State of the Art in Mathematics - A Panoramic View

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More than one hundred papers are posted every day on the Mathematics arXiv; it is therefore nigh impossible even for practitioners of mathematics like all of us gathered here, to keep abreast of the mathematical developments in more than one area.

In this talk, I shall try to provide a hawk-like (panoramic) view of certain mathematical problems which were open for a long time, and were either resolved or, crucial developments were accomplished, in the last two decades, say.

Humans have one fovea in each eye, responsible for our central vision and sharpness; hawks have two foveae in each eye: one primary and one peripheral.

The central fovea gives them high visual insight, which is seeing fine details and resolving small objects.

The peripheral fovea gives them a wide field of view, which is the ability to see a large area of the surroundings; this helps them to scan the horizon for potential prey or threats.

Venn is it Possible

While depicting intersections and unions of sets in class, we often use Venn diagrams consisting of circles; however, from 4 circles, one can get only a maximum of $14 = 4^2 - 4 + 2$ regions, as can be proved using the famous Euler formula.

One can use other types of simple, closed curves though, to get 2^n regions from n curves.

In such a Venn diagram with n curves, there are $\binom{n}{r}$ regions of 'rank' (the number of curves that contain the region) r.

A Venn diagram with n curves $C_1, \cdots, C_n$ is *symmetric*, if there is a point (the center of the diagram actually) such that rotations of any C_i by an angle of $2\pi/n$ produces the other curves C_j one after the other.

For the existence of a symmetric n-Venn diagram, it is necessary and sufficient that n is prime; the necessity is simply because the number $\binom{n}{r}$ of regions of rank r must be distributed symmetrically, and hence, must a multiple of n for each $0 < r < n$.

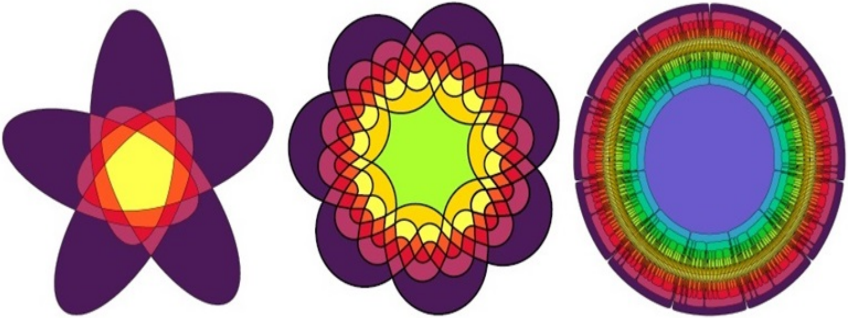


Figure : Symmetric Venn diagrams: (a) $n = 5$, (b) $n = 7$, (c) $n = 11$.

Although the necessity of primality of n was noticed by Henderson in 1963, the existence of symmetric Venn diagrams was known until 2004 only for the primes $3, 5, 7, 11$.

Finally, in 2004, Griggs, Killian and Savage constructed for each prime n, a symmetric n-Venn diagram in the Electronic Journal of Combinatorics 11 (2004).

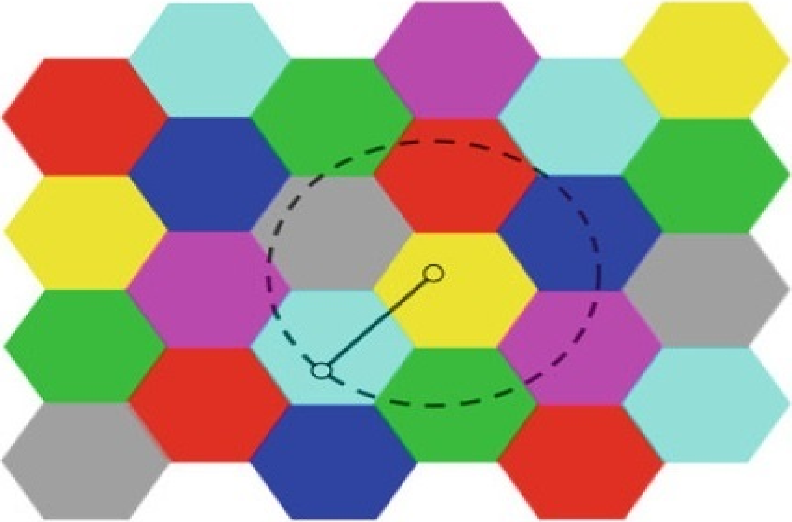
Chromatic Number of Plane

If we color each point of the plane in such a way that no two points at distance 1 from each other have the same color, then what is the minimum number of colors that needed?

This minimum number CNP - called the chromatic number of the plane - was first discussed by Nelson in 1950 (though not in print).

Looking at the vertices of any equilateral triangle with side length 1, clearly at least 3 colors are needed.

