $j_\lambda(\lambda \in P^{++} + \rho)$ form a $\mathbb{Z}$-basis for the subgroup

$$J^1 = \left\{ y \in \mathbb{Z}[\tilde{W}_{aff}] \,\middle|\, \left(\sum_{w \in W} (-1)^{\ell(w)} \cdot w^{-1} \right) y = y \left(\sum_{w \in W} w \right) = |W| \cdot y \right\}.$$

It follows that K^1 is a subring of $\mathbb{Q}[\tilde{W}_{aff}]$ with unit element $\frac{1}{|W|} \cdot \sum\limits_{w \in W} w$ and that $J^1 \cdot K^1 \subseteq J^1$ and that the map $k \mapsto j_\rho \cdot k$ of K^1 to J^1 is an isomorphism of right K^1-modules.

Then, for $\lambda \in P^{++}$ and $C'^1_\lambda = \sum\limits_{\mu \in P^{++}} d_{\mu,\lambda} \cdot k_\mu \in K^1$, Weyl character formula says that C'^1_λ is the unique element in K^1 satisfying $j_\rho C'^1_\lambda = j_{\lambda+\rho}$.

The q-analogues of the elements j_λ, k_λ are the elements J_λ, K_λ of $\tilde{H}$ defined as $K_\lambda = \frac{1}{\sum\limits_{w \in W_0} q^{\ell(w)}} \sum\limits_{w \in W p_\lambda W} T_w,\ \lambda \in P^{++}$ and

$$J_\lambda = \left(\sum_{w \imath W} (-q)^{\ell(w)} \cdot T_w^{-1} \right) q^{-\ell(n_\lambda)/2} T_{n_\lambda} \left(\sum_{w \in W} T_w \right),\ \lambda \in P^{++} + \rho$$

where W_0 denotes the stabilizer of 0 in W and n_λ is the shortest element in $W p_\lambda W$.

Note that K_λ, J_λ etc. reduce to k_λ, j_λ on putting $q = 1$. It is, then, shown for any $\lambda \in P^{++}$, there is a unique element $C'_\lambda \in \tilde{H} \otimes \mathbb{Q}(q^{1/2})$ such that

$$J_\rho \cdot C'_\lambda = J_{\lambda+\rho} \text{ and } \overline{J_{\lambda+\rho}} = J_{\lambda+\rho} \tag{5.3}$$

and it is of the form $C'_\lambda = q^{-\ell(p_\lambda)/2} \cdot \sum\limits_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} d_{\mu,\lambda}(q) K_\mu$ where $d_{\mu,\lambda}(q)$ are in $\mathbb{Z}[q]$ with deg. $d_{\mu,\lambda}(q) < \frac{\ell(p_\lambda) - \ell(p_\mu)}{2}$ if $\mu < \lambda$ and $d_{\lambda,\lambda}(q) \equiv 1$.

Thus, it is finally shown that for $\mu \leq \lambda$ in P^{++},

$$d_{\mu,\lambda}(q) = P_{m_\mu, m_\lambda}(q). \tag{5.4}$$

Using (5.4), a q-analogue of the Kostant partition function is defined.

In fact, if $\mu \leq \lambda$ in P and $\tau \in P$ satisfies $\langle \tau, \check{\alpha}_s \rangle > 0\ \forall\ s \in S$, the polynomial $P_{m_{\mu+\tau}, m_{\lambda,\tau}}$ is independent of τ i.e. depends only on $\lambda - \mu$. Here S is the set of reflections of W and $\check{\alpha}_s \in h$ are the corresponding simple coroots. Thus, there is a well-defined function

$$\hat{P} = \{k \in Q | k \geq 0\} \to \mathbb{Z}[q^{-1}]$$

which, by (5.4), is such that for any $\mu \leq \lambda$ in P with $\lambda - \mu = k$, satisfies

$$\hat{P}(k) = q^{-\langle k,\rho\rangle} d_{\mu+\tau,\lambda+\tau}(q) \text{ for any } \tau \in P \text{ such that } \langle \tau, \check{\alpha}_s\rangle > 0 \ \forall\ s \in S. \quad (5.5)$$

From (5.3) $\hat{P}(k)$ are seen to satisfy a recurrence relation and it is shown that

$$\hat{P}(k) = \sum_{\substack{n_1,\ldots,n_\nu \geq 0 \\ n_1\alpha_1+\ldots+n_\nu\alpha_\nu = k}} q^{-(n_1+\ldots+n_\nu)} \quad (5.6)$$

where $\{\alpha_1, \ldots, \alpha_\nu\}$ is the set of all positive roots. For $q = 1$, this is the usual Kostant partition function.

It is conjectured here that for $\mu \leq \lambda$ in P^{++}, we have

$$q^{-\langle\lambda-\mu,\check{\rho}\rangle} \cdot d_{\mu,\lambda}(q) = \sum_{w\in W} (-1)^{\ell(w)} \cdot \hat{P}(w(\lambda+\rho) - (\mu+\rho)).$$

This was shown to be true by Kato in [K2].

So, we have actually a formula for $P_{y,w}$ as $\mu \leq \lambda$ in P^{++} :

$$P_{m_\mu,m_\lambda}(q) = q^{\langle\lambda-\mu,2\check{\rho}\rangle} \sum_{w\in W} (-1)^{\ell(w)} \hat{P}(w(\lambda+\rho) - (\mu+\rho)). \quad (5.7)$$

The right-hand side of (5.7) with the special case $\mu = 0$, appears in the work (unpublished) of D. Peterson, regarding the $\mathcal{F}$-module structure of the (graded) co-ordinate ring of the nilpotent variety of $\mathcal{J}$.

In fact, if A is the co-ordinate ring of the nilpotent variety of $\mathcal{J}$, then $A = \bigoplus_{i\geq 0} A_i$, each A_i being a finite-dimensional representation of $\mathcal{J}$. So, given $\lambda \in P^{++}$, Kostant defines $[L_\lambda : A_i]$ to be the generalized exponents of $\mathcal{F}$ and defines the polynomial $F_\lambda(q) = [L_\lambda : A_i]q^i$.

(For L_λ = adjoint representation of $\mathcal{J}, F_\lambda(q) = \sum_{i=1}^{rk\mathcal{J}} q^{e_i-1}$, where e_i are the exponents of $\mathcal{J}$.)

