How to count - an exposition of Polya's theory of enumeration

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Consider the following problem: Mr. A has a cube whose faces he wants to paint either red or green. He wants to know how many such distinct coloured cubes he can make.

Now, since the cube has 6 faces, and he has 2 colours to choose from, the total number of possible coloured cubes is 2^6. But, painting the top face red and all the other faces green produces the same result (aesthetically speaking), as painting the bottom face red and all the other faces green. That is why Mr. A is so confused!

Trial and error is not the best way to solve this problem. We want to find ageneral method. Consider the set of all possible coloured cubes (in this case, these are 2^6 in number). The rotational symmetries transform the cube and, evidently, we would consider two colouring patterns to be different only ifeither cannot be obtained from the other by a rotation. Infact, weconsider two coloured cubes to be equivalent precisely if a rotation is all that distinguishes them. To find the various possible colour patterns which are inequivalent, we shall exploit the fact that the rotational symmetriesof the cube have the structure of a group. Let us explain the above in precise terms. Let D denote a set of objects to be coloured (in our case, the 6 faces of the cube) and R denote the range of colours (in the above case {red, green}). By a colouring of D, one means a mapping $\phi : D \to R$.

Let X be the set of colourings. If G denotes a group of permutations of D,

we can define a relation on the set of colourings as follows:
$\phi_1 \sim \phi_2$ if, and only if, there exists some $g \in G$ such that $\phi_1 g = \phi_2$.
By using the fact that G is a group, it is easy to prove that $\sim$ is an equivalence relation on X, and so it partitions X into disjoint equivalence classes.

Now for each $g \in G$, consider the map $\pi_g : X \to X$ defined as $\pi_g(\phi) = \phi g^{-1}$; it is a bijection from X to itself. In other words, for each $g \in G$, we have $\pi_g \in \text{ Sym } X$, where Sym X = the group of all permutations on X.

Let us define $f : G \to \text{ Sym } X$ as $f(g) = \pi_g$.

Now,

$$\begin{aligned}\pi_{g_1 g_2}(\phi) &= \phi(g_1 g_2)^{-1} = \phi g_2^{-1} g_1^{-1} = \pi_{g_1}(\phi g_2^{-1}) \\ &= \pi_{g_1}(\pi_{g_2}(\phi)) = \pi_{g_1}\pi_{g_2}(\phi)\end{aligned}$$

Therefore f is a homomorphism from G to the group of permutations on X i.e., G can be regarded as a group of permutations of X.
Recall that one says that a group G *acts* on a set X if there is a homomorphism from G to the group of all permutations of the set X. It is clear that the orbits of the action described above are precisely the different colour patterns i.e., the equivalence classes under $\sim$. Therefore, we need to find the number of inequivalent colourings, i.e. the number of equivalence classes of $\sim$, i.e. the number of orbits of the action of G on X. Note that, like in the example of the cube we shall consider only finite sets D, R. The answer will be provided by a famous theorem of Polya. Polya's theorem was published first in a paper of J.H.Redfield in 1927 and, apparently no one understood this paper until it was explained by F.Harary in 1960. Polya's theorem is considered one of the most significant papers in 20th-century mathematics. The article contained one theorem and 100 pages of applications. Before stating this theorem we will recall what has come to be generally known as Burnside's lemma and which will be needed in the proof. Apparently, it is due to Cauchy but attributed to Burnside by Frobenius (see $[M]$ for this bit of history).

Burnside's lemma
Let G be a group of permutations of a set X. Then, number of orbits = $\frac{1}{|G|}\sum\limits_{x \in G}^{X} |X^g|$ *where* $X^g = \{x \in X | x \cdot g = x\}$*, the set of points of X fixed under*

g.