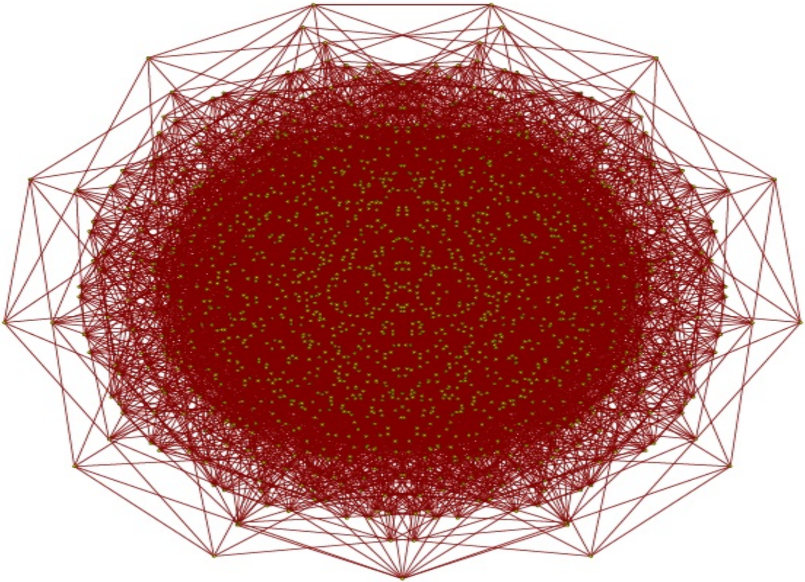
Looking at a rhombus $ABCD$ with unit side length and angle $BAD = 60$ degrees, it is easy to deduce that CNP is at least 4.



CNP is at most 7 - Isbell 1950

By the way, one can look forward to the January 2025 issue of Resonance, which features Wolfgang Haken.

For almost seven decades the bounds $4 \leq CNP \leq 7$ could not be bettered; finally, in 2018, Aubrey D.N.J. de Grey proved that the chromatic number of the plane is at least five (Geombinatorics 28 (2018) pp., 5-18); he did this by producing a 1581-vertexed unit distance graph that is not four-colorable.



The 1581-vertex, non-4-colourable unit-distance graph

God Settles a Score

In 1974, Ernö Rubik invented perhaps the best-selling toy in the world.

The least number g of rotations needed to ensure that every position can be solved, has come to be known as God's number.

In 2013, it was proved that $g = 20$ (hence justifying my assertion that God settles a 'score'!

The fact that g is at least 20, was proved in 1995 by M. Reid, when he showed that the "superflip" position - all corners solved, all edges flipped in their home positions - does require 20 moves to resolve.

Considering that the number of positions possible is about 4.3×10^{19}, the number 20 is indeed remarkable.

The group of four mathematicians Tomas Rokicki, Herbert Kociemba, Morley Davidson and John Dethbridge achieved this ('The Diameter of the Rubik Cube is Twenty', SIAM Journal of Discrete Mathematics, Vol. 27 (2013) pp. 1082-1105) by:

Partitioning all the positions into about two billion cosets of a carefully selected subgroup of the Cube group (the group of ALL moves), using symmetry to significantly reduce the number of cosets that need to be treated, and then writing a highly efficient computer program to search for at most 20-move solutions for the positions in each coset.

In 1953, Mordell had asked for ‘other’ solutions of 3 as a sum of three integer cubes, beyond the obvious ones

$3 = 1^3 + 1^3 + 1^3 = 4^3 + 4^3 + (-5)^3.$

Here's the answer - what is the question

The question of expressibility of integers that are not congruent to ± 4 modulo 9 as a sum of three integer cubes, is old; up to 100, the only elusive case 42 remained.

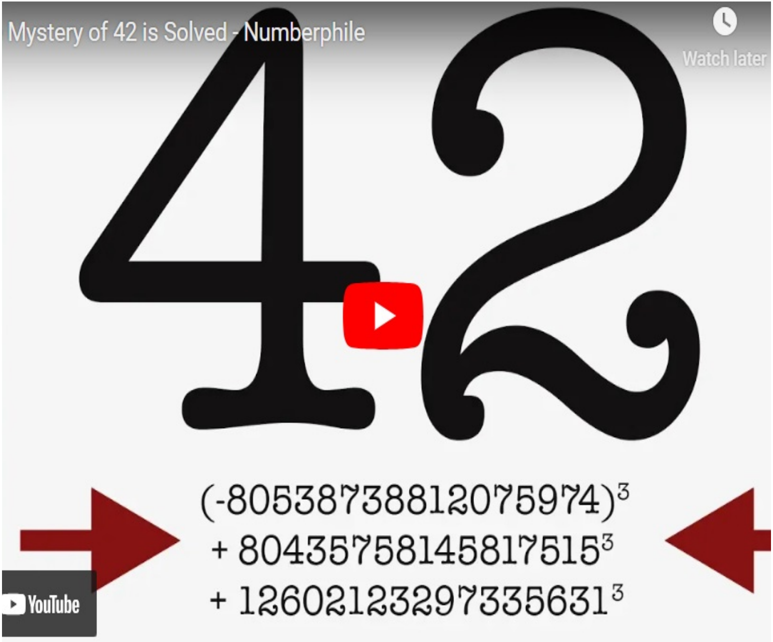
Finally, in 2021, settled by a team led by Andrew Booker from Bristol, and Andrew Sutherland from MIT - authorities on parallel computations (see 'On a question of Mordell', PNAS, Vol. 118 (2021) e2022377118, https://doi.org/10.1073/pnas.2022377118).