It can be seen by Frobenius reciprocity that $F_\lambda(1) = d_{0,\lambda}$. Peterson showed that

$$F_\lambda(q) = q^{\langle\lambda,2\check{\rho}\rangle} \sum_{w\in W} (-1)^{\ell(w)} \cdot \hat{P}(w(\lambda+\rho)-\rho).$$

Thus, we have

$$F_\lambda(q) = P_{m_0,m_\lambda}(q). \tag{5.8}$$

5.3 Representations of real Lie groups

Note that the idea of the statement of Theorem 3.7 was to use the parametrizing set W of $\hat{\mathcal{J}}$ to define subvarieties $\overline{\mathcal{B}_w}$ of $\mathcal{B}$, the set of Borel subgroups, whose geometry was related to the multiplicities $M_{y,w}$ of Theorem 2.4. There are ways to try to do this for real groups. One, is to use the orbits of the real group G on $\mathcal{B}$ (see §2.3]) but this has the disadvantage that the cells produced are only components of (real) algebraic varieties. We will follow another way.

The proper setting is Harish-Chandra's category of reductive groups. We fix a reductive algebraic group $G_{\mathbb{C}}$ defined over R and assume that G has finite index in the set of real points of $G_{\mathbb{C}}$. Then G is a connected real semisimple linear Lie group. Let K be $G_{\mathbb{C}}^\theta$, where θ is a Cartan involution on G, let $\mathcal{B}$ denote the flag manifold of G. K acts on $\mathcal{B}$ and has finitely many orbits. (Unlike the complex case, the orbits here are not simply connected.) If $x \in \mathcal{B}$, we write K_x for the isotropy group.

A K-equivariant local system on the orbit $K\cdot x$ is specified by a representation of $Kx/(Kx)_0$ on the stalk at x. Thus, one-dimensional stalks play a central role. We define $\mathcal{D}$ to be the set of all pairs $(\theta, \mathcal{L})$, with θ an orbit of K on $\mathcal{B}$, and $\mathcal{L}$ an 1-dim. K-equivalent local system on θ. For $(\theta,\mathcal{L}) \in \mathcal{D}$, we write $\ell((\theta,\mathcal{L}))$ = Length of $(\theta,\mathcal{L}) \overset{d}{=} dim\ \theta$.

Example 5.1 *([LV]) Consider $G_{\mathbb{C}} \times G_{\mathbb{C}}$ with the involution $\theta(x,y) = (y,x)$. Then, the fixed point set K is the diagonal subgroup of $G_{\mathbb{C}} \times G_{\mathbb{C}}$ and its orbits* *on $\mathcal{B}\times\mathcal{B}$ are in one-to-one correspondence with the Weyl group W of $G_{\mathbb{C}}$. Thus, in this case $\mathcal{D}$ may be identified with W and the problem that we will consider will turn out, in the case of this example, to be equivalent to the first aspect which we have already discussed in §3.*

Example 5.2 *([LV]) Suppose $G_{\mathbb{R}} = SL(2,\mathbb{R})$, and*
$\theta(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *on $G_{\mathbb{C}}$.*
Then $K = G_{\mathbb{C}}^{\theta}$ = the torus $\left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} / t \in \mathbb{C}^* \right\}$.
The flag manifold $\mathcal{B} \cong \mathbb{P}' = \mathbb{C} \cup \{\infty\}$. K acts by $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cdot y = t^2 y, t \in \mathbb{C}^*, y \in \mathbb{C} \cup \{\infty\}$. K has 3 orbits $\{0\}, \{\infty\}$, and $\mathbb{C}^*$; the isotropy groups being K, K and $\{\pm I\}$ respectively. Thus, the set $\mathcal{D}$ has four elements: three of them the pairs with the orbit and constant sheaves on them, and a Möbius band coming from the double cover of $\mathbb{C}^*$.

Then, one has the

Proposition 5.1 *([V1]) The set of infinitesimal equivalence classes of irreducible admissible representations of G, having same infinitesimal character as the unit representation, is in a natural 1-1 correspondence with the set $\mathcal{D}$.*

Let us fix a representation (irreducible, finite-dimensional) F of G. Let $(\theta, \mathcal{L}) \in \mathcal{D}$. Let $\bar{X}(\theta, \mathcal{L})$ denote the irreducible $(\mathcal{J}, K)$-module corresponding to $(\theta, \mathcal{L})$ (i.e. the Harish-Chandra module with character defined by F).

Theorem 5.2 *([V1]) (a) $\bar{X}(\theta, \mathcal{L})$ has a finite composition series, and all of its irreducible subquotients are in $\hat{G}$, the set of infinitesimal equivalence classes of irreducible admissible representations of G, on which the centre of $U(\mathcal{F})$ acts as it does in the trivial representation.*

(b) $\bar{X}(\theta, \mathcal{L})$ has a unique irreducible subrepresentation $\tilde{X}(\theta, \mathcal{L})$.

(c) $\tilde{X}(\theta, \mathcal{L})$ exhaust $\hat{G}$.

Let $\bar{\Theta}(\theta, \mathcal{L})$ and $\tilde{t}heta(\theta, \mathcal{L})$ denote the characters of $\bar{X}(\theta, \mathcal{L})$ and $\tilde{X}(\theta, \mathcal{L})$ respectively, in the Grothendieck group of $(\mathcal{J}, K)$-modules of finite length.

Writing $\bar{\Theta}(\theta,\mathcal{L}) = \sum_{(\theta',\mathcal{L}')\in\mathcal{D}} \alpha_{(\theta,\mathcal{L}),(\theta',\mathcal{L}')}\tilde{\Theta}(\theta',\mathcal{L}')$, we have the :

Theorem 5.3 *([V1]) The above formulae can be inverted to give*

$$\tilde{\Theta}(\theta,\mathcal{L}) = \sum_{(\theta',\mathcal{L}')\in\mathcal{D}} \beta_{(\theta,\mathcal{L}),(\theta',\mathcal{L}')}\bar{\Theta}(\theta',\mathcal{L}')$$

for unique integers $\beta_{(\theta,\mathcal{L}),(\theta',\mathcal{L}')}$.

