1. Ramanujan's mathematics—some glimpses

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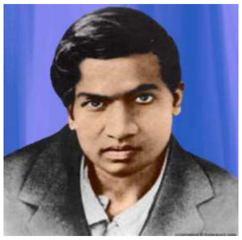


In India, if the common man is asked to name great mathematicians from our country, it is almost certain that the first name that would appear is that of S. Ramanujan (22 December 1887-26 April 1920). Ramanujan's initial struggles in India followed by his departure for England to collaborate with G. H. Hardy, one of topmost mathematicians of those times, have been well-chronicled. The book by Robert Kanigel and the feature film made on his life have splendidly showcased these aspects. However, understandably, they do not feature technical descriptions of his mathematical work in detail. In a short life span, Ramanujan made deep contributions which are still of significance in present day mathematics. In this article, we provide a brief glimpse into a wide variety of topics that he contributed to significantly.

1.1 A Continued Fraction of Ramanujan

$$\frac{e^{-2\pi/5}}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-6\pi}}{1+}\cdots = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$

This continued fraction appeared in Ramanujan's very first letter to Hardy written on January 16, 1913. Of this and some other formulae in that letter, Hardy said in 1937:

"They defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."

We mention briefly what a simple continued fraction is. It is an expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

A less cumbersome notation is

$$l = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\cdots$$

That is, $l = a_0 + 1/l_1, l_1 = a_1 + 1/l_2, l_2 = a_2 + 1/l_3$ etc. and l is the limiting value which can be shown to exist.

For instance, $1 + \frac{1}{1+}\frac{1}{1+}\cdots$ equals the golden ratio. Akin to decimal expansions, this is a device to express any positive real number in terms of positive integers. Decimal expansions of rational numbers are recurring and, an analogous thing happens for real numbers satisfying a quadratic equation over integers.

For instance, $\sqrt{7} = 2 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\cdots$ where the $1,1,1,4$ will keep repeating; one writes briefly as

$$\sqrt{7} = [2; \overline{1,1,1,4}, \cdots].$$

Actually, writing continued fractions for rational numbers p/q is familiar to all if one realizes that it is essentially equivalent to the Euclidean division algorithm of finding the GCD of p and q. In fact, it is also equivalent to very efficiently solving the linear Diophantine equation $px + qy = (p,q)$. For example:

Suppose we wish to solve $72x + 5^7y = 1$ in integers x, y. Look at the following divisions:

$5^7 = 78125 = 1085 \times 72 + 5; \quad 72 = 14 \times 5 + 2; \quad 5 = 2 \times 2 + 1; \quad 2 = 2 \times 1 + 0$

Then $\frac{78125}{72} = [1085;\ 14,\ 2,\ 2]$.

[1]This is an expanded version of talks delivered to undergraduate students of science

The penultimate convergent $[1085; 14, 2] = 1085 + 1/(14 + \frac{1}{2}) = 1085 + \frac{2}{29} = \frac{31467}{29}$ provides us with a solution of $72x + 5^7 y = 1$; viz. $(x, y) = (-31467, 29)$!
It is easy to write the above as an algorithm to solve linear equations.
Indeed, if $m < n$ and we wish to solve $mx + ny = (m, n)$, their GCD.

1. Write $\frac{n}{m} = [a_0; a_1, \cdots, a_k]$.
2. Compute $[a_0; a_1, \cdots, a_{k-1}] = \frac{u}{v}$.
3. Then $(x, y) = ((-1)^k u, (-1)^{k-1} v)$ is a solution of $mx + ny = (m, n)$.

The continued fraction of Ramanujan quoted in the beginning can be proved using the so-called Rogers-Ramanujan identities which are, in turn, intimately connected to the theory of partitions to which Ramanujan made fundamental contributions.

1.2 A quick peek at partitions

Given a natural number n, the number $p(n)$ of ways of partitioning n as a sum of natural numbers seems simple enough to study but turns out to be deceptively difficult. The 7 partitions of 5 are: $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$.
The first few values

$$p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ p(4) = 5,\ p(5) = 7$$

do not seem to give a clue as to either a formula or even of how these numbers grow astronomically. For instance, $p(200)$ is almost 4×10^{12}.
So, it would be impossible to enumerate big numbers like $p(200)$ actually.
Ramanujan first observed empirically, then conjectured the following amazing congruences:

$$p(5n+4) \equiv 0 \ (mod \ 5);\ p(7n+5) \equiv 0 \ (mod \ 7);\ p(11n+6) \equiv 0 \ (mod \ 11).$$

He proved the first two and the last one was proved by Atkin. Ramanujan also conjectured that congruences modulo powers of 5, 7, 11 must also hold.
The partition function has a nice generating function : $\sum_{n=0}^{\infty} p(n) q^n = \prod_{r=1}^{\infty} \frac{1}{1-q^r}$, where the convention is to put $p(0) = 1$.
The above identity is formally seen to be true as an identity in q by expanding each term of the right hand side as a geometric series.
The following wonderful algebraic identities have reformulation in terms of the partition functions.
Rogers-Ramanujan identities:
If $|q| < 1$, then

$$1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n \geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and

$$1 + \sum_{n \geq 1} \frac{q^{n(n+1)}}{(1-q)\cdots(1-q^n)} = \prod_{n \geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

These identities are equivalent forms of the partition-theoretic statements (see P. A. MacMahon, Combinatory Analysis, vol. 2, Cambridge University Press, New York, NY, USA, 1916):

(i) *The number of partitions of n into parts, any two of which differ by at least* 2*, equals the number of partitions of n into parts congruent to* ± 1 *modulo* 5.