They used 'Charity Engine', a world-wide computer that harnessed idle, unused computing power from over 500000 home PCs to create a crowd-sourced platform.

The most difficult case of 42 resolved using a computing platform reminiscent of ‘Deep Thought’, the giant machine which gave the answer 42 in Douglas Adams’s Hitchhiker’s Guide to the Galaxy.

The Earth was actually a giant supercomputer, created by another supercomputer, Deep Thought.
‘Deep Thought’ built by its creators to give the answer to the “Ultimate Question of Life, the Universe, and Everything”.
After eons of calculations, the answer was given simply as 42.

Deep Thought was then instructed to design the Earth supercomputer to determine what the Question actually is!


Mystery of 42 is Solved - Numberphile
Watch later
42
(-80538738812075974)³
+ 80435758145817515³
+ 12602123297335631³
YouTube


569936821221962380720³
+
(-569936821113563493509)³
+
(-472715493453327032)³
=
3

Challenge for the clairvoyant

"*It occurs to me that these sorts of questions would be excellent challenge questions to pose to any psychics who claim to be in contact with super-intelligent aliens, since the solutions are already expected to be produced by computer search in a few years but would be instantly verifiable evidence of some extraordinary computational or intellectual resource if produced sooner*".

Kakeya Problems

The following geometric problem was proposed in 1917 by Soichi Kakeya: *find a figure of the least area on which a needle of length 1 can be turned continuously through* 360 *degrees*.

This seemingly innocuous problem was solved not long after (in 1926) by A S Besicovitch, but became the launching pad for what is known today as the Kakeya problem.

In general, a Kakeya set (also referred to as a Besicovitch set) is defined to be a Borel subset of $\mathbb{R}^n$, which contains a unit line segment in every direction (no requirements of 'continuous movement' of needle etc.).

Already, in 1919, Besicovitch showed the existence in of a Kakeya set in two dimensional space which has Lebesgue measure zero - this example has become a cornerstone of modern developments.

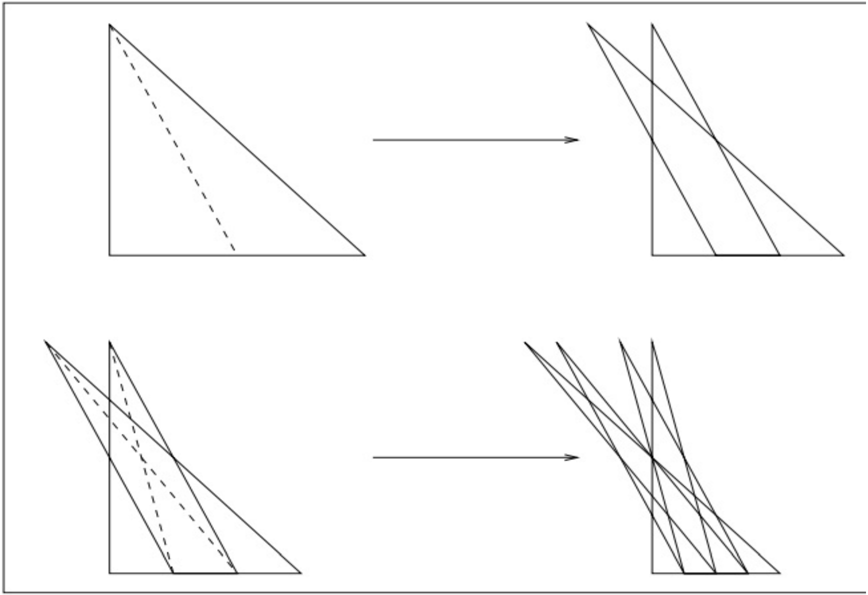


Figure . . The iterative construction of a Besicovitch set. Each stage consists of the union of triangles. To pass to the next stage, the triangles are bisected and shifted together to decrease their area.

The main problem is to decide how small a Kakeya set can be - classically, 'smallness' is in terms of the Hausdorff dimension.

There are a number of variations of the so-called Kakeya conjecture, one of which asserts that every Kakeya set in $\mathbb{R}^n$ must have Hausdorff dimension n (even if it has Lebesgue measure zero - note that sets of positive Lebesgue measure necessarily have Hausdorff dimension n).