Example 5.3 *([V1]) Let us take the case of $SL(2,\mathbb{R})$ as in example 5.2. The four irreducible $(\mathcal{J},K)$-modules (i.e. the four elements of $\hat{G}$) corresponding to the four elements of $\mathcal{D}$ are $X_d(1), X_d(-1), \bar{X}_c(1)(0)$ and $\bar{X}_c(1)(1)$, where $X_d(\pm 1)$ are discrete series representations, $\bar{X}_c(1)(0)$ is the trivial representation, and $\bar{X}_c(1)(1)$ is the irreducible principal series representation (this corresponds to the 'Mobius band' in $\mathcal{D}$).*

In §3, we studied the Hecke algebra and constructed the polynomials $P_{y,w}$ etc. We try to copy the construction in our case by taking (in place of the Hecke algebra $\mathcal{H}$ of the Weyl group of G), the $\mathcal{H}$-module $\mathcal{M}$ which is a free $\mathbb{Z}[q^{1/2},q^{-1/2}]$-module with basis $\mathcal{D}$ and the $\mathcal{H}$-module structure defined as follows:-

For $s\in S$, the set of simple reflections, we have a natural projection $\pi_s : \mathcal{B}\to\mathcal{P}_s =$ variety of parabolic subgroups of G of type s.
Then $L_x^2 = \pi_s^{-1}(\pi_s(x)) \cong \mathbb{P}^1 (x\in\mathcal{B})$ is the line of type s through x. Suppose $(\theta,\gamma)\in\mathcal{D}$. Fix $x\in\theta$. Let us denote $\hat{\theta} = \bigcup_{y\in\theta} L_y^s$. Then,

1. If $L_x^s\subseteq\theta$, define $T_s\gamma = q^{1/2}$,
2. If $L_x^s\cap\theta = \{x\}$ and $\hat{\theta}-\theta$ is a single K-orbit, then define $T_s\gamma = \hat{\gamma}|\hat{\theta}-\theta$ where $\hat{\gamma}$ is the unique locally constant extensions of γ to $\hat{\theta}$.
3. If $L_x^s\cap\theta = L_x^s$-{point}, then (it being necessarily true that $\hat{\theta}-\theta$ is a single K-orbit) define $T_s\gamma = (q^{1/2}-1)\gamma + q^{1/2}(\hat{\gamma}|\hat{\theta}-\theta)$,
4. If $L_x^s\cap\theta = \{x,y\}$, then it follows that $\hat{\theta}-\theta$ is a single K-orbit, and γ has two distinct extensions $\hat{\gamma}_i$ to $\hat{\theta}$ and so, we define $T_s\gamma = \gamma + (\hat{\gamma}_1 + \hat{\gamma}_2)|(\hat{\theta}-\theta)$,

5. If $L_x^s \cap \theta = L_x^s$ - {two points in one K-orbit} and γ extends to $\hat{\theta}$, then writing $\gamma' = \hat{\gamma}|\hat{\theta} - \theta$ and $\hat{\gamma}_2$ for the other extension of γ' to $\hat{\theta}$, we define $T_s\gamma = (q^{1/2} - 1)\gamma - \hat{\gamma}_2|\theta + (q^{1/2} - 1)\gamma'$,

6. If $L_x^s \cap \theta = \{x\}$, and $\hat{\theta} - \theta$ is the union of two orbits θ' and θ'' labelled so that dim $\theta = dim\ \theta'' = dim\ \theta' - 1$, we define $T_s\gamma = \hat{\gamma}|\theta' + \hat{\gamma}|\theta''$,

7. If $L_x^s \cap \theta = L_x^s$ - {two points in two K-orbits} and γ extends to $\hat{\theta}$, calling the orbits θ', θ'', we define $T_s\gamma = (q^{1/2} - 1)\gamma + (q^{1/2} - 1)(\hat{\gamma}/\theta' + \hat{\gamma}/\theta'')$, and finally

8. If $L_x^s \cap \theta = L_x^s$ - {two points} and, γ *does not* extend to $\hat{\theta}$, we define $T_s\gamma = -\gamma$.

That the endomorphisms T_s make $\mathcal{M}$ a module for $\mathcal{H}$ can be checked (cf. Prop. 5.5 of [LV]).

The function $\ell(\theta, \gamma) = dim\ \theta$ on $\mathcal{D}$ plays the role of the length function on W in the complex case. The Bruhat-ordering on $\mathcal{D}$ is the smallest order such that:

If $(\theta', \delta') \in \mathcal{D}$ and δ appears in $T_s\delta'$ with a non-zero coefficients, and $\ell(\delta) + \ell(\delta') + 1$; and if γ and γ' have the same relationship (with the same s) and if $\gamma' \leq \delta'$, then we require that $\gamma \leq \delta$ and $\delta' \leq \delta$. This does reduce to the Bruhat order in Example 5.1.

We can also define an anti-linear (with respect to $q^{1/2} \mapsto q^{-1/2}$) automorphism of M compatible with the Hecke algebra action and the Bruhat ordering. (This is done using Verdier duality). In fact :

Theorem 5.4 *([LV]) There is a unique $\mathbb{Z}$-linear map $D : \mathcal{M} \to \mathcal{M}$, subject to the following conditions:*
(a) $D(q^{1/2}m) = q^{-1/2}D(m)\ \forall\ m \in M$,
(b) $D((T_s + 1)m) = q^{-1/2}(T_s + 1)D(m)\ \forall\ m \in M, s \in S$ and
(c) If $\delta \in \mathcal{D}$, then $D(\delta) = q^{-\frac{\ell(\delta)}{2}} \cdot \left[\delta + \sum_{\gamma < \delta} R_{\gamma,\delta}(q) \cdot \gamma\right]$.

The $R_{\gamma,\delta}$ are actually polynomials in $q^{1/2}$ of degree atmost $\ell(\delta) - \ell(\gamma)$.

Then, the analogue of Theorem 2.3 is true (cf Theorem 1.11 of [LV]). Once the $R_{\gamma,\delta}$ are known, $P_{\gamma,\delta}$ may be computed exactly as for the Hecke algebra case.