(ii) *The number of partitions of n into parts* > 1*, any two of which differ by at least* 2*, equals the number of partitions of n into parts congruent to* ± 2 *modulo* 5.

Partition identities are intimately related to many subjects like statistical mechanics, representation theory, modular forms etc.
A natural question is whether there is a formula to express the exact number of partitions. The remarkable exact formula below is due to Rademacher but it was based on an asymptotic formula of Hardy and Ramanujan.

$$p(n) = \frac{1}{\sqrt{2}\pi}\sum_{q=1}^{\infty} A_q(n)\sqrt{q}[\frac{d}{dx}\frac{\sinh((\pi/q)(\frac{2(x-1/24)}{3})^{1/2})}{(x-1/24)^{1/2}}]_{x=n}$$

where $A_q(n) = \sum \omega_{p,q} e^{-2np\pi i/q}$, the last sum being over p's prime to q and less than it, $\omega_{p,q}$ is a certain $24q$-th root of unity.
Here is an interesting aspect which may not be well-known! More than 30 years back, a conference was held in TIFR Bombay to celebrate Ramanujan's centenary, where the mathematician Atle Selberg (who won the Fields medal for his elementary proof of the prime number theorem) spoke thus:
"If we look at Ramanujan's first letter to Hardy, there is a statement which has relation to his later work on the partition function. He claims an approximate expression for a certain coefficient of a reciprocal of a theta series. This is the exact analogue of the leading term in Rademacher's formula. Ramanujan, in whatever way, had been led to the correct term. It must have been, in a way, Hardy who did not fully trust Ramanujan's insight and intuition when he chose another expression which they developed into an asymptotic formula $p(n) \sim exp(c\sqrt{n})/4n\sqrt{3}$ where $c = \pi\sqrt{2/3}$. If Hardy had trusted Ramanujan more, they would have inevitably ended with the Rademacher series."
He also went on to say: "One might speculate, although it may be somewhat futile, about what would have happened if Ramanujan had come in contact not with Hardy but with a great mathematician of more similar talents, someone who was more inclined in the algebraic directions, for instance, Erich Hecke in Germany. This might perhaps have proved much more beneficial and brought out new things in Ramanujan that did not come to fruition by his contact with Hardy. But Hardy deserves greatest credit for recognizing Ramanujan's originality and assisting him and his work in the best way he could."

1.3 Ramanujan's Tau function

Ramanujan's work on the tau function (named after him) is among the most influential in several parts of mathematics. We discussed the partition function $p(n)$ which has the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{r=1}^{\infty}\frac{1}{1-q^r}$$

Related to $p(n)$ is the function

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

where $q = e^{2i\pi z}$ and $z = x + iy$ with $y > 0$.

$\Delta(z)$ has strong transformation properties under the transformations $z \mapsto z+1$ and $z \mapsto -1/z$; indeed $\Delta(z+1) = \Delta(z)$. So $\Delta(z)$ has a Fourier expansion in powers of $q = e^{2i\pi z}$:

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

where $\tau(n)$ is now known as Ramanujan's tau function.
Ramanujan predicted remarkable properties of the tau function and they have been proved much later leading to some more deep discoveries. Ramanujan's tau function takes integer values and

he conjectured:
$\tau(mn) = \tau(m)\tau(n)$ if m, n are coprime;
$\tau(p^{r+1}) = \tau(p^r)\tau(p) - p^{11}\tau(p^{r-1})$ for $r > 0$ and p prime;
$|\tau(p)| \leq 2p^{11/2}$ for prime p.
The first two conjectures were proved by Mordell not very long after they were made but the third one was proved by Pierre Deligne who won a Fields medal for that work in 1974 (see "La conjecture de Weil I" Pierre Deligne, Publications Mathématiques de l'IHÉS, Volume 43 (1974), p. 273-307 and "Formes modulaires et représentations l-adiques" by Pierre Deligne, Séminaire Bourbaki, vol. 1968/69, exposé 355).
An elementary (but perhaps strange-looking) implication is that for any natural number n, the value $\tau(n)$ differs from $\sigma_{11}(n)$ by a multiple of the prime 691. Here $\sigma_{11}(n)$ denotes the sum of the 11-th powers of the divisors of n. A famous unsolved conjecture of D. H. Lehmer from 1947 asserts that Ramanujan's tau function never vanishes! In fact, even the question whether p divides $\tau(p)$ for infinitely many primes p is open.
One curious recent result shows that each integer $N \neq 0, 1, -1$ is expressible as $\sum_{i=1}^{148000} \tau(n_i)$ where $\max n_i << |N|^{2/11} exp(-c \log|N| / \log\log|N|)$. This is optimal in view of Deligne's result proving Ramanujan's conjecture that $|\tau(n)| \leq d(n)n^{11/2}$.