An analogy from the realm of physics is to think of the Lebesgue measure of a set as its volume and we can consider its Hausdorff dimension to be a quantification of its mass.

A Kakeya set K of zero Lebesgue measure would appear similar to a system of photons with so much energy as to be equivalent to the mass of an actual rubber ball - K has no 'volume', still has "mass" equivalent to that of the unit ball.

The Kakeya conjecture has been solved affirmatively for $n = 2$ (in 1971 by Davies) but is still open for $n > 2$.

In 2019, it has been shown for $n = 3$ that the Hausdorff dimension is bounded below by $\frac{5}{2} + \epsilon$ for each $\epsilon > 0$; this is due to Nets Hawk Katz and Joshua Zahl ('An improved bound on the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^3$', Journal of the American Mathematical Society, Vol. 32 (2019) pp. 195-259).

Over a finite field F, a Kakeya set $K \subset F^n$ is a subset such that for each $x \in F^n$, there exists $y \in F^n$ for which the entire line $\{y + ax : a \in F\}$ lies inside K.

In 2009, Dvir ('On the Size of Kakeya Sets in Finite Fields', J. Amer. Math. Soc., Vol. 22 (2009) pp. 1093-1097) proved that there exists a constant $C_n > 0$ such that, Kakeya sets $K \subset \mathbb{F}_q^n$ have size at least $C_n q^n$.

Despite its modest origins, the Kakeya conjecture has numerous applications to diverse areas such as arithmetic combinatorics, harmonic analysis etc.

One expository article by Terence Tao appears in Notices of the AMS, Vol. 48, No.1 (2001) pp.294-303 ('From Rotating Needles to Stability of Waves: Emerging Connections between Combinatorics, Analysis, and PDE').

Closer home, there is a 4-part survey article in 2015, in the Mathematical Newsletter of the RMS by Edward Kroc, who worked on this topic in his M.Sc. project at UBC under the tutelage of Malabika Pramanik.

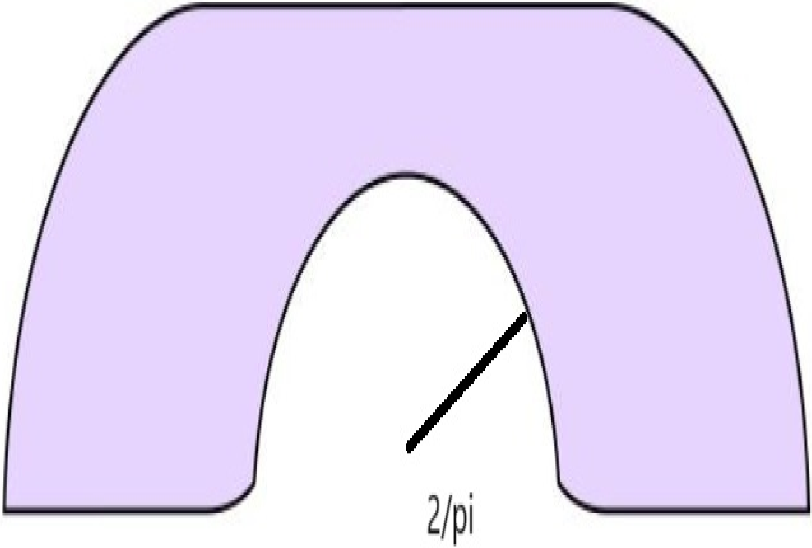
Sofa So Good

In 1966, Leo Moser posed the question of determining the shape of largest area in the plane that can be moved around a right-angled corner in a hallway of unit width.

Of course, a unit square-shaped sofa or a semicircle of unit radius can clearly be moved around such a corner; the latter has larger area $\pi/2$, but we can do better.

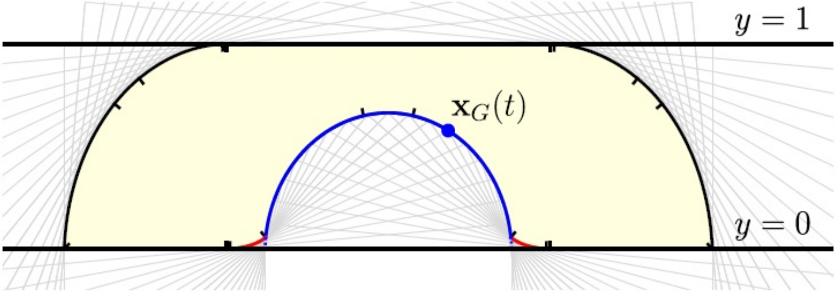
In 1968, John Hammersley noticed that if the semicircle is cut into two quarter-circles, pulled apart and the gap between them filled with a rectangular block, we get a larger sofa shape, which can be moved around a corner provided, a smaller semicircular hole is removed from the rectangular part; the resulting shape looks more like an actual sofa.