Proof.
Consider the subset S of $X \times G$ consisting of elements (x, g) such that $x \cdot g = x$. Then, $|S| = \sum_{g \in G} |X^g|$ as is apparent from counting over the various x's corresponding to a particular g and then summing over the g's. Also, counting over the g's corresponding to a particular x and then summing over x gives us $|S| = \sum_{x \in X} |G_x|$, where $G_x = \{g \in G | g \cdot x = x\}$, the so-called stabiliser of x. Note that each G_x is a subgroup. Let the orbits in X be $X_1, X_2, \ldots X_k$. But, the stabilizers of elements in the same orbit have the same cardinality as they are conjugate subgroups.

$$\text{Therefore } |S| = \sum_{i=1}^{k} \sum_{x_i \in X_i} |G_{x_i}|.$$

The assertion on stabilisers holds because, if $y = xg$, then

$$G_y = \{h \in G : yh = y\} = \{h \in G : xgh = xg\}$$
$$= \{h \in G : xghg^{-1} = x\} = \{h \in G : ghg^{-1} \in G_x\} = g^{-1} G_x g.$$

Equating the two expressions for S, we get $k = \frac{1}{|G|} \sum_{g \in G} |X^g|$.

To use this lemma for permutation groups, we need the notion of a cycle index. First, recall that any permutation σ in S_n has a disjoint cycle decomposition viz.,

$$\sigma = (i_1, \cdots, i_{r_1}) \cdots (i_{r_d+1}, \cdots i_{r_{d+1}})$$

where the cycles have no index common.
For instance, in S_6, the permutation which interchanges 1 and 3 and interchanges 2 and 4 can be written as (13)(24)(5)(6).

Definition : The cycle index
Let G be a group of permutations on a set of n elements. Let $\{s_1, s_2, \ldots, s_n\}$ be variables. For $g \in G$, let $\lambda_i(g)$ denote the number of i-cycles in the disjoint cycle decomposition of g. Then, the cycle index of G, denoted by $z(G; s_1, s_2, \ldots, s_n)$ is defined as the polynomial expression

$$z(G; s_1, s_2, \ldots, s_n) = \frac{1}{|G|} \sum_{g \in G} s_1^{\lambda_1(g)} s_2^{\lambda_2(g)} \ldots s_n^{\lambda_n(g)}$$

Examples.

1. $G = \{e, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$. Then,

$$z(G; s_1, s_2, s_3, s_4) = \frac{s_1^4 + 2s_1^2 s_2 + s_2^2}{4}$$

2. $G = S_3 = \{e, (1\ 2), (1\ 3), (2\ 3)(1\ 2\ 3), (1\ 3\ 2)\}$

$$z(G; s_1, s_2, s_3) = \frac{1}{6}(s_1^3 + 3s_1 s_2 + 2s_3)$$

In fact,

$$z(S_n; s_1, s_2, \ldots, s_n) = \sum_{\lambda_1 + 2\lambda_2 + \ldots + k\lambda_k = n} \frac{s_1^{\lambda_1} s_2^{\lambda_2} \ldots s_k^{\lambda_k}}{1^{\lambda_1}\lambda_1! 2^{\lambda_2}\lambda_2! \ldots k^{\lambda_k}\lambda_k!}$$

To see this, look at the number of permutations in S_n of the type $(\lambda_1, \lambda_2, \cdots, \lambda_k)$. The i-cycles can be arranged amongst themselves in $\lambda_i!$ ways giving rise to the same permutation. Also, in each i-cycle, one can write any one of the i symbols first and, therefore, we must also divide by i^{λ_i}.

3. In our example of the cube, G is the group of rotations of a cube induced on the set of 6 faces. The rotations of the cube which leave it invariant are (see figure):

(I) 90 degree (clockwise or anti-clockwise) rotations about the axes joining the centres of the opposite faces - there are 6 such;
(II) 180 degree rotations about each of the above axes - there are 3 such;
(III) 120 degree (clockwise or anti-clockwise) rotations about the axes joining the opposite vertices - there are 8 such;
(IV) 180 degree rotations about the axes joining the midpoints of the opposite edges and;
(V) the identity.