1.4 Ramanujan primes

A beautiful theorem about primes which goes under the name Bertrand's postulate asserts that there is always a prime $n < p \leq 2n$ for any $n > 1$. Several proofs are known including an elegant one due to Ramanujan. The wandering mathematician Paul Erdős wrote his first paper on a proof of this. It happens to be close to Ramanujan's proof. But, Ramanujan went further and analyzed the number of primes between n and $2n$-this increases with n. Indeed, for each r, if n_r is the smallest positive integer such that there are at least r primes between $N/2$ and N for any $N \geq n_r$, then clearly n_r is itself a prime-called the r-th Ramanujan prime[2]. The first few such 'Ramanujan primes' are $2, 11, 17, 29, 41, 47$.
It is a consequence of the prime number theorem (PNT) that the n-th Ramanujan prime is between the $2n$-th prime and the $4n$-th prime for every n. We mention that the PNT is the statement that the number of primes up to x is asymptotic to $x/\log(x)$; equivalently, the n-th prime is asymptotic to $n\log(n)$. Recently, it has been proved that the n-th Ramanujan prime R_n is asymptotic to the $2n$-th prime.
There are interesting open conjectures like there are arbitrarily long strings of primes which consist entirely of Ramanujan primes etc.!

1.5 A Diophantine equation with primes

A beautiful observation due to Ramanujan is:

$$2 + (1/2)^2 = (3/2)^2; \quad 2.3 + (1/2)^2 = 5/2)^2; \quad 2.3.5 + (1/2)^2 = (11/2)^2;$$
$$2.3.5.7 + (1/2)^2 = (29/2)^2; \quad 2.3.5.7.11.13.17 + (1/2)^2 = (1429/2)^2.$$

The natural question arises as to whether there are other solutions to the equations $p_1 p_2 \cdots p_k + r^2 = y^2$ where $r,\ y$ are non-zero rational numbers and p_i's are certain primes.
Diophantine equations seeking prime number solutions are notoriously difficult to solve compared to solutions in arbitrary integers. Very recently, new techniques from subjects like dynamical systems, ergodic theory and 'thin subgroups of arithmetic groups' have enabled us to obtain results such as there are infinitely many Pythagorean triangles whose area is an 'almost' prime for some r (an r-almost prime is a product of at the most r primes).

[2]The name Ramanujan prime in this context was given by J. Sondow; there is another very different notion of Ramanujan prime coming from automorphic forms

1.6 Ramanujan and denesting

It is elementary to prove that the 'nested' radical converges and gives

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = 3.$$

Ramanujan had posed similar, more complicated problems of "*de-nesting radicals*"-that is replacing an expression with radicals by another with fewer radical symbols. In this regard, he proved the following very beautiful theorem:
If m, n are arbitrary, then

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}} =$$

$$\pm\frac{1}{3}(\sqrt[3]{(4m+n)^2}+\sqrt[3]{(4m-8n)(4m+n)}-\sqrt[3]{2(m-2n)^2}).$$

Of course, as in the case of many algebraic or number-theoretic identities, it is easy to verify simply by manipulation; in this case, by squaring both sides! However, a priori, it is neither clear how this formula was arrived at by Ramanujan nor how general it is. Are there more general formulae? In fact, it turns out that Ramanujan was absolutely on the dot here; the following result shows Ramanujan's result cannot be bettered :
Let $\alpha, \beta \in \mathbf{Q}^*$ such that α/β is not a perfect cube in $\mathbf{Q}$. Then, $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ can be denested if and only if there are integers m, n such that $\frac{\alpha}{\beta} = \frac{(4m-8n)m^3}{(4m+n)n^3}$.
For instance, it follows by this theorem that $\sqrt{\sqrt[3]{3}+\sqrt[3]{2}}$ cannot be denested.
In fact, the proof of theorems like the one above uses Kummer theory of Galois extensions. Ramanujan probably did not know this theory but then he had this uncanny ability to unearth a special result which turns out each time to be the only one of its kind!

1.7 House numbers

Every one has heard the story of the famous taxi-cab number 1729. We know this story thanks to P. C. Mahalanobis who was a contemporary of Ramanujan at the Cambridge university.
About Ramanujan's thought process or his familiarity with specific natural numbers, G. H. Hardy wrote:
"I have often been asked whether Ramanujan had any special secret; whether his methods differed in kind from those of other mathematicians; whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction; but, I do not believe it. My belief is that all mathematicians think, at bottom, in the same kind of way, and that Ramanujan was no exception. He had, of course, an extraordinary memory. He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Mr Littlewood (I believe) who remarked that every positive integer was one of his personal friends. I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."

Lest this anecdote gives the impression that Ramanujan came up with this amazing property of 1729 on the spot, let me add that he had actually written it down even before going to England!

Ramanujan discovered the identity

$$(6A^2-4AB+4B^2)^3 = (3A^2+5AB-5B^2)^3+(4A^2-4AB+6B^2)^3+(5A^2-5AB-3B^2)^3$$

from which the identity $12^3 = (-1)^3 + 10^3 + 9^3$ follows. With a modern point of view where geometric methods are often used to solve arithmetic problems, the equation $x^3 + y^3 = z^3 + w^3$ represents a rational elliptic surface. One may interpret that Ramanujan discovered a $K3$ surface with Picard number 18, one which can be used to obtain infinitely many cubic twists over $\mathbb{Q}$ with rank at least 2 (see Ken Ono & Sarah Trebat-Leder, *The 1729 K3 surface*, Res. Number Theory (2016) 2; 26). Incidentally, $K3$ surfaces are fundamental objects not only in arithmetic geometry, but also in string theory, and remarkably, they were defined first by André Weil in 1958!