If the radius of the semicircular hole is taken to be $2/\pi$ as in the figure here, one gets the maximum possible area $\pi/2+2/\pi$ which is approximately 2.2074. Hammersley also noted that a sofa with area as large as 2.8284 cannot be moved.

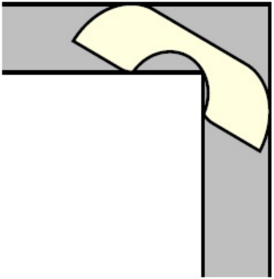

2/pi

In 1992, James Gerver obtained a better shape with a slightly bigger area - approximately 2.2195. His idea was:

The area of the shape is in equilibrium when making little perturbations to the path through which the shape is moved around a corner; this gives raise to differential equations that can be solved, and the different pieces of the shape consists of 3 line segments and 15 curved pieces, each described by its own formula. This looks like the following:



$y = 1$
$\mathbf{x}_G(t)$
$y = 0$



Less than a month back, Jineon Baek has posted on the arXiv, a 119-page long manuscript which proves that Gerver's sofa is indeed one with the largest area that can be moved (arXiv:2411.19826v1 [math.NG] 29 Nov 2024).

The problem may be theoretical but an application could possibly be in understanding how fibers suspended in a fluid behave when they pass through a nozzle. Like most things in mathematics, ideas from the abstract have a way of showing up in the real world sooner or later.

Primes with missing digits

Problems dealing with prime numbers are often notoriously difficult.

For instance, it is only in 2019 that James Maynard proved that the number of primes missing any given digit is infinite.

The set of positive integers with a missing digit is a sparse set; the cardinality up to x grows at the most like x^c, where $c = \ln(9)/\ln(10)$ which is approximately 0.954.

Usually, one can show that certain sparse sets contain infinitely many primes when it possesses some additional multiplicative structure.

The sparse set above does not have such a multiplicative structure, but its Fourier transform has a convenient explicit analytic description.

The 92-page paper by Maynard appeared in Inventiones Mathematicae, Vol. 217 (2019) pp. 127-218.

His proof applies the Hardy-Littlewood circle method, and involves de-correlating Diophantine conditions which dictate when the Fourier transform of the primes is large from digital conditions which dictate when the Fourier transform of numbers with restricted digits is large.

The arguments rely on a combination of the geometry of numbers, the large sieve and moment estimates obtained by comparison with a Markov process.

Maynard's methods are not powerful enough to show the existence of infinitely many primes with two digits not appearing in their decimal expansion, but they can show that there are infinitely many primes with d digits excluded in base b provided b is large enough in terms of d.

The following topic has been copiously written about by many people, and I will just quickly skirt over it.

Ben Green and Terence Tao proved that given an arbitrary positive integer k, there are arithmetic progressions consisting of length k that consist only of primes; this appeared in Annals of Mathematics, Vol. 167 (2008) pp. 481-547.

Green-Tao also proved other results on simultaneous prime values of polynomials; I do not discuss this, but the interested auditor can consult 'Linear equations in primes', Annals of Mathematics, Vol. 171 (2010) pp. 1753-1850, 'The Möbius function is strongly orthogonal to nilsequences', Annals of Mathematics, Vol. 175 (2012) pp. 541-566, and (with T. Ziegler) 'An inverse theorem for the Gowers norm, Annals of Mathematics, Vol. 176 (2012) pp. 1231-1372.

Green-Tao's work implies the 1937 theorem of Vinogradov that shows that every 'large enough' odd number is a sum of three odd prime numbers.

Green-Tao's results are asymptotic, and do not deal with small odd numbers; hence the odd case of Goldbach's conjecture was out of reach by their methods.

Helfgott proved the odd case of Goldbach's conjecture in 2014; in other words, each odd $n > 5$ is expressible as a sum of three prime numbers; the even case of Goldbach's conjecture that every even $n > 2$ is expressible as a sum of two primes, is still open.

In a recent paper posted on the arXiv, James Leng and Mehtaab Sawhney (arXiv:2409.06894v2 [math.NT] 15 Sep 2024) prove a generalization of Vinogradov's and Maynard's theorems by showing that for a sufficiently large base b (a conservative estimate being 2^{100000}), and any digit d from that base, all large odd numbers are sums of three primes whose digits in base b do not contain d.

Primes expressible as $p^2 + 4q^2$

Even more recently, Ben Green and Mehtaab Sawhney have posted on Math ArXiv (arXiv:2410.04189v2 [math.NT] 12 Oct 2024) a paper containing, among other things, the following stunning result:

For any $n \equiv 0$ or 4 mod 6, there exist infinitely many primes are expressible in the form $p^2 + nq^2$ with p, q primes.

We note that, already in 1882, Weber had shown that for any $n > 0$, there are infinitely many primes of the form $x^2 + ny^2$ with x, y positive integers.