Let $(\theta, \mathcal{L}) \in \mathcal{D}$. One can construct the Deligne-Goresky-Macpherson intersection cohomology complex of sheaves $IC(\bar{\theta}, \mathcal{L})$ on $\bar{\theta}$ as an extension of $\mathcal{L}$ to $\bar{\theta}$. It can be characterized as the unique K-equivariant constructible complex $\hat{F}$ of sheaves on $\bar{\theta}$ satisfying the following:
(i) $\hat{F}$ is self-dual,
(ii) $\hat{F}^i = 0$ for $i < 0$, where $\hat{F}^i$ are the cohomology sheaves of $\hat{F}$,
(iii) $\hat{F}^0|\theta \cong \mathcal{L}$, and
(iv) if $i > 0$, then $\text{supp}(\hat{F}^i)$ has codimension $\geq i+1$ in $\bar{\theta}$.

Let us write $\tilde{\mathcal{L}}$ instead of $IC(\bar{\theta}, \mathcal{L})$ for simplicity. We regard its cohomology sheaves $\tilde{\mathcal{L}}^i$ to be defined on all of $\mathcal{B}$ by extending it by zero outside of $\bar{\theta}$. Given $(\theta, \mathcal{L})$ and $(\theta', \mathcal{L}')$ in $\mathcal{D}$, we write $[\mathcal{L} : \tilde{\mathcal{L}}'^i]$ for the multiplicity of $\mathcal{L}$ in the Jordan-Holder series for $\tilde{\mathcal{L}}'^i$. These are measures of the singularity of $\bar{\theta}$.

The main theorem *proved in* [LV] is

Theorem 5.5 *Let $(\theta, \gamma), (\theta', \delta) \in \mathcal{D}$. Then,*
(a) $\tilde{\delta}^i = 0$ for odd i, and
(b) $P_{\gamma,\delta}(q) = \sum_i [\gamma : \tilde{\delta}^{2i}] \cdot q^i$.

Also, [V2] shows

Theorem 5.6 *For γ, δ as before, the integers $\beta_{\gamma,\delta}$ are given by*

$$\beta_{\gamma,\delta} = (-1)^{\ell(\delta)-\ell(\gamma)} \cdot P_{\gamma,\delta}(1).$$

5.4 Luszting's conjecture for p-adic groups

G is a simply connected, almost simple group $/Q_p$. I is an Iwahori subgroup. Then, there is a bijection $I \backslash G/I \leftrightarrow W_{aff}$, the affine Weyl group. Let $F(G/I)$ be the set of locally constant functions on G which are I-invariant. Then $End_I F(G/I) =$ Double coset algebra of G with respect to $I =$ Hecke algebra

H of W_{aff} with coefficients in $\mathbb{C}$.
Borel-Matsumoto proved :

Theorem 5.7 *There exists a one-one correspondence between the set of irreducible admissible representations V of G such that $V^I \neq 0$ and the set of irreducible finite-dimensional representations of H.*

Thus, it is useful to construct some irreducible representations of H and we endeavour to do so.

Therefore let $E = \bigoplus_{w \in W_{aff}} \mathbb{C}e_w, E^{\geq i} = \bigoplus_{\substack{w \\ a(w) \geq i}} \mathbb{C}e_w, E^i = E^{\geq i}/E^{\geq i+1}$. These are left H-modules as well as right W-modules.

Note that though these two structures do not commute on E or $E^{\geq i}$, they do commute on E^i just as in the proof of Theorem 4.1 since Corollary 4.10 carries through to affine Weyl groups.

Thus, if V is an irreducible right $\mathbb{C}[W_{aff}]$-module, then $(E^i \bigotimes_{\mathbb{C}} V)_{W_{aff}}$ is an irreducible left H-module.

We claim that there is a canonical choice for i such that $(E^i \bigotimes_{\mathbb{C}} V)_{W_{aff}} \neq 0$.

For this, we consider $0 \to E^{\geq i}/E^{\geq i+1} = E^i \to E/E^{\geq i+1} \to E/E^{\geq i} \to 0$. We have then

$$(E^i \bigotimes_{\mathbb{C}} V)_{W_{aff}} \to (E/E^{\geq i+1} \bigotimes_{\mathbb{C}} V)_{W_{aff}} \xrightarrow{\alpha_i} (E/E^{\geq i} \bigotimes_{\mathbb{C}} V)_{W_{aff}} \to 0.$$

Now, by lemma 4.7, $E^{\geq i} = 0$ for large i and hence $\exists$ a unique n such that $\alpha_i \neq 0$ is an isomorphism for $i > n$ and $\alpha_i = 0$ for $i < n$. For the choice $i = n$, we clearly have $(E^n \bigotimes_{\mathbb{C}} V)_{W_{aff}} \neq 0$.

We call this n to be a_v.

Thus if $\hat{V} = (E^{a_v} \bigotimes_{\mathbb{C}} V)_{W_{aff}}$, then $\hat{V} \to V \to 0$ as W-modules.

(In the case of finite Weyl groups, $\hat{V} \cong V$ i.e. we get a representation of H with the same dimension as that of the given representation of W.)

We have the

Conjecture 5.2 *(Lusztig)*

(i) The H-module $\hat{V}$ has a unique irreducible quotient $\tilde{V}$. All other composition factors are of the form $\tilde{V}'$ with $a_{V'} < a_V$.

(ii) $V \to \tilde{V}$ is a bijection between irreducible representations of W_{aff} and irreducible representations of H.

