CROSS-JUMP NUMBERS

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Consider any n-digit integer expressed in the base b. Divide it into a right part of r digits and a left part of $n-r$ digits. To the left part add a number $L<b$ and to the right part add some $R<b$. The addition is done modulo b and the "carry-over" is ignored. Transfer the left part to the right of the right part and we again get an n-digit number. Apply this same process (which we call "cross-jumping") to the new number. Iterating this several times, we can ask if we get the original number back, and, if so, what is the least number N of steps required? We prove that

$$N=\frac{bn}{(b,L+R)\cdot(n,r)}$$

where (a,b) denotes the G.C.D. of two numbers a and b. We first illustrate this by an example.

Example: We take $b=10, n=8, r=2, L=4, R=2$. Starting with the number 56240317, the iteration gives

56240317	26051556	07175426
19562407	58260519	28071758
09195628	11582609	50280711
20091950	01115820	13502801
52200913	22011152	03135022
15522003	54220115	24031354
05155224	17542205	56240317

which gives back the original number in the 20th step.

Let us prove our claimed formula for n. We denote the positions of the n digits from left to right by 1, 2, .., n, respectively. The positions change as $a\to a+r\to a+2r\ldots$ for each $a\le n$, where + is addition modulo n. For repetition of the original number, we should have some $k>0$ so that $a+kr\equiv a \mod n$. Clearly then, $k=n/(n,r)$ is the least such k. The choice of k only ensures that the positions of the original digits are the same after every k steps. Now, for any $m\le k=n/(n,k)$, there is a corresponding a_0 such that $a_0+mr=n$. We have

$$a_0\to a_0+r\cdots\to a_0+(m-1)r=n-r\xrightarrow{L}a_0+mr=n\xrightarrow{R}a_0+(m+1)r\cdots a_0+kr=a_0,$$

where we have written L, R over an arrow to indicate an increase in the value of that digit by L, R, etc. Thus, we have an increment of $L+R$ in the value of each digit for every k steps. For repetition of the original number, this increment should be a multiple of b and, therefore, N must be a multiple of k as well as of $kb/(L+R)$. This gives $N=$ L.C.M. of k and $kb/(L+R)$, i.e.,

$$N=\frac{bn}{(b,L+R)\cdot(n,r)}.$$

In our example, $N=20$.