Elliptic Curves

In simple language, elliptic curves over $\mathbb{Q}$ are curves of the form $y^2 = x^3 + ax + b$ where a, b are integers and the cubic has distinct roots.

The main property they have is that the set of rational points $E(\mathbb{Q})$ forms a finitely generated abelian group; the torsion part is easy to compute but the rank (the number of copies of $\mathbb{Z}$) is mysterious; the positivity of the rank often solves some classical number theoretic problem.

The million-dollar Birch and Swinnerton-Dyer conjecture claims an analytic avataar of the rank.

On the one hand, the existence of elliptic curves of arbitrarily high rank cannot be ruled out (the present record is 29 by Noam Elkies and Zev Klagsbrun) but, on the other hand, some heuristics tell us that the density of curves of ranks 0 and 1 are $1/2$ each!

Manjul Bhargava and Arul Shankar showed in 'Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves, Annals of Mathematics 181 (2015) pp. 191-242, and 'Ternary cubic forms having bounded invariants, and the existence of a positive proportion of elliptic curves having rank 0, Annals of Mathematics 181 (2015) pp. 587-621 that:

The ‘average rank’ of elliptic curves is bounded above by $7/6$, and a positive proportion of curves do have rank 0.

The approach to these results is via counting the equivalence classes of integral three variable cubic forms having bounded invariants; an idea going back to Gauss.

We note that in 2002, Ulmer showed that the rank of elliptic curves over functions fields in one variable over a finite field, is unbounded.

Periodic Billiards

Consider a billiard ball that moves inside a polygon, along a straight line; when it reaches the boundary, it changes direction according to the mirror law.

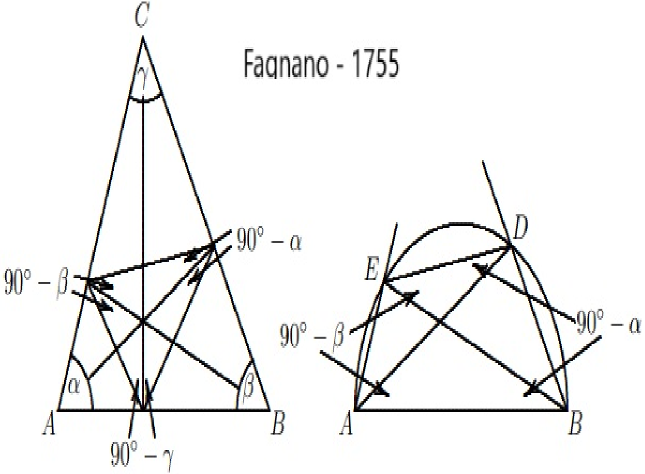
It continues in this manner unless it reaches a corner when it stops.

It is an open question for at least two centuries whether there exists a periodic billiard orbit in any polygon.

Even for triangles, this is unsolved, although it was proved already in 1755 by Fagnano for acute-angled triangles.

It is also easy to solve positively for general polygons when the angles are rational multiples of π - using Teichmüller theory, one can even show that there is a dense set of periodic orbits.

Fagnano - 1755



Let ABC be an acute triangles, and let AD, BE, and CF be the three altitudes. Then DEF is a closed billiard trajectory. The right hand side shows the proof.

Only in 2005, Serge Troubetzkoy proved the existence of a dense set of periodic billiard paths on any right-angled triangle ('Periodic billiard orbits in right triangles', Annales de l'institut Fourier, Vol. 55 (2005) pp. 29-46).

For obtuse-angled triangles where the largest angle is at the most 100 degrees, the existence of a periodic billiard path has been proved in 2009 by R. E. Schwartz ('Obtuse triangular billiards II: One hundred degrees worth of periodic trajectories', Expositiones Mathematicae, Vol. 18 (2009) pp. 137-171); the general obtuse-angled triangle case is still open.

The classical Birkhoff conjecture claims that the boundary of a strictly convex 'integrable' billiard table is necessarily an ellipse (or a circle as a special case); this is still open.

In 2016, Artur Avila, Jacopo De Simoi, and Vadim Kaloshin showed (Annals of Mathematics, Vol. 184 (2016), pp. 527-558) that a version of this conjecture is true for billiard tables bounded by small perturbations of ellipses of small eccentricity.

In 2018, Vadim Kaloshin and Alfonso Sorrentino extended and completed the above result, by proving a complete, local version (Annals of Mathematics, Vol. 188 (2018), pp. 315-380).

Hot Spots

The so-called ‘Hot Spots’ conjecture posed by Rauch in 1975 asserts roughly that if a flat piece of metal is given some initial heat distribution which then flows through the metal, then after some time, the hottest point on the metal will lie on its boundary.

The hot spots conjecture can be justified by appealing to our physical intuition; heuristically speaking, ‘heat’ and ‘cold’ are substances that annihilate each other,and so, it is easy to believe that the hottest and coldest spots lie as far as possible from each other - hence, on the boundary of the domain.

