Some exercises for tutorials
AIS on Representation theory
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The first 5 problems are on elementary representation theory.
The problems 6 to 18 are on semisimple rings and group algebras.
The problems 19 and 20 are on linear groups and Burnside lemma.
The problems 21 (repeating number 3) to 28 are on character theory.

1. *If $t \neq 0$ is a real number, show that* $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ *is conjugate to* $\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ *in $GL_2(\mathbf{R})$ but not in $SL_2(\mathbf{R})$.*

2. *Show that the only abstract homomorphism from $SL_2(\mathbf{R})$ to $U(n)$ is the trivial one.*

3. *Let $G \leq GL_n(\mathbf{C})$ be a finite group. If $\sum_{g \in G} trace(g) = 0$, prove that $\sum_{g \in G} g$ is the zero matrix.*

4. *Prove that the map*

$$B : M_n(\mathbf{C}) \times M_n(\mathbf{C}) \to \mathbf{C}; (X, Y) \mapsto trace(XY)$$

is a nondegenerate bilinear form.
'Nondegenerate' means that if $B(X, Y) = 0$ for all Y, then $X = 0$. This is used in the proof of Burnside's lemma.

5. Consider the 4-dimensional real representation of the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ on the real vector space $\mathcal{H} := \{a + bi + cj + dk : a, b, c, d \in \mathbf{R}\}$ given by left multiplication. Prove that it is irreducible.

6. *Let $\alpha, \beta : M \to N$ be A-module homomorphisms for a commutative ring A. Assume N is semisimple and that* $\ker \alpha \subseteq \ker \beta$. *Then, prove that there exists an A-module homomorphism $\theta : N \to N$ such that $\beta = \theta \circ \alpha$.*

7. (a) *Is $\mathbf{Z}$ a semisimple ring? Is $\mathbf{Q}$ a semisimple ring?*
(b) *What are all the semisimple $\mathbf{Z}$-modules? What are all the semisimple A-modules for a PID A?*

8. *Prove that the ring* $A = C([0,1], \mathbf{R})$ *of continuous real-valued functions on* $[0,1]$ *is not a semisimple ring by showing that the ideal* $I = \{f \in A : f(0) = 0\}$ *is not a direct summand.*

9. (a) *Show that the center of a left simple ring is a field.*
(b) *Prove that a ring* A *is the direct sum of left ideals* $I_1, \cdots, I_r$ *if, and only if, there are 'idempotents'* $e_i \in I_i$ *(that is, elements satisfying* $e_i^2 = e_i$*) such that* $1 = e_1 + \cdots + e_r$ *and* $e_i e_j = 0$ *for* $i \neq j$.

10. *Let* $A = \mathbf{Q}[X]$ *and let* $M = \mathbf{Q}^2$ *be the* A*-module where any polynomial* f *acts on a vector* (x, y) *as*

$$f(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Is M *a semisimple* A*-module?*

11. *Let* G *be any group and* H*, a subgroup of finite index. Let* K *be a field whose characteristic does not divide the index* $[G : H]$*. Modify the proof of Maschke's theorem to show that any left* $K[G]$*-module which is semisimple as a* $K[H]$*-module is also semisimple as a* $K[G]$*-module.*

12. *Prove that for any finite group* G *and any ring* A*, the group ring* $A[G]$ *contains zero divisors.*

13. *Prove that for any infinite group* G *and any ring* A*, the group ring* $A[G]$ *is never semisimple.*

14. *For any two groups* G_1 *and* G_2*, show that there is a ring* A *such that* $A[G_1] \cong A[G_2]$.
Hint : Get a group G_0 such that $G_0 \times G_1 \cong G_0 \times G_2$.

15. *Find* $m \neq n$ *and rings* $A \ncong B$ *such that* $M_m(A) \cong M_n(B)$.

16. Show that the group algebra $\mathbf{C}[\mathbf{R}^{>0}]$ of the group $\mathbf{R}^{>0}$ is not a semisimple ring.
Hint : Consider $\mathbf{C}^2$ as a module where $t \in \mathbf{R}^{>0}$ acts by sending (z, w) to $(z + (logt)w, w)$.

17. (a) *If* G *is the cyclic group of order* n *and* K *is any field, prove that the group algebra* $K[G] \cong K[X]/(X^n - 1)$.

(b) *If* $G =< g >$ *is cyclic of order* n*, consider the 'left regular representation'*

$$\rho : G \to GL(\mathbf{C}[G]); g \mapsto (\sum_x \alpha_x x \mapsto \sum_x \alpha_x gx).$$

Find the matrix of $\rho(g^i)$ *with respect to the ordered basis* $\{1, g, \cdots, g^{n-1}\}$ *of* $\mathbf{C}[G]$.
(c) *If* $G =< g >$ *is the group or order* 5*, show that* $1 - g - g^4$ *and* $1 - g^2 - g^3$ *are units in the group ring* $\mathbf{Z}[G]$.