During one of the times Mahalanobis visited Ramanujan, he mentioned that in the Strand magazine, he had seen the following problem:

Imagine that you are on a street with houses marked 1 through n. There is a house in between such that the sum of the house numbers to the left of it equals the sum of the house numbers to its right. If n is between 50 and 500, what are n and the house number?

Ramanujan thought for a moment and replied "Take down the solution" and dictated a continued fraction saying that it contained the solution. Evidently, Ramanujan wanted to have some fun instead of directly giving the answer! So, what is behind this?

If the house number is r, then we have

$$1 + 2 + \cdots + (r-1) = (r+1) + \cdots + n$$

The LHS is $\frac{(r-1)r}{2}$ and if we add $1 + 2 + \cdots + r = \frac{r(r+1)}{2}$ to both sides, we have $r^2 = \frac{n(n+1)}{2}$. Multiplying by 8 and adding 1, we have $8r^2 + 1 = (2n+1)^2$.

The equation above is a special case of the equation $x^2 - Ny^2 = 1$ where N is a square-free positive integer (popularly and erroneously known as Pell's equation-it has nothing to do with Pell!)

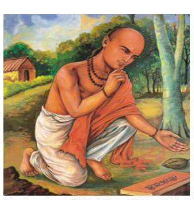
Brahmagupta lived during 598-670 AD

The ancient Indian mathematicians Brahmagupta and Bhaskara II had completely solved them and their solution-the 'CHAKRAVALA' method-can be expressed using the (more modern) continued fractions as we will presently recall. Before that, I very briefly recall some history of this problem. Brahmagupta (6th century) and Bhaskara (12th century) not only solved equations of the form $x^2 - Ny^2 = \pm 1$, as mentioned above, they also gave the marvellous chakravala algorithm to do that. In 1150 AD, Bhaskara II gave the explicit solutions (this is the smallest solution)

$$1766319049^2 - 61(226153980)^2 = 1$$

$$158070671986249^2 - 109(15140424455100)^2 = 1!$$

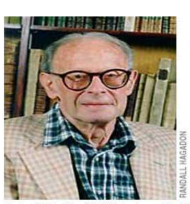
In fact, Brahmagupta had already solved this equation in 628 AD for several values like $N = 83$ and $N = 92$. He is said to have remarked, "*a person who is able to solve these two cases within a year is truly a mathematician*"!



Bhaskaracharya II lived from 1114 to 1185 AD.

Amazingly, a 1657 challenge of Fermat "*to the English mathematicians and all others*" was posed in a letter to his friend Frenicle; he posed the problem of finding a solution of $x^2 - Ny^2 = 1$ "*pour ne vous donner pas trop de peine*" like $N = 61, 109$. André Weil, one of the greatest mathematicians of the 20th century, who is also an Indophile, says of this:

"What would have been Fermat's astonishment if some missionary, just back from India, had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier?".



Andre Weil : May 6, 1906 to August 6, 1998

The continued fraction solution to Brahmagupta-Pell equation goes as follows. For a positive integer N which is not a perfect square, the continued fraction expansion of $\sqrt{N}$ looks like $\sqrt{N} = [b_0; \overline{b_1, b_2, \cdots, b_r, 2b_0}]$.

Further, the penultimate convergent $[b_0; b_1, b_2, \cdots, b_r]$

(in fact, each of the convergents $[b_0; b_1, b_2, \cdots, b_r, 2b_0, b_1, b_2, \cdots, b_r]$ etc.) gives a solution of $x^2 - Ny^2 = -1$ or of $x^2 - Ny^2 = 1$ according as to whether the period $r+1$ above is odd or even.

Let us return to our problem of the Strand magazine. As $\sqrt{8} = [2; \overline{1,4}]$, the convergents $\frac{3}{1}, \frac{17}{6}, \frac{99}{35}, \frac{577}{204}, \frac{3363}{1189}, \cdots$ give solutions of $x^2 - 8y^2 = 1$.
In the above problem, the house number r and the total number n of houses are related by the equation $(2n+1)^2 - 8r^2 = 1$; this means $(n, r) = (1,1), (8,6), (288, 204), (1681, 1189), \cdots$.
The unique solution for n between 50 and 500 is $n = 288$ and the house number is $r = 204$ but there are infinitely many solutions all given by the continued fraction of $\sqrt{8}$ that Ramanujan dictated!

1.8 Ramanujan's 6-8-10 theorem

Ramanujan proved the following beautiful identity for arbitrary numbers a, b, c, d with $ad = bc$:

$$\begin{aligned}
&64[(a+b+c)^6 + (b+c+d)^6 - (c+d+a)^6 - (d+a+b)^6 + (a-d)^6 - (b-c)^6] \\
&\times[(a+b+c)^{10} + (b+c+d)^{10} - (c+d+a)^{10} - (d+a+b)^{10} + (a-d)^{10} - (b-c)^{10}] \\
&= 45[(a+b+c)^8 + (b+c+d)^8 - (c+d+a)^8 - (d+a+b)^8 + (a-d)^8 - (b-c)^8]^2.
\end{aligned}$$

Only several decades later, more general identities have been discovered.