In slightly more precise terms, it concerns Lipschitz domains $\Omega \subset \mathbb{R}^2$, and the second eigenvalue μ_2 for the Neumann boundary condition; Rauch's conjecture concerns the extrems of a smooth eigenfunction u satisfying $\Delta u = -\mu_2 u$ on Ω and the normal derivative of u vanishing on the smooth points of $\partial\Omega$.

K. Burdzy and W. Werner showed in 1999 that the conjecture holds for some domains and fails for certain domains in $\mathbb{R}^2$.

In 2020, Chris Judge and Sugata Mondal ('Euclidean triangles have no hot spots', Annals of Mathematics, Vol. 191 (2020) pp. 167-211) proved the conjecture for triangles; their proof for acute triangles had errors which were rectified by them later in 2021 (Annals, Vol. 195).

Quadratic Forms

Over a field F of characteristic $\neq 2$, a non-degenerate quadratic form is of the form $a_1x_1^2 + \cdots + a_nx_n^2$ with $a_i \in F^*$.

It is said to be isotropic, if it takes the value 0 at some non-zero tuple $(x_1, \cdots, x_n)$; else, it is called anisotropic over F.

Questions on quadratic forms often reduce to those on the anisotropic part; therefore, the largest n for which we have an anisotropic form as above plays a significant role.

Define $u(F)$, the u-invariant of a field F to be the largest possible dimension of an anisotropic quadratic form over F.

We have:

$$u(\mathbb{R}) = \infty = u(\mathbb{Q}), u(\mathbb{C}) = 1,$$

$$u(\mathbb{F}_q) = 2, u(\mathbb{Q}_p) = 4 = u(\mathbb{Q}(i)).$$

Also $u(\mathbb{C}(t_1, t_2, \cdots, t_n)) = 2^n$ and $u(F) = 4$ if F is an algebraic number field which is not real.

In 1953, Kaplansky conjectured that the u-invariant can only be a power of 2, if it is finite.

38 years later, this was disproved by Merkurjev, who produced fields with u-invariant, any even number!

It was known that $u(F)$ cannot be $3, 5$ or 7 and it was not known if $u(F)$ could ever be odd.

In 2001, Oleg Izhboldin explicitly constructed a field F with $u(F) = 9$ ('Fields of u-invariant 9', Annals of Mathematics, Vol. 154 (2001), pp. 529-587). His arsenal includes some big results including the proof of Milnor conjecture due to Voevodsky.

Ore Conjecture

In 1951, Oystein Ore made the remarkable conjecture that every element of every finite, simple, nonabelian group is a single commutator $xyx^{-1}y^{-1}$.

This can be false for quasi-simple groups - perfect groups G for which $G/Z(G)$ is simple.

Ore's conjecture defied the efforts of a large number of mathematicians despite the CFSG having been completed; finally, in 2010, as a culmination of several mathematicians' efforts, it was completed by Martin Liebeck, E. A, O'Brien, Aner Shalev, and Pham Huu Tiep ('The Ore Conjecture', Journal of the European Mathematical Society, Vol. 12 (2010) pp. 939-1008).

They used several deep results including Deligne-Lusztig theory, the theory of dual pairs and Weil characters of classical groups.

Differentiability in $\mathbb{R}^n$

If a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz (that is, there exists $C > 0$ so that $||f(x) - f(y)|| \leq C||x - y||$ for all $x, y \in \mathbb{R}^m$, where the norms on the left and the right are in the respective Euclidean spaces), Rademacher proved in 1919 that the set of points where f is not differentiable, is meagre.

A converse statement would be whether given a Lebesgue null subset S, there exists SOME Lipschitz function f that is not differentiable at any point of S.

However, this is false in general; if $m > 1$, then any Lebesgue null G_δ-set $S \subset \mathbb{R}^m$ containing all lines passing through distinct points with rational coordinates has the property that EVERY Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable at SOME point of S.

Following the works of several people, finally it has been proved in 2015 that the converse holds if and only if $n \geq m$.

Csörnyei and Jones had shown in 2011 that the converse holds good if $n \geq m$.

Finally, in 2015, David Preiss and Gareth Speight ('Differentiability of Lipschitz functions in Lebesgue null sets', Inventiones Mathematicae, Vol. 199 (2015) pp. 517-559) proved that the answer is 'No' if $n < m$; that is, they proved that if $m > n \geq 1$, then there exists a Lebesgue null set $S \subset \mathbb{R}^m$ containing a point of differentiability of EVERY Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$. In fact, their proof produces a set S of Hausdorff dimension arbitrarily close to n with the above property.

The Invariant Subspace Problem

This problem has a most murky history; one form of the problem asks if every bounded operator on an infinite-dimensional, separable Hilbert space admit a non-trivial, (closed) invariant subspace?

For compact operators, it was proved (unpublished) by von Neumann in the 1930's and others published proofs later.