The permutations of the 6 *faces* induced by these rotations are as follows. The rotations of type (I) are of the form $(1, 2, 3, 4)$ etc. where we have

numbered the faces from 1 to 6. The 6 permutations of this type give the term $6s_1^2s_4$ in the cycle index of G.
Similarly, the types (II),(III),(IV) and (V) give the terms $3s_1^2s_2^2, 8s_3^2, 6s_2^3$ and s_1^6 respectively. Therefore, the cycle index of G is

$$z(G; s_1, \cdots, s_6) = \frac{1}{24}(6s_1^2s_4 + 3s_1^2s_2^2 + 8s_3^2 + 6s_2^3 + s_1^6).$$

4. Let $G = C_n$ = cyclic group of order n.
The cyclic group C_n is regarded as the group of permutations of the vertices of a regular n-gon. That is, it is the subgroup of S_n generated by an n-cycle $(1, 2, \cdots, n)$. Note that, for a generator g of S_n, the element g^i has the same cycle structure as that of $g^{(i,n)}$. Therefore, the cycle index is

$$\begin{aligned} z(C_n; s_1, s_2, \ldots, s_n) &= \frac{1}{n}\sum_{d/n} \phi\left(\frac{n}{d}\right) s_{n/d}^d \\ &= \frac{1}{n}\sum_{d/n} \phi(d) s_d^{n/d} \end{aligned}$$

Here, ϕ is Euler's totient function defined by $\phi(n)$ being the number of m upto n which are coprime to n.

5. For $n > 2$, the dihedral group D_n is defined as the group of rotations of the regular n-gon given by n rotations about an n-fold axis perpendicular to the plane of the n-gon and reflections about the n two-fold axes in the plane of the n-gon like the spokes of a wheel, where the angle between consecutive spokes is $\frac{2\pi}{n}$ or $\frac{\pi}{n}$ according as n is odd or even. It has order $2n$.
It can be regarded as a subgroup of S_n as follows. The n rotations corresponding to the powers of $\sigma = (1, 2, \cdots, n)$ and the group D_n is the subgroup

$$\{Id, \sigma, \cdots, \sigma^{n-1}, \tau, \tau\sigma, \cdots, \tau\sigma^{n-1}\}$$

where $\tau = (2, n)(3, n-1) \cdots$ The cycle index of D_n is

$$z(D_n; s_1, \cdots, s_n) = \frac{1}{2n}\sum_{d|n} \phi(\frac{n}{d}) s_{n/d}^d + \frac{1}{4}(s_1^2 s_2^{\frac{n}{2}-1} + s_2^{\frac{n}{2}})$$

if n is even and

$$z(D_n; s_1, \cdots, s_n) = \frac{1}{2n}\sum_{d|n}\phi(\frac{n}{d})s_{n/d}^d + \frac{1}{2}s_1 s_2^{\frac{n-1}{2}}$$

if n is odd.
So, the dihedral group D_6 is the symmetry group of the hexagon. One can represent it as the subgroup of S_6 generated by (16)(25)(34) and (123456).
Thus, $z(D_6) = \frac{1}{12}(s_1^6 + 3s_1^2s_2^2 + 4s_2^3 + 2s_3^2 + 2s_6)$.
We shall need this later.

6. The cycle index of the group of symmetries of the *vertices* of the regular octahedron can be obtained as for the cube and, is

$$\frac{1}{24}(6s_1^2s_4 + 3s_1^2s_2^2 + 8s_3^2 + 6s_2^3 + s_1^6.$$

Note that this is the same as that of the group of symmetries of the *faces* of the cube.

Now, we are in a position to state Polya's Theorem.

Polya's Theorem
Suppose D is a set of m objects to be coloured using a range R of k colours. Let G be the group of symmetries of D. Then, the number of colour patterns $= \frac{1}{|G|}z(G; k, k, \ldots, k)$.

Proof.
As explained before, G acts on X, the set of all possible colourings. Clearly, $|X| = k^m$. The number of colour patterns is computed using Burnside's lemma as the number of orbits of this action. This equals $\frac{1}{|G|}\sum_{g\in G}|X^g|$ where $X^g = \{\phi \in X | \phi g = \phi\}$.

So, now we need to find the number of colourings fixed by g. But, a colouring is fixed by g precisely when it is fixed by all the cycles in the disjoint cycle representation of g. Therefore, number of colourings fixed by g equals

$$k^{\lambda_1(g)}k^{\lambda_2(g)}\ldots k^{\lambda_m(g)}.$$

This is evidently equal to $k^{n(g)}$ where $n(g)$ is the number of cycles in g. Therefore, the number of patterns

$$\begin{aligned} &= \frac{1}{|G|}\sum_{g\in G} k^{n(g)} \\ &= \frac{1}{|G|} z(G; k, k, \ldots, k) \end{aligned}$$

This proves the theorem.