1.9 Ramanujan and fast convergents to Pi

Ramanujan wrote a paper, 'Modular equations and approximations to π' where one of his amazing formulae reads

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

In a Ramanujan centenary conference volume, J. M. Borwein and P. B. Borwein assert that the partial sums in the above infinite series converge to the true value more rapidly than any other calculation of π until the 1970's. Each successive term adds roughly eight more correct digits. The Borweins bettered Ramanujan's result in 1987. In an article in Scientific American of 1988, they say:
"Iterative algorithms (where the output of one cycle is taken as the input for the next) which rapidly converge to pi were, in many respects, anticipated by Ramanujan, although he knew nothing of computer programming.
Indeed, computers not only have made it possible to apply Ramanujan's work but have also helped to unravel it.
Sophisticated algebraic manipulations software has allowed further exploration of the road Ramanujan travelled alone and unaided 75 years ago."
A sense of incredulity prevails on reading these words when one pictures Ramanujan sitting and writing on a slate and erasing with his elbow!

1.10 Highly Composite Numbers and Probablistic Number Theory

A number n is highly composite if the number of divisors $d(m)$ of m satisfies $d(m) < d(n)$ for all $m < n$. Ramanujan wrote a long paper on 'highly composite numbers' which inspired Erdös for his beginning work. Erdös recalls that he got access to a manuscript of Ramanujan on this topic which was not completely published because "*during the first world war, paper was expensive*"!
Ramanujan proved-in collaboration with Hardy-that most positive integers have at least $\log\log\ n$ distinct prime factors. This means essentially that however slow a function $\psi(n)$ goes to infinity, the difference between the number $\omega(n)$ of distinct primes dividing n and the number $\log\log n$ is at the most as large as $\psi(n)\sqrt{\log\log n}$ for 'almost' all n-in the sense that the proportion of n up to N satisfying this inequality tends to 1 as N tends to infinity.

It is now well-documented that this was the key result that inspired Erdös and Mark Kac to come up with a new subject now called probabilistic number theory. In fact, Erdös and Kac proved that for any real number t, the proportion of $n \leq N$ satisfying

$$\omega(n) \leq \log\log n + t\sqrt{\log\log n}$$

tends to $\frac{1}{2\pi}\int_{-\infty}^{t} e^{-x^2/2}dx$ as $N \to \infty$. This statement can be interpreted as saying:
The number of prime factors of a random positive integer n behaves like a normally distributed random variable with mean $\log\log n$ and standard deviation $\sqrt{\log\log n}$.
Mark Kac also wrote an article titled 'Can one hear the shape of a drum?" This intriguing title means whether the acoustics produced by a drum (read this as eigenvalues of the Laplace-Beltrami operator of that drum) can be the same for two differently-shaped drums. It is known now there are many examples of such 'Isospectral but not isometric' spaces.

1.11 15 Theorem of Bhargava

Although infinitely many positive integers are not expressible as the sums of three integer squares, Lagrange proved that four integer squares always suffice. That is, the 'quadratic form' $x^2 + y^2 + z^2 + w^2$ is 'universal' for positive integers (that is, the set of values covers all positive integers). Ramanujan was the first to put forth a study of universal forms in 1916 when he wrote down the list of all integral, positive-definite, 4-dimensional diagonal forms $ax^2 + by^2 + cz^2 + dw^2$ which are universal!
Ramanujan's list of 55 forms was later shown to be accurate and exhaustive excepting one form $x^2 + 2y^2 + 5z^2 + 5w^2$ which was observed to take all values excepting the value 15. Manjul Bhargava proved the remarkable "15 theorem" asserting that an integral quadratic form which takes all the values between 1 and 15 takes all integer values! This shows how optimal Ramanujan's result is.

Ramanujan's list of 55 4-dimensional universal diagonal forms

1, 1, 1, 1	1, 2, 3, 5	1, 2, 4, 8
1, 1, 1, 2	1, 2, 4, 5	1, 2, 5, 8
1, 1, 2, 2	~~1, 2, 5, 5~~	1, 1, 2, 9
1, 2, 2, 2	1, 1, 1, 6	1, 2, 3, 9
1, 1, 1, 3	1, 1, 2, 6	1, 2, 4, 9
1, 1, 2, 3	1, 2, 2, 6	1, 2, 5, 9
1, 2, 2, 3	1, 1, 3, 6	1, 1, 2, 10
1, 1, 3, 3	1, 2, 3, 6	1, 2, 3, 10
1, 2, 3, 3	1, 2, 4, 6	1, 2, 4, 10
1, 1, 1, 4	1, 2, 5, 6	1, 2, 5, 10
1, 1, 2, 4	1, 1, 1, 7	1, 1, 2, 11
1, 2, 2, 4	1, 1, 2, 7	1, 2, 4, 11
1, 1, 3, 4	1, 2, 2, 7	1, 1, 2, 12
1, 2, 3, 4	1, 2, 3, 7	1, 2, 4, 12
1, 2, 4, 4	1, 2, 4, 7	1, 1, 2, 13
1, 1, 1, 5	1, 2, 5, 7	1, 2, 4, 13
1, 1, 2, 5	1, 1, 2, 8	1, 1, 2, 14
1, 2, 2, 5	1, 2, 3 8	1, 2, 4, 14
1, 1, 3, 5		

1.12 Ramanujan sums

In 1918, Ramanujan published a famous paper titled 'On certain trigonometrical sums and their applications in the theory of numbers' in the Transactions of the Cambridge Philosophical Society. He proved several nice properties of certain finite sums which are now known as Ramanujan sums. Even though Dirichlet and Dedekind had already considered these sums in the 1860's, according to G. H. Hardy, "Ramanujan was the first to appreciate the importance of the sum and to use it systematically."