A.V. Zelevinskii [Z] obtained all the irreducible representations of the Hecke algebra in the case of GL_n. In fact he obtained a classification of irreducible complex representations of the groups $GL_n(\mathbb{Q}_p)$. These are parametrized by collections 'a' of 'segments' in the set of cuspidal representations. With each collection'a' of 'segments', there is associated the induced module π_a and the irreducible module $\langle a \rangle$ that is the only irreducible submodule of π_a. Particular cases of π_a are the representations of the principal series. At present, the computation of the multiplicity $m_{b,a}$ with which the irreducible representation $\langle b \rangle$ occurs in the Jordan-Hölder series of the module pi_a is an open problem. He formulates the p-adic analogue of the Kazhdan-Lusztig hypothesis for this set-up as follows:

The role of G/B is taken by the variety $E = E(V)$ of linear operators of degree 1 acting on a fixed graded finite-dimensional vector space V over the field $\mathbb{Q}_p$, The automorphism group Aut V of V, preserving the gradation, acts naturally on E, and its orbits on E are the analogues of the Schubert cells. They are parametrized by the collections of segments in $\mathbb{Z}$. If X_a denotes the orbit in E corresponding to a collection 'a' of segments in $\mathbb{Z}$, then the study of X_a is connected with that of the irreducible representations of $GL_n(\mathbb{Q}_p)$ as seen from *Langlands reciprocity.*

He then formulates the

Conjecture 5.3 (Conjectural Hypothesis) *All sheaves $\mathcal{H}^i(\bar{X}_b)$ are equal to 0 for odd i, and*

$$m_{b,a} = \sum_i dim\ \mathcal{H}^{2i}(\bar{X}_b)_{X_a}.$$

In the above, of course, the intersection cohomology sheaves etc. are the usual ones (i.e over $\mathbb{C}$); also one takes V over $\mathbb{C}$, in the conjecture. He verifies the hypothesis in some cases when $m_{b,a}$ are known (for eg. the determinant varieties - the varieties of rank not exceeding a preassigned value). In these cases, he computes the sheaves $\mathcal{H}^i(\bar{X}_b)$ starting from the explicit construction of the resolution of singularities of $\bar{X}_b$. He also shows that, in general, $m_{b,a} \neq 0 \Leftrightarrow X_a \subseteq \bar{X}_b$. (This would follow from the hypothesis above, once it is shown.)

5.5 Representations of Weyl groups

Springer shows in 'good' characteristic (i.e. large characteristic - he worked over Lie Algebras and used the non-degeneracy of the Killing form) a connection between the Étale cohomology of $\mathcal{B}_u$, the variety of Borel subgroups containing a unipotent u and representations of Weyl group by showing one between Representations of Weyl group and Unipotent conjugacy classes. (Note that Luzstig showed that in all cases the # of unipotent conjugacy classes is finite.) He did not use any Intersection homology techniques in Schubert varieties. We give an alternative approach which works over any characteristic. We have an important

Lemma 5.8 *(Gorseky-Macpherson) [GM2] Suppose $f : X' \to X$ be a 'small' map which is birational and generically one-one where X' is an irreducible non-singular variety and X is a singular variety.*

(By definition, 'small' means that $\forall\ i$,

$$Codim\ \{x \in X/dim\ f^{-1}(x) = i\} > 2i,$$

so that the fibres will have dimension $\leq \frac{1}{2}dim\ x$.) Then, $IC(X) = Rf_*(\mathbb{C})$.

Let $X = G$, a semi-simple, connected algebraic group over $k = \bar{k}$. Let $X' = \tilde{G} = \{(g, B)|g \in B\}$.

Both X and X' are non-singular and the first projection $X' \xrightarrow{p_1} X$ satisfies the conditions of the lemma.
In fact, $p_1^{-1}(g) = \mathcal{B}_g =$ set of Borels containing g has dimension $\leq \frac{1}{2}\{dim\ C_G(g) - rank\ G\}$.

In fact, the equality holds in 'good' characteristic [St 1], [Sp 2]. Therefore by the lemma 5.8,

$$(p_1)_*(\mathbb{C}) = IC(G, \mathcal{L}), \text{ where } \mathcal{L} \text{ is a local system on } G.$$

Since there is a canonical action of W on $\mathcal{L}$, there is a canonical extension to $IC(G, \mathcal{L})$ also.

Thus, we have

$$\mathcal{H}^i(IC(G, \mathcal{L}))_g = H^i(\mathcal{B}_g), \text{ the ordinary cohomology } \forall\, g \in G. \qquad (5.9)$$

5.6 Representations of Chevalley groups over finite fields

Let G be a connected reductive algebraic group $/\mathbb{F}_q$. Let $F : G \to G$ be the Frobenius map. We can imbed G in $GL_n(\mathbb{F}_q)$ in such a way that $F = \pi|G$, where $\pi : (x_{ij} \mapsto (x_{ij}^q)$. If T is a maximal turns$/\mathbb{F}_q \subseteq \mathrm{B}$, a Borel subgroup $/\mathbb{F}_q$, $U =$ radical of B and W is the Weyl group, we define a decomposition of $\mathcal{B} = G/B$ as $\mathcal{B} = \bigcup\limits_{w \in W} X_w$, where $X_w = \{gB \in \mathcal{B} : g^{-1}f(g) \in BwB\}$ is a locally closed subvariety which is smooth of dimension $\ell(w)$. $\bar{X}_w$ are locally isomorphic to the Schubert varieties $\bar{\mathcal{B}}_w$. $G(\mathbb{F}_q) = \mathrm{G}^{\mathrm{F}}$ acts on X_w by conjugation. Define $\tilde{X}_w = \{g(U \cap w \cup w^{-1}) \in \mathcal{B} : g^{-1} \cdot F(g) \in wU\} \subseteq G/(U \cap wUw^{-1})$.

Then $T_w := \{t \in T : F(t) = w^{-1}tw\}$ acts on $\tilde{X}_w$ by right multiplication and has no fixed points.
We have $\tilde{X}_w/T_w = X_w$ i.e. $\tilde{X}_w \to X_w$ is a finite étale covering.
If $\theta : T_w \to \bar{Q}_\ell^*$, the $\mathrm{I\!H}^{\mathrm{i}}_{\mathrm{c}}(\tilde{\mathrm{X}}_{\mathrm{w}}, \bar{\mathrm{Q}}_\ell)_\theta$ is a finite-dimensional vector-space over $\bar{Q}_\ell$ on which G^F acts. (Here $\mathrm{I\!H}^{\mathrm{i}}_{\mathrm{c}}(\tilde{\mathrm{X}}_{\mathrm{w}}, \bar{\mathrm{Q}}_\ell)_\theta$ is the subspace of $H^i_c(\tilde{x}_w, \bar{Q}_\ell)$ on which $\mathbb{F}_q$ acts by the character θ).

Then, $R_{T_w,\theta} = \sum\limits_i (-1)^i H^i_c(\tilde{X}_w, \bar{Q}_\ell)_\theta$ is called a virtual representation of $G(\mathbb{F}_q)$.