So, in our example of the cube, the number of distinct coloured cubes

$$\begin{aligned} &= \frac{1}{24}[2^6 + 6\cdot 2^3 + 8\cdot 2^2 + 3\cdot 2^2\cdot 2^2 + 6\cdot 2^2\cdot 2] \\ &= \frac{1}{24}\times 240 = 10. \end{aligned}$$

There are 10 distinct cubes in all.

Now, our problem of equivalence of colourings has been disposed of. But, a second problem often encountered in counting is that sometimes not all objects are counted with same weight. So, for instance, if Mr. A did not merely wish to know how many cubes he could paint, but how many would have precisely 2 red faces and 4 green faces. Then the above is not good enough. So we will proceed to state and explain a more general form of Polya's theorem which can handle both the above problems.

For that, we will make use of the following concepts: Consider all maps from D to R as before. But, now each $r \in R$ has a weight $w(r)$ attached to it. The $w(r)$'s can be thought of as independent variables and polynomial expressions in them with Q-coefficients can be manipulated formally like polynomials. (In other words, they are elements from a commutative algebra over Q). The weight of a colouring $\phi : D \to R$ is defined as $w(\phi) = \prod_{d\in D} w(\phi(d))$.
$\sum w(r)$ is called the inventory of R and $\{\sum w(\phi) : \phi \in X\}$ is called the inventory of X.

Now, we notice some useful facts about weights, viz. :-

Proposition.

(i) $\sum\limits_{\phi\in\delta} w(\phi) = \left[\sum\limits_{r} w(r)\right]^{|D|}$
(ii) *If* $D_1, D_2, \ldots, D_k$ *partition* D *and,*
if $S = \{\phi \in X|\phi(d) = \textit{constant}\ \forall\ d \in D_i, \forall\ i = 1, 2, \ldots, k\}$*, then,* $\sum\limits_{\phi\in S} w(\phi) = \prod\limits_{i=1}^{k} \sum\limits_{r\in R} w(r)^{|D_i|}$.

Proof.
(i) Let $D = \{d_1, d_2, \cdots, d_n\}$, and $R = \{r_1, r_2, \cdots, r_m\}$.
Then, the right hand side is $(w(r_1) + w(r_2) + \cdots + w(r_m))^n$.
Any term here is of the form $w(r_{i_1})w(r_{i_2})\cdots w(r_{i_n})$. This is equal to $w(\phi)$ for that map ϕ which takes d_1 to r_{i_1}, d_2 to r_{i_2}, and so on. Conversely, any $w(\phi)$ from the left side is of theform $w(r_{j_1})w(r_{j_2})\cdots w(r_{j_n})$ which gives a unique term of the right side. This proves (i).
We prove (ii) now. A term of the right hand side has the form $w(r_{i_1})^{|D_1|}w(r_{i_2})^{|D_2|}\cdots w(r_{i_k})^{|D_k|}$ which is precisely the weight of a function which assumes the value r_{i_1} on D_1, r_{i_2}, on D_2 and so on. Conversely, every function has such a weight and the result follows.

Along with these concepts, we will also use the following generalisation of Burnside's Lemma known as the weighted form of Burnside's lemma :-

Suppose G *a finite group acting on a finite set* S*. Let us write* $s_1 \approx s_2$ *if, and only if,* $\exists g \in G$ *such that* $s_1 \cdot g = s_2$*. Let a weight function* w *be defined on* S *with values in a commutative algebra over the rationals. Suppose that elements in the same orbit have the same weight i.e.* $s_1 \approx s_2 \Rightarrow w(s_1) = w(s_2)$*. Let* ζ *be the set of equivalence classes of* S*. Let* $w(\bar{S})$ *denote the weight of any element in the equivalence class* $\bar{S}$*. Then,*

$$\sum_{\bar{S}\in\zeta} w(\bar{S}) = \frac{1}{|G|}\sum_{g\in G}\sum_{s\in S^g} w(s)$$

Note that, by putting $w(s) = 1 \forall s \in S$ we get the statement of the earlier form of Burnside's lemma.

Proof.
The proof is very similar to the proof of Burnside's lemma when we consider the subset Y of $S\times G$, consisting of elements (s, g) such that $s\cdot g = s$. Instead

of finding the cardinality of Y, we find $\sum_{(g,s)\in Y} w(s)$ proceeding in the same way as the earlier proof and, the asserted result follows.