In 1973, Lomonosov proved that an operator which commutes with a non-zero compact operator also has a non-trivial invariant subspace; a really simple proof of this was given by Hilden.

However, the status of the invariant subspace problem for Banach spaces is unclear (‘embarrassingly unclear’ as some experts mention.

In a recent paper on Math ArXiv (‘On the Invariant Subspace Problem in Hilbert Spaces’, arXiv:2305.15442v2 [math.FA] 6 Apr 2024), Per Enflo has announced a proof for Hilbert spaces.

For Banach spaces, Per Enflo had announced a counter-example in 1975 - the preprint of 87 pages is still unverified by the mathematical community.

In 1973, Per Enflo had solved an old problem (in Acta Mathematica, Vol. 130 (1973) pp. 309-317) by constructing a Banach space B, and a compact operator from a Banach space into B which cannot be approximated in the norm topology for operators by finite rank operators; this gives some experts confidence that Per Enflo's counter-example to the Invariant Subspace Problem for certain Banach Spaces might be correct; others have now given counter-examples too.

Dynamics of Maps and Mordell-Lang

If $f \in \mathbb{C}[X]$ and $a \in \mathbb{C}$, then by the orbit of a under f, we mean the set $O_f(a) := \{a, f(a), f(f(a)), \cdots\} = \{f^n(a) : n \geq 0\}$ where we write f^n for the n-fold iterate of f.

Evidently, if f, g are two polynomials of the same degree such that $f^n = g^n$ for some $n > 0$, and $a, b \in \mathbb{C}$, then $O_f(a) \cap O_g(b)$ is infinite.

Dragos Ghioca, Thomas Tucker, and Michael Zieve proved the converse ('Intersections of polynomial orbits, and a dynamical Mordell-Lang conjecture', Inventiones Mathematicae, Vol. 171 (2008) pp. 463-483).

More precisely, they prove:
Let K be a field of characteristic 0; let $a, b, u, v \in K$ with $u \neq 0$, and let $f, g \in K[X]$ have the same degree > 1. If $(O_f(a) \times O_g(b)) \cap \{uX + v = 0\}$ is infinite, then there exists $n > 0$ such that $g^n(uX + v) = uf^n(X) + v$.

They are able to deduce a special case of a conjectured dynamical analogue of the Mordell-Lang conjecture for the case of a line in $\mathbb{A}^2$.

Interestingly, one of the main ingredients of the proof is a 1920 theorem of Ritt that classifies all polynomials f, g with $f^m = g^n$ for some positive integers $m, n > 0$ (the authors need only the case $m = n$ as degrees are the same in their case).

Random Matrices

Starting with the work of von Neumann in the 1960's on approximate solutions to systems of linear equations, the least singular value $\sigma_n(A) = min\{||Ax||_2 : ||x||_2 = 1\}$ of an $n \times n$ matrix A has been at the forefront of research on random matrices.

A conjecture due to Spielman and Teng made in the ICM 2002 survey talk asserts that for a Rademacher random $n \times n$ matrix B (that is, when the entries are iid uniform in $\{-1, 1\}$), one has $P(\sigma_n(B) \leq \epsilon/\sqrt{n}) \leq \epsilon + e^{-\Omega(n)}$ for all $\epsilon > 0$.

In a paper posted in May (arXiv:2405.20308v12 [math.PR] 30 May 2024), Ashwin Sah, Julian Sahasrabuddhe and Mehtaab Sawhney have proved this conjecture up to a $1 + o(1)$ factor; that is, the conjectured upper bound with $(1 + o(1))\epsilon$ in place of ϵ.

Small Cancellation Theory

For a long time, the basic question as to the existence of a finitely generated group with exactly two conjugacy classes (other than the 2-element group) was wide open.

In 2010, Denis Osin used the so-called small cancellation theory over relative hyperbolic groups to prove the astonishing theorem ('Small cancellations over relatively hyperbolic groups and embedding theorem', Annals of Mathematics, Vol. 172 (2010), pp. 1-39):

Every countable, torsion-free group can be embedded into a finitely generated, torsion-free group with exactly two conjugacy classes.

The famous HNN-construction shows that any countable group can be embedded into a 2-generated group such that elements of the same order are mutually conjugate in the extension.

Osin actually proves the generalization that any countable group G can be embedded into a 2-generated group such that elements of the same order are mutually conjugate in the extension C, and the sets of primes dividing the orders of G and C are the same. He deduces the above-mentioned result from this as a corollary.

Yet another corollary is the assertion that for any $n \geq 2$, there is an uncountable set of pair-wise non-isomorphic finitely generated groups with exactly n conjugacy classes.

We end with a limerick that encapsulates what the outside world thinks of the mathematical community and what we think of their idea.

Here, we speak with pleasure, immense

a language that may seem nonsense

to some, who make fun of us.

But then, none of us

really mind - we are mathematicians!

THANK YOU FOR LISTENING!