It is known that

Theorem 5.9 *(1) $\pm R_{T_w,\theta}$ is irreducible for 'almost all' θ.*

(2) Any irreducible representation of $G(\mathbb{F}_q)$ appears with non-zero coefficients in some $R_{T_w,\theta}$.

A proof may be found in Lusztig's book on character sheaves.

Assume from now on that G is *split* $/\mathbb{F}_q$. Then, $\exists$ a maximal torus $T/\mathbb{F}_q$ such that $F(t) = t^q \ \forall\ t \in T$. Call $R_w = R_{T_w,1}$. If ρ is a unipotent representation of $G(\mathbb{F}_\mathrm{q})$, then :

Proposition 5.10 *(i)* $dim\ \rho = \frac{1}{|W|} \sum\limits_{w\in W} \langle \rho, R_w\rangle dim\ R_w$

(ii) $\langle \rho, R_w\rangle = Tr(s,\rho)$ *for some semiregular semisimple element in* T_w*.*

Suppose E is a irreducible representation of W.
Define $R_E = \frac{1}{|W|} \sum\limits_{w\in W} Tr(w,E)R_w \in$ (Grothendieck group of G^F) $\otimes Q$.
Two such E and E' are said to be in the same family (we say $E \sim E'$) if $\exists$ a unipotent representation ρ of G^F such that $\langle \rho, R_E\rangle \neq 0 \neq \langle \rho, R_{E'}\rangle$. Similarly, we can define a family of unipotent representations of G^F. Then, we will have a bijection

$$\mathcal{U} \leftrightarrow \{Rep(W)\}$$

where $\mathcal{U}$ denotes the set of families of unipotent representations of $G(\mathbb{F}_\mathrm{q})$ and the association is $\rho \leftrightarrow E$ iff $\langle \rho, R_E\rangle \neq 0$.

In the case of GL_n, the equivalence relations are trivial i.e. each family contains exactly one element.

We will show that the above two sets are also in bijection with the set of two-sided cells in W.

For this, first write

$$W = \coprod_{\substack{\mathcal{C} \\ 2-sided\ cell}} \mathcal{C}.$$

Here any $[\mathcal{C}]$ is regarded as a W-module by regarding it as the quotient $\bigoplus\limits_{\mathcal{C}'\subseteq\mathcal{C}} [\mathcal{C}'] / \bigoplus\limits_{\mathcal{C}'\neq\mathcal{C}} [\mathcal{C}']$ of two-sided ideals.

Then, we have $Q[W] = \bigoplus\limits_{\mathcal{C}} [\mathcal{C}]$. But, we can also write $Q[W]$ (since it is a semi-simple algebra) as a sum $Q[W] = \bigoplus\limits_{E} I_E$ of two-sided simple ideals.

Thus, $\bigoplus_{\mathcal{C}'\subseteq\mathcal{C}}[\mathcal{C}']$ and $\bigoplus_{\mathcal{C}'\not\subseteq\mathcal{C}}[\mathcal{C}']$ are also subdirect sums of this form.

So, for any two-sided cell $\mathcal{C}, [\mathcal{C}] \cong \bigoplus_E I_E$. If we define two representations E and E' of W to be equivalent, if I_E and $I_{E'}$ occur in the same two-sided cell, then it can be checked that this is the same equivalence relation defined before i.e. we have bijections

$$\mathcal{U} \leftrightarrow \{Rep(W)\} \leftrightarrow 2 - \text{sided cells}.$$

With help of this identification, one can show :

Theorem 5.11

$$\bar{R}_w = \sum_i (-1)^i \mathbb{H}^{\mathrm{i}}(\bar{\mathrm{X}}_{\mathrm{w}}, \mathbb{Q}_\ell) = \sum_{\mathrm{y}\leq\mathrm{w}} \mathrm{P}_{\mathrm{y,w}}(1)\left(\sum_{\mathrm{i}}(-1)^{\mathrm{i}}\mathbb{H}^{\mathrm{i}}(\mathrm{X}_{\mathrm{y}}, \mathbb{Q}_\ell)\right).$$

In fact, we can obtain $\mathbb{H}^{\mathrm{i}}(\bar{\mathrm{X}}_{\mathrm{w}}, \mathbb{Q}_\ell)$ *as an explicit linear combination of* $\bar{R}_y, y \leq w$.

Suppose, now E is an irreducible representation of W. Then, we have a_E as in §5.4 i.e. $a_E = a(w)\ \forall\ w \in \mathcal{C}$ where E appears in the two-sided cell $\mathcal{C}$.

Then, if $E(q)$ denotes the corresponding representation of H as in §4.8, we have $\forall\ x \in W$,

$$Tr(\tilde{T}_x, E(q)) = C_{x,E}q^{-\frac{a_E}{2}} + \text{higher powers of } q^{1/2} \text{ where } C_{x,E} \in \mathbb{Z}.$$

If F is the family of representations of W corresponding to a 2-sided cell $\mathcal{C}$, then $\exists$ a left cell $\mathcal{L} \subseteq \mathcal{C}$ satisfying the property that the matrix $(C_{x,E})_{\substack{x\in\mathcal{L}\\ E\in F}}$ is the character table of a finite group $\Gamma_{\mathcal{C}}$.

(In the above matrix, we delete rows or columns which are identically zero.)

Then, the main theorem is :

Theorem 5.12 *There is a bijection*

$$\mathcal{U} \leftrightarrow \sqcup_{\mathcal{C}}\mathcal{M}(\Gamma_{\mathcal{C}}$$

where $\mathcal{M}(\Gamma) := \{(x, \sigma) | x \in \Gamma$ *upto conjugacy and* σ *is an irreducible representation of* $C_\Gamma(x)\}$.

The main lemma which makes everything work is

Lemma 5.13 $\forall\ x \in W$, *if* $\alpha_x := (-1)^{\ell(x)} \sum\limits_E C_{x,E} E$, *then* $R_{\alpha_x} = (-1)^{\ell(x)} \sum\limits_E C_{x,E} R_E$ *is an actual representation of* $G(\mathbb{F}_q)$.

The proof uses intersection cohomology.

It should be remarked that each unipotent representation of $G(\mathbb{F}_q)$ occurs in some R_{α_x} and if we know all the R_{α_x} and their inner-products, we can recover all unipotent representations.