Ramanujan sums play a key role in the proof of a famous result due to Vinogradov asserting that every large odd number is the sum of three primes. These sums have numerous other applications in combinatorics, graph theory and even in physics; they have applications in the processing of low-frequency noise and in the study of quantum phase locking-subjects about which Ramanujan had no remarkable knowledge! So, what are these sums?
For integers $n \geq 1, k \geq 0$, the sum $c_n(k) = \sum_{(r,n)=1} e^{2ikr\pi/n}$ is called a *Ramanujan sum*. In other words, it is simply the sum of the k-th powers of the primitive n-th roots of unity-'primitive' here means that the number is not an m-th root of unity for any $m < n$. Note that the primitive n-th roots of unity are the numbers $e^{2ikr\pi/n}$ for all those $r \leq n$ which are relatively prime to n.
The first remarkable property the Ramanujan sums have is that they are integers-indeed

$$c_n(m) = \sum_{d|(n,\, m)} d\mu(n/d).$$

Recall that the Möbius function is defined by
$\mu(n) = 1$, if $n = 1$;
$\mu(n) = (-1)^k$ if $n = p_1 \cdots p_k$, a product of k distinct primes;
$\mu(n) = 0$ otherwise.
Ramanujan showed that several arithmetic functions (that is, functions defined from the set of positive integers to the set of complex numbers) have 'Fourier-like' of expansions in terms of the sums; hence, nowadays these expansions are knows as Ramanujan expansions. They often yield very pretty elementary number-theoretic identities. Recently, mathematicians have used the theory of group representations of the permutation groups (the so-called supercharacter theory) to re-prove the old identities in a quick way and also discover new identities (see C. F. Fowler, S. Ramon Garcia & G. Karaali, *Ramanujan sums as supercharacters*, Ramanujan J. 35 (2014) p.205-241).
Look at the sum of the r-th powers of divisors function

$$\sigma_r(n) = \sum_{d|n} d^r$$

If $\zeta(s)$ denotes the sum of the series $\sum_{l=1}^{\infty} \frac{1}{n^s}$ for any $s > 1$, we have:

$$\sigma_r(k) = k^r \zeta(r+1) \sum_{n=1}^{\infty} \frac{c_n(k)}{n^{r+1}}$$

Note that these give expansions for σ_r when $r \geq 1$.
The expansion for the divisor function $d(k) = \sigma_0(k)$ can also be deduced from the above as

$$d(k) = \sum_{n=1}^{\infty} -c_n(k) \frac{\log(n)}{n}$$

For any $m \geq 1$, a generalization of the Euler's totient function is $\phi_m(k) = k^m \prod_{p|k}(1 - p^{-s})$ where the product on the right is over all the prime divisors of k; ϕ_1 is the phi function. Ramanujan showed for any $m \geq 1$ that

$$\phi_m(k) = \frac{k^m}{\zeta(m+1)} \sum_{n=1}^{\infty} \frac{\mu(n) c_n(k)}{\phi_{m+1}(n)}$$

Let $r_m(k) = |\{(a,b) : a,b \in \mathbf{Z}, a^m + b^m = k\}|$, the number of ways to write k as a sum of two m-th powers. Ramanujan obtained expressions for r_2, r_4, r_6, r_8 and a few other related arithmetic functions.
For $r_2(k)$, this is:

$$r_2(k) = \pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} c_{2n-1}(a).$$

Here is a curiosity: a form of the famous prime number theorem is the assertion that $\sum_n \frac{\mu(n)}{n} = 0$ and this is also equivalent to the assertion that $\sum_{n \geq 1} \frac{c_n(k)}{n} = 0$ for all k !

1.12.1 Ramanujan sums and cyclotomic polynomials

For convenience, if we write

$$\Delta_n = \{e^{2ir\pi/n} : (r,n) = 1\},$$

then the set of all n-th roots of unity $\{e^{2ik\pi/n} : 0 \leq k < n\}$ is a union of the disjoint sets Δ_d as d varies over the divisors of n. Consider the 'characteristic' function $\delta_{k|n}$ which has the value 1 when k divides n and the value 0 otherwise. Here are some properties of Ramanujan's sums which are left as easy exercises to verify.

Properties of $c_k(n)$.

(i) $c_n(k) = c_n(-k) = c_n(n-k)$;

(ii) $c_n(0) = \phi(n)$ and $c_n(1) = \mu(n)$;

(iii) $c_n(ks) = c_n(k)$ if $(s,n) = 1$; in particular, $c_n(s) = \mu(n)$ if $(s,n) = 1$;

(iv) $c_n(k) = c_n(k')$ if $(k,n) = (k',n)$; in particular, $c_n(k) \equiv c_n(k') \bmod n$ if $k \equiv k' \bmod n$;

(v) $\sum_{k=0}^{n-1} c_n(k) = 0$;

(vi) $\sum_{d|n} c_d(k) = \delta_{n|k} n$ and $c_n(k) = \sum_{d|n} d\mu(n/d)\delta_{d|k} = \sum_{d|(n,k)} d\mu(n/d)$;

(vii) $c_{mn}(k) = c_m(k)c_n(k)$ if $(m,n) = 1$.

The equality $c_n(k) = \sum_{d|n} d\mu(n/d)\delta_{d|k}$ is very useful; even computationally the defining sum for $c_n(k)$ requires approximately n operations where as the other sum requires roughly $\log(n)$ operations.