Our next aim is to obtain a weighted version of Polya's theorem. Again, suppose D is a finite set of objects to be coloured using a finite range R of colours. As before, let $X = \{\phi : D \to R\}$ be the set of all colourings. Then, the group G of permutations of D and, hence, it acts on X in the same way as explained before. Suppose now that each $r \in R$ is given a weight $w(r)$ with the property that equivalent colourings have the same weight (here, as before, the weight of any colouring ϕ is $w(\phi) = \prod_{d\in D} w(\phi(d))$). Let us write $w(\Phi)$ for the weight of any colouring belonging to a particular pattern Φ.

Then, the weighted version of Polya's theorem is:

Polya's Theorem (weighted form)
The inventory of patterns is given by

$$\sum_{\Phi} w(\Phi) = z(G; \sum w(r), \sum (w(r))^2, \ldots).$$

Proof.
Using the weighted form of Burnside's lemma, we get

$$\sum_{\Phi} w(\Phi) = \frac{1}{|G|} \sum_{g\in G} \sum_{\phi\in X^g} w(\phi)$$

where $X^g = \{\phi \in X | \phi g = \phi\}$.

So, we need to find ϕ's that are fixed by g. Let g split D into cycles $D_1, D_2, \ldots, D_n$. These are clearly disjoint and partition D. An element g fixes ϕ precisely if all the cycles of g fix ϕ, i.e., $\phi(d)$ is constant $\forall d \in D_i, \forall i = 1, 2, \ldots n$. Therefore, $X^g, D_1, D_2, \ldots, D_n$ satisfy the conditions of proposition (ii). In fact, we note that
$|D_i| = 1$ for $1 \le i \le \lambda_1$;
$|D_i| = 2$ for $\lambda_1 < i \le \lambda_1 + \lambda_2$ etc.
By propositon (ii)

$$\sum_{\phi\in X^g} w(\phi) = \prod_{i=1}^{n} \sum_{r\in R} w(r)^{|D_i|}$$

$$
\begin{aligned}
&= (\sum w(r))^{\lambda_1(g)} (\sum (w(r))^2)^{\lambda_2} \dots (\sum (w(r)^n))^{\lambda_n} \\
\text{Therefore } \sum_{\Phi} w(\Phi) &= \frac{1}{|G|} \sum_{g\in G} \prod_{i=1}^{n} \left[\sum_{r\in R} (w(r))^i \right]^{\lambda_i(g)} \\
&= \frac{1}{|G|} z(G; \sum w(r), \sum (w(r))^2, \dots)
\end{aligned}
$$

which completes the proof.

To illustrate the above, let us come back to the same example of the cubes. Let weight (red) = r, w (green) = g. Then,

$$
\begin{aligned}
\sum w(r) &= r + g \\
\sum (w(r))^2 &= r^2 + g^2 \\
&\vdots \\
\text{and } \sum (w(r))^k &= r^k + g^k
\end{aligned}
$$

Also, we saw that

$$z(G; s_1, s_2, \dots, s_6) = \frac{1}{24}[s_1^6 + 6s_2^3 + 8s_3^2 + 3s_1^2 s_2^2 + 6s_1^2 s_4]$$

Using Polya's theorem, $\sum w(\Phi)$ is

$$\frac{1}{24}[(r+g)^6 + 6(r^2+g^2)^3 + 8(r^3+g^3)^2 + 3(r+g)^2(r^2+g^2)^2 + 6(r+g)^2(r^4+g^4)]$$

$$= r^6 + r^5 g + 2r^4 g^2 + 2r^3 g^3 + 2r^2 g^4 + r g^5 + g^6.$$

So, from the above inventory of patterns, it is easy to see that there are exactly 2 patterns with precisely 2 red faces and 4 green faces (the coefficient of $r^2 g^4$).

We also note that on putting $r = 1 = g$, we get 10, i.e. the total number of patterns. *Thus, in the weighted form of Polya's theorem, by putting* $w(r) = 1 \ \forall \ r \in R$, *we get the total number of patterns.*

Polya's famous paper begins with the following example (Recall example 3.8 on p.129 without proof but only the answer.)