Let us see through some examples as to which $x \in W$ give non-zero α_x.

(i) In S_n, each left cell contains a unique x such that $\alpha_x \neq 0$; this will turn out to be the unique involution in the left cell.

(ii) In B_n, C_n, D_n we have $\alpha_x \neq 0 \Leftrightarrow x$ is an involution.

Also, each left cell contains some involution; the number of involutions in a left cell is a power of 2 (this fact can be proved a priori only using representation theory techniques which, in turn, use intersection cohomology techniques!).

(iii) In exceptional groups, there are non-involutions x with $\alpha_x \neq 0$. Finally, we have the

Conjecture 5.4

$$\alpha_x \neq 0 \Leftrightarrow x \underset{L}{\sim} x^{-1}.$$

5.7 Conjectural representation theory of Kac-Moody algebras

We will be following [DGK].

Definition 5.1 *Let $A = (a_{ij})_{n\times n}$ be a matrix over $\mathbb{C}$. We can associate a Lie algebra $G(A)$ over $\mathbb{C}$ which is uniquely defined upto an isomorphism by:*

(i) $G(A)$ contains an Abelian diagonalizable subalgebra H such that $G(A) = \bigoplus_{\alpha\in H^*} G_\alpha$, where

$$G_\alpha = \{x \in G(A) : [h, x] = \alpha(h)x \ \forall\ h \in H\}$$

and $G_0 = H$,

(ii) there exists a linearly independent set of linear functions $\alpha_1, \cdots, \alpha_n \in H^*$ and elements $e_1, \ldots, e_n, f_1, \ldots, f_n$ in $G(A)$ such that

1. *$G_{\alpha_i} = \mathbb{C}e_i, G_{-\alpha_i} = \mathbb{C}f_i (1 \le i \le n)$,*
2. *$[e_i, f_j] = 0$ for $i \ne j$,*
3. *$\{e_1, \ldots, e_n, f_1, \ldots, f_n] \cup H$ generates $G(A)$ as a Lie algebra,*
4. *the elements $h_i = [e_i, f_i] (1 \le i \le n)$ are linearly independent,*
5. *$\alpha_j(h_i) = a_{ij} (1 \le i, j \le n)$,*
6. *if $h \in H$ is such that $\alpha_i(h) = 0 \ \forall\ 1 \le i \le n$, then $h \in \sum_{i=1}^{n} \mathbb{C}h_i$, and*
7. *any ideal of $G(A)$ which intersects H trivially is zero.*

$G(A), H$ and A are called a contragradiant Lie algebra, a Cartan subalgebra of $G(A)$ and the Cartan matrix of $G(A)$ respectively.

Denoting by Γ, the lattice in H^* generated by $\{\alpha_1, \ldots, \alpha_n\}$ and by Γ^+ the set $\{\Sigma k_i\alpha_i \in \Gamma | k_i \ge 0, 1 \le i \le n\}$, we define, for $\lambda \in H^*$,

$$D(\lambda) = \lambda - \Gamma^+ = \{\lambda - \nu | \nu \in \Gamma^+\}.$$

For $\lambda, \mu \in H^*$, we say $\lambda \ge \mu$ if $\mu \in D(\lambda)$. Thus, we can define positive and negative roots etc. in the obvious manner. Also, we write $N_+ = \sum_{\alpha\in\Delta^+} G_\alpha, N_- = \sum_{\alpha\in\Delta^-} G_{-\alpha}$. The Cartan matrix A is said to be symmetrizable iff there exists a non-degenerate matrix $D = diag.(d_1, \ldots, d_n)$ such that $D \cdot A$ is symmetric. We, then have the following:

Theorem 5.14 *If A is symmetrizable, then $\exists$ a non-degenerate $\mathbb{C}$-valued symmetric bilinear form (,) on $G(A)$ such that:*
(i) (,) is $G(A)$-invariant,
(ii) the restrictions of (,) to H and to $G_\alpha \oplus G_{-\alpha}(\alpha \in \Delta^+)$ are non-degenerate,
(iii) $(G_\alpha, G_\beta) = 0$ if $\alpha + \beta \neq 0$,
(iv) $\forall\ \alpha \in \Delta^+$, one has $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha}) \cdot h_\alpha$ for any $e_\alpha \in G_\alpha, e_{-\alpha} \in G_{-\alpha}$ where $h_\alpha = \Sigma k_i d_i h_i$ if $\alpha = \Sigma k_i \alpha_i$, and
(v) $(h_i, h_j) = d_j^{-1} a_{ij} = d_i^{-1} a_{j_i}\ \forall\ i, j$.

If $A = (a_{ij})$ is symmetrizable and such that (1) $a_{ii} = 2$, (2) a_{ij} are non-positive integers for $i \neq j$ and (3) $a_{ij} = 0 \Rightarrow a_{ji} = 0$, then the associated Lie algebra $G(A)$ is called a *Kać-Moody Lie algebra*.

In the above discussion, we choose (as we can) the d_i to be positive rational numbers. We define s_i on H^* by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i\ \forall\ \lambda \in H^*$. Let W be the group generated by these reflections $\{s_i\}_{i=1}^n$; it keeps the set Δ of roots invariant and $\dim G_\alpha = dim\ G_{w(\alpha)} \forall\ \alpha \in \Delta, w \in W$.
W is called the 'Weyl group' of $G(A)$.

Let $G(A)$ be a Kać-Moody Lie algebra.

We consider the category Θ whose objects are $G(A)$-modules M satisfying:

(a) M is H-semisimple with finite-dimensional weight spaces and
(b) $\exists$ finitely many elements $\mu_1, \ldots, \mu_k \in H^*$ such that any weight of M belongs to some $D(\mu_i)$.

The morphism of Θ are $G(A)$-module homomorphisms. The Verma modules are highest modules in Θ defined as $M(\lambda) \simeq U(G(A))/I_\lambda$ where $\lambda \in H^*$ and I_λ is the left ideal of $U(G(A))$ generated by $\{h - \lambda(h) | h \in H\}$ and N_+.