Recall that $c_n((k,n)) = c_n(k)$; thus, for each fixed n, one may say that the function $k \mapsto c_n(k)$ is "*even modulo* n". This is in analogy with even functions which are 'even modulo 2'. The following beautiful general theorem holds good.

Let n be a fixed positive integer and let f be any arithmetic function which is even modulo n. Then, there exists unique numbers a_d for each $d|n$ which satisfy

$$f(k) = \sum_{d|n} a_d c_d(k)$$

In fact, for each $d|n$, we have

$$a_d = \frac{1}{n}\sum_{e|n} f(n/e)c_e(n/d)$$

Just as eigenfunctions have orthogonality properties like the classical Legendre, and Hermite polynomials etc., the finite Ramanujan sums have the remarkable orthogonality relations:

- $\sum_{r|n} \phi(r)c_d(n/r)c_e(n/r) = n\phi(d)$ or 0 according as to whether $d = e$ or not.
- $\sum_{r|n} \frac{1}{\phi(r)} c_r(n/d)c_r(n/e) = \frac{n}{\phi(d)}$ or 0 according as to whether $d = e$ or not.
- If $(mu, nv) = 1$, then $c_{mn}(uv) = c_m(u)c_n(v)$.
- $\sum_{d|n} c_d(n/d) = \sqrt{n}$ or 0 according as to whether n is a perfect square or not.
- $c_d(n/e)\phi(e) = c_e(n/d)\phi(d)$ if d, e are divisors of n.
- $\sum_{d,e|n} c_d(n/e)c_e(n/d) = nd(n)$ for divisors d, e of n.

In addition, there are mixed orthogonality relations such as:

- For divisors d, e of n, we have $\sum_{r|n} c_d(n/r)c_r(n/e) = n$ or 0 according as to whether $d = e$ or not.
- For a divisor d of n, we have $\sum_{r|n} c_d(n/r)\mu(r) = n$ or 0 according as to whether $d = n$ or not.

These novel developments allow us to discover new identities for power sums such as the following:
Let $n = \prod_{i=1}^r p_i^{\alpha_i}$ and $d = \prod_{i=1}^r p_i^{\beta_i}$ be a divisor; that is, $\beta_i \leq \alpha_i$ for all i. Then, for all $m \geq 0$

$$\sum_{k|n} c_d(k)^m = \prod_{i=1}^r \left((\alpha_i - \beta_i + 1)\phi(p_i^{\beta_i})^m + (-1)^m (p_i^{\beta_i - 1})^m \right).$$

One of the sources for the proofs of such properties and their generalizations is the paper "Ramanujan sums as supercharacters" by Christopher F. Fowler, Stephan Ramon Garcia, and Gizem Karaali in Ramanujan J. 35 (2014) P. 205-241.
To give an idea as to how ubiquitous Ramanujan sums are in many counting problems like the number of solutions of congruences etc., here is one recent result (see K. Bibak, B. M. Kapron, Venkatesh Srinivasan, R. Tauraso & L. Toth, *Restricted linear congruences*, Journal of Number Theory 171 (2017) p.128-144):
Let $b, n \geq 1, t_i | n (1 \leq i \leq k)$ be given integers. The number of solutions of the linear congruence $x_1 + \ldots + x_k \equiv b \pmod{n}$ with $(x_i, n) = t_i (1 \leq i \leq k)$, is

$$\frac{1}{n} \sum_{d|n} c_d(b) \prod_{i=1}^k c_{\frac{n}{t_i}}\left(\frac{n}{d}\right)$$

This result has been used in counting coverings (see K. Bibak, B. M. Kapron and Venkatesh Srinivasan, *Counting surface-kernel epimorphisms from a co-compact Fuchsian group to a cyclic group with motivations from string theory and QFT*, Nuclear Physics B 910 (2016) p.712-723). For instance, a co-compact Fuchsian group Γ with signature $(g; n_1, \cdots, n_k)$ has a presentation

$$< x_1, \cdots, x_k, a_1, b_1, \cdots, a_g, b_g | x_1^{n_1}, \cdots, x_k^{n_k}, [a_1, b_1] \cdots [a_g, b_g] > .$$

It can be seen without much difficulty that for every positive divisor d of the lcm of n_i's, the number of surface-kernel homomorphisms from Γ to the cyclic group of order d equals the number of solutions of the restricted linear congruence $x_1 + \cdots + x_k \equiv 0 \pmod{d}$, with $(x_i, d) = d/n_i$ for all $i \leq k$.

1.13 Modular forms and Mock theta functions

There are several other topics like Ramanujan graphs and the circle method which we have not alluded to. We just look at one final topic-mock theta functions-which Ramanujan mentioned in his last letter to Hardy three months before his death and which is proving to be of deep interest today in conformal field theory, the theory of black holes and quantum invariants of some special 3-dimensional manifolds. In this last letter, Ramanujan talks excitedly about some functions called 'mock theta functions'. He does not define these functions but gives 17 examples and observes a certain key property they possess.
A modern-day motto in mathematics that has been gaining popularity is that there are 5 basic operations in mathematics-addition, subtraction, multiplication, division and modular forms! Lest it sound incredible, one must add that most of the time, modular forms are present somewhat below the surface making things work! Properties of arithmetic nature like the analysis of the number of divisors of an integer, the number of partitions or the number of expressions of a