18. (a) *For any finite group* G *and any field* K*, prove that*

$$I = \{\alpha \sum_{g \in G} g : \alpha \in K\}$$

is a two-sided ideal in $K[G]$.
(b) *If the characteristic of* K *divides* $O(G)$*, prove that* I *above is not a direct summand of* $K[G]$.
(c) *If characteristic of* K *divides* $O(G)$*, show that there exists an element* $\alpha \in K[G]$ *such that* $\alpha^2 = 0$.

19. *Let* $G \leq GL_n(\mathbf{C})$ *be a finite group. If* G *has noncyclic center, prove that the underlying action of* G *on* $\mathbf{C}^n$ *is reducible.*

20. *Let* $\rho : G \to GL_n(\mathbf{C})$ *be a representation of a (not necessarily finite) group* G*. Suppose there exists a natural number* N *so that* $\rho(g)^N = Id$ *for all* $g \in G$*. Prove that the image of* ρ *is finite.*

21. *Let* $G \leq GL_n(\mathbf{C})$ *be a finite group. If* $\sum_{g \in G} trace(g) = 0$*, prove that* $\sum_{g \in G} g$ *is the zero matrix.*

22. Consider the following character table of a finite group G:

	w	x	y	z
χ_{ρ_1}	1	1	1	1
χ_{ρ_2}	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$	1
χ_{ρ_3}	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$	1
χ_{ρ_4}	3	0	0	-1

Find: (a) $O(G)$, (b) which one of w, x, y, z is central, (c) cardinalities of the conjugacy classes and (d) Dimensions of the irreducible representations.
Can you also determine G up to isomorphism in this case?

23. Let G be a finite abelian group and $\hat{G}$ denote the group $Hom(G, \mathbf{C}^*)$.
(a) Prove that $\{\sum_g \chi(g)g : \chi \in \hat{G}\}$ is a basis of $\mathbf{C}[G]$.
(b) For any element $\alpha = \sum_g a_g \cdot g$ in $\mathbf{C}[G]$, compute the matrix of α with respect to the two bases $\{g : g \in G\}$ and $\{\sum_g \chi(g)g : \chi \in \hat{G}\}$ and conclude that

$$det(a_{gh^{-1}}) = \prod_{\chi \in \hat{G}} \sum_{g \in G} \chi(g)a_g.$$

24. Deduce from Burnside's $p^\alpha q^\beta$-theorem the following generalization(!) due to Philip Hall :
If $O(G) = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and G contains subgroups of orders $O(G)/p_i^{\alpha_i}$ for each $i \leq r$, then G must be solvable.

25. (i) Let $C_1, \cdots, C_s$ denote the conjugacy classes of a finite group G. Prove that the number of solutions of the equation $x_1 x_2 \cdots x_s = 1$ in G is given by the sum

$$\frac{|C_1| \cdots |C_s|}{|G|} \sum_{\chi \ irred} \frac{\chi(x_1) \cdots \chi(x_s)}{\chi(1)^{s-2}}.$$

(ii) Deduce that an element $g \in G$ is a commutator $xyx^{-1}y^{-1}$ if, and only if, $\sum_{\chi \ irr} \frac{\chi(g)}{\chi(1)} = 0$ where the sum is over all irreducible complex characters.
Hint : Write $\mathcal{C}_i = \sum_{g \in C_i} g$ and to find a_1 in $\mathcal{C}_1 \cdots \mathcal{C}_s = \sum_i a_i \mathcal{C}_i$, apply various irreducible characters and use Schur's second orthogonality relations.

26. Let G be a finite group and let C denote the 'Casimir element' in $\mathbf{C}[G]$ defined as $\sum_{x,y \in G} xyx^{-1}y^{-1}$. (a) Prove, for each $g \in G$, $\sum_{x,y} xygx^{-1}y^{-1} = gC$.
(b) For each $g \in G$, if $N_n(g)$ denotes the number of solutions $(x_1, \cdots, x_n)$ with $x_i \in G$ such that $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} = g$, then show in $\mathbf{C}[G]$ that

$$\sum_{g \in G} N_{2k}(g)g = C^k$$

and

$$\sum_{g \in G} N_{2k+1}(g)g = C^k |G|.$$

27. Prove Burnside's result that every nontrivial irreducible complex character χ of a finite group G satisfies $\chi(g) = 0$ for some $g \in G$.

28. Let G be a finite group and $f, g : G \to \mathbf{C}$ be class functions. Prove Plancherel's formula: $< f, g >= \sum_{i=1}^{s} < f, \chi_i >< \chi_i, g >$ where $\chi_1, \ldots, \chi_s$ are the irreducible characters of G.