They have the following properties which are easily verifiable:

(a) If v_λ is the image of 1 in $U(G(A))/I_\lambda$, then $N_+ \cdot v_\lambda = 0$ and $h \cdot v_\lambda = \lambda(h) \cdot v_\lambda\ \forall\ h \in H$,

(b) $M(\lambda)$ is a free $U(N_-)$-module of rank 1 with $\{v_\lambda\}$ as a basis;

(c) The Γ-gradation of $U(N_-)$ induces a weight space decomposition of $M(\lambda)$: $M(\lambda) = \bigoplus_{\nu \in \Gamma^+} M(\lambda)_{\lambda-\nu}$, $dim M(\lambda)_{\lambda-\nu} = P(\nu)$, the partition function of $G(A)$ at ν,

(d) For any $G(A)$-module M containing a vector v of highest weight λ such that $N_+ \cdot v = 0$, there exists a unique $G(A)$-module homomorphism. $\varphi : M(\lambda) \to M$ such that $\varphi(v_\lambda) = v$,

(e) $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$ and

(f) Any irreducible module L in Θ is isomorphic to $L(\lambda)$ for a unique $\lambda \in H^*$.

We recall the notion of formal characters in this setup.

Let $\mathcal{A}$ be the set of all functions $f : H^* \to \mathbb{Z}$ such that f vanishes outside a finite union of $D(\lambda_i)$'s. Then $\mathcal{A}$ is a ring under pointwise addition and convolution $*$ given by $(f * g)(\lambda) = \sum_{\mu+\delta=\lambda} f(\mu) \cdot g(\delta), \lambda \in H^*$.

We will call a family $\{f_i\}_{i \in I}$ in $\mathcal{A}$ summable iff
(i) $\exists\ \mu_1, \ldots, \mu_k \in H^*$ such that each f_i vanishes outside $D(\mu_1) \cup \ldots \cup D(\mu_k)$,

(ii) for any $\lambda \in H^*, f_i(\lambda) = 0$ for all but finitely many $i \in I$.

Thus, the function $f = \sum_{i \in I} f_i$ will be a well-defined function in $\mathcal{A}$. The formal character $ch\ M$ of M in Θ is defined as the element of $\mathcal{A}$ such that

$$ch\ M(\lambda) = dim\ M_\lambda\ \forall\ \lambda \in H^*.$$

We have :

Proposition 5.15 *([DGK]) Given $M \in \Theta$, there exists a unique set $\{a_\lambda\}_{\lambda \in H^*}$ of non-negative integers such that the family $\{a_\lambda \cdot ch\ L(\lambda)\}_{\lambda \in H^*}$ is summable with sum $= ch\ M$. Moreover, $a_\lambda \neq 0$ iff $L(\lambda) \simeq$ a subquotient of M. We write $a_\lambda = [M : L(\lambda)]$.*

Kać and Kazhdan [KK] showed :

Theorem 5.16 *Let $\lambda, \mu \in H^*$. Then $L(\mu)$ occurs in $M(\lambda)$ iff the ordered pair $\{\lambda, \mu\}$ satisfies the condition:*

There exists a sequence $\beta_1, \ldots, \beta_k$ of positive roots and a sequnce $n_1, \ldots, n_k$ of positive integers such that
(i) $\lambda - \mu = \sum_{i=1}^{k} n_i \beta_i$ *and*
(ii) $2(\lambda + \rho - n_1\beta_1 - \ldots - n_{j-1}\beta_{j-1}, \beta_j) = n_j(\beta_j, \beta_j)\ \forall\ 1 \leq j \leq k$ *where* $\rho \in H^*$ *is an element satisfying* $\rho(h_i) = \frac{1}{2}a_{ii}\ \forall\ 1 \leq i \leq n$.

To make a meaningful conjecture about $[M_\lambda : L(\mu)]$ we have to consider a 'good' subcategory Θ^g of Θ which has a decomposition into a direct sum of subcategories Θ^g_Ω, where the objects of Θ^g_Ω are modules for which the highest weights of all ireeducible subquotients, translated by ρ, lie on the same orbit Ω of the Weyl group W. This is the same decomposition as in the theory of characters in the finite-dimensional case. This has helped in describing the components of the Verma modules in terms of the Weyl group.

For a complex number c we write $c \geq 0$ if either $Re(c) > 0$ or else $Re(c) = 0$ and $Im(c) \geq 0$. We write $c < 0$ if $c \not\geq 0$.

Let C be the set of elements $\lambda \in H^*$ which satisfy:
$(\lambda, \alpha_i) \geq 0$ for $i = 1, \ldots, n$ and
$(\lambda, \alpha) \neq 0$ for $\alpha \in \Delta^+$ such that $(\alpha, \alpha) = 0$.
Set $K = \bigcup_{w \in W} w(C)$. Then, we have

Proposition 5.17 *(i)* $\{W; s_1, \ldots, s_n\}$ *is a Coxeter system.*

(ii) Every orbit of W *in* K *contains a unique element of* C.

(iii) W *is finite* $\Leftrightarrow K = H^* \Leftrightarrow$ *dim* $G(A) < \infty$.

(iv) If $\lambda \in C$ *and* $w \in W$, *then* $\lambda - w\lambda = \sum_i c_i\alpha_i, c_i \geq 0$, *and*
$\lambda = w\lambda \leftrightarrow w \in$ *the group generated by* $\{s_i | (\alpha_i, \lambda) = 0\}$.

If we set $K^g = -\rho + K, C^g = -\rho + C$, we can define a subcategory Θ^g of Θ whose objects are those $M \in \Theta$ whose components have their highest weights in K^g.

The main point is

Lemma 5.18 *Let $\lambda \in K^g$ and $L(\mu)$ occur in $M(\lambda)$. Then $\exists\ \sigma \in W$ such that $\sigma(\lambda+\rho) = \mu+\rho$. In particular, $\mu \in K^g$.*

Corollary 5.19

$$\lambda \in K^g \Rightarrow M(\lambda) \in \Theta^g.$$

One can show that for $\lambda_0 \in C^g$ which is integral and such that $(\lambda_0+\rho, \alpha) \neq 0\ \forall\ \alpha \in \Delta^+$, and for $x \leq y$ in W, $[M(x(\lambda_0+\rho)-\rho) : L(y(\lambda_0+\rho)-\rho)]$ is independent of λ_0 and can conjecture that this is $= P_{x,y}(1)$.

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