number as a sum of squares of integers are 'ruled' by modular forms. Modular forms are functions which have a lot of symmetry in them due to their transforming nicely under some natural transformations like the so-called Möbius transformations.
A classical example is Jacobi's theta function $\theta(x) = \sum_{n \in \mathbf{Z}} e^{i\pi n^2 x}$; it transforms nicely under $x \mapsto -1/x$ and $x \mapsto x+1$. It is effective in determining the number of expressions of a positive integer as a sum of four squares, thereby going much further than just proving Lagrange's result on 4 squares.
Ramanujan gave examples of functions which were not modular forms (which he called mock theta functions) but which asymptotically behaved like theta functions when the argument approached a root of unity. Since Ramanujan's death, several mathematician have studied his examples but there was no unified theory behind them. Almost 82 years later in 2002, Zwegers, in his Bonn Ph.D. work done under the supervision of the versatile mathematician Don Zagier, uncovered such a theory. It is outside the scope here to explain the theory but we can definitely give some elementary consequences which we proceed to do now.
Recall Ramanujan's three congruences for partitions:

$$p(5n+4) \equiv 0 \pmod 5; \; p(7n+5) \equiv 0 \pmod 7; \; p(11n+6) \equiv 0 \pmod{11}$$

In order to understand these, the physicist Freeman Dyson came up with the following conjecture which was proved to be correct for the first 2 congruences by Atkin and Swinnerton-Dyer. Call the rank of a partition to be the largest part minus the number of parts. Dyson's (conjectural) refinement of the fact that $p(5n+4)$ is a multiple of 5 is that the number of partitions of $5n+4$ falls into 5 equal classes-the partitions whose rank is a given residue mod 5. The same explanation works for $p(7n+6)$ being a multiple of 7. It does not work for the 3rd congruence and Dyson later defined something called the 'crank' which we don't go into here.
The generating function for the rank of a partition is:

$$\sum_{n \geq 1} \Big(\sum_{\lambda \in P(n)} w^{rank(\lambda)} \Big) q^n = \sum_{n \geq 1} \frac{q^{n^2}}{\prod_{m \leq n} (1 - wq^m)(1 - w^{-1} q^m)}$$

Here, the notation $\lambda \in P(n)$ means λ is a partition of n.
When $w = -1$, this gives the first of Ramanujan's examples of a 'mock theta function' of order 3(!)
In fact, if one multiplies the above expression (for $w = -1$) by $q^{1/24}$, the resulting function behaves like a modular form of weight $1/2$.
We end with a recent beautiful result due to Ken Ono and Kathryn Bringmann which comes out of refining Zwegers's work on Ramanujan's mock theta functions-it shows the partition function indeed admits of a large number of congruences. Here are many:
Let t be a positive integer and Q be a prime power which is co-prime to 6. Then, there exists a positive integer A and a residue class B modulo A such that for any residue class r modulo t, and any positive integer $n \equiv B$ modulo A, the number $N(r,t,n)$ of partitions of n which have rank congruent to r modulo t is a multiple of Q.
The Nobel Laureate S. Chandrasekhar wrote 25 years ago:
"It must have been a day in April 1920, when I was not quite ten years old, when my mother told me of an item in the newspaper of the day that a famous Indian mathematician, Ramanujan by name, had died the preceding day; and she told me further that Ramanujan had gone to England some years earlier, had collaborated with some famous English mathematicians and that he had returned only very recently, and was well-known internationally for what he had achieved.
Though I had no idea at that time of what kind of a mathematician Ramanujan was, or indeed what scientific achievement meant, I can still recall the gladness I felt at the assurance that one brought up under circumstances similar to my own, could have achieved what I could not grasp. I am sure that others were equally gladdened.

The fact that Ramanujan's early years were spent in a scientifically sterile atmosphere, that his life in India was not without hardships, that under circumstances that appeared to most Indians as nothing short of miraculous, he had gone to Cambridge, supported by eminent mathematicians, and had returned to India with every assurance that he would be considered, in time, as one of the most original mathematicians of the century-these facts were enough, more than enough, for aspiring young Indian students to break their bonds of intellectual confinement and perhaps soar the way that Ramanujan did."

Another great mathematician Harish-Chandra once said:

"I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two, and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore, a certain naivete, unburdened by conventional wisdom, can sometimes be a positive asset."

Combine this with what Hardy wrote about Ramanujan:

"The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations and theorems of complex multiplication to orders unheard of . . . who had found for himself the functional equation of the zeta function . . . ; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was."

Given these views by eminent mathematicians, we cannot help but wonder how Ramanujan's mind really worked. I end with the following light-hearted description of Ramanujan's mathematics and methods:

Ramanujan did mathematics somehow;
we still can't figure out even now.
He left his mark on 'p of n',
wrote π in series quite often.
The theta functions he called 'mock'
are subject-matter of many a talk.
He died very young–yes, he too !
He was only thirty–two !
His name prefixes the function tau.
Truly, that was his last bow !

□ □ □

2. Some Unlikely but Amazing Men of Mathematics I have met

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The choice of the topic of this article is based on a trend I have noticed in my cursory study of the history of Mathematics in India. My observation is that prior to the advent of Western methods of Education in India, the emergence of great creative mathematicians in India was rather sporadic. We did not have original mathematicians in every generation. The Guru-Shishya tradition ensured the emergence of great musicians and singers in almost every generation but such was not the case with Science and Mathematics.

[1]This is an expanded version of the talk given by the author in the 56th annual conference of Gujarat Ganit Mandal, October, 2019, at Jamnagar.