Another example: How many distinct circular necklace patterns are possible with n beads, these being available in k different colours?

So, we need to find out how many of the k^n possible necklaces are distinct. Clearly, the group G of rotational symmetries here is C_n, the cyclic group of order n.

We have already computed

$$z(C_n, s_1, s_2, \ldots, s_n) = \frac{1}{n} \sum_{d/n} \phi(d) s_d^{n/d}$$

Special case: n is prime.
Then, number of patterns $= k + \frac{k^n - k}{n}$.

Let us consider the case when only white and black beads are allowed (i.e. $k = 2$) and n is prime, say $n = 5$.

$$\begin{aligned}
\sum w(\phi) &= z(C_5; w + b, w^2 + b^2, \ldots, w^5 + b^5) \\
&= \frac{1}{5}[(w + b)^5 + (w^5 + b^5)] \\
&= w^5 + w^4 b + 2w^3 b^2 + 2w^2 b^3 + wb^4 + b^5
\end{aligned}$$

In fact, these patterns are shown in the figure here.

Yet another very useful form of Polya's theorem uses the concept of 'content'. Here, it is convenient to think of R not as a range of colours but of 'figures'. Maps from D to R are called configurations. Especially, this is useful in counting isomers of chemical compounds as we shall see. Every figure in R has a 'content' which is a non-negative integer. The figure counting series is

$$c(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k + \ldots$$

where c_k is the number of figures in R with content k.

Content of a configuration is the sum of contents of figures which occur as images (taking into account the multiplicity) i.e.,
content of ϕ equals $\sum_{d \in D} (c(\phi(d))$.

So, if we introduce some equivalence of maps by the action of a group G on D, (which induces, therefore, an action of G on configurations), then equivalent configurations have the same content.

The generating function to count all configurations is defined as the formal power series

$$F(x) = 1 + F_1 x + F_2 x^2 + \ldots + F_k x^k + \ldots$$

where F_k = number of configurations with content k.

Now, let a figure $r \in R$ with content k be considered as having weight x^k.

$$\begin{aligned}
\sum_r w(r) &= c_0 + c_1 x + c_2 x^2 + \ldots = c(x) \\
\sum_r [w(r)]^2 &= c_0 + c_1 x^2 + c_2 x^4 + \ldots = c(x^2) \\
&\vdots \\
\sum_r [w(r)]^k &= c_0 + c_1 x^k + c_2 x^{2k} + \ldots = c(x^k).
\end{aligned}$$

Then, given $\phi : D \to R$,

$$\begin{aligned}
w(\phi) &= \prod_{d \in D} w(\phi(d)) \\
&= \prod_{d \in D} x^{(\text{contents of } \phi(d))} \\
&= x^{\sum_d \text{ content of } \phi(d)} \\
&= x^{\text{content of } \phi}
\end{aligned}$$

Therefore the configuration counting series is

$$\begin{aligned}
F(x) &= 1 + F_1 x + F_2 x^2 + \ldots + F_k x^k + \ldots \\
&= \sum_\phi w(\phi).
\end{aligned}$$

We call $F(x)$, the inventory of all confugurations.

Therefore a group G acts on that and, hence on the set of configurations, the inventory of inequivalent configurations,

$$\sum w(\Phi) = 1 + \Phi_1 x + \Phi_2 x^2 + \ldots + \Phi^k x^k + \ldots$$

where Φ_k = number of in equivalent configurations with content k.

$$\begin{aligned}\sum w(\Phi) &= z(G; \sum w(r), \sum [w(r)]^2, \ldots) \\ &= z(G; c(x), c(x^2), \ldots)\end{aligned}$$

In other words, we have proved the 'content' version of Polya's theorem:

Let there be a permutation group G acting on the domain D, and hence, on the set of maps into the set R of 'figures'. Let the figure counting series be $c(x)$. Then, the inequivalent configuration counting series $\Phi(x)$ is obtained by substituting $c(x^r)$ for s_r in the cycle index of G i.e., $\Phi(x) = z(G; c(x))$.

Examples.

One of the important applications of the content version of Polya's theorem is the finding of different possible isomers of a chemical compound. Recall that isomers are chemical compounds with the same chemical formula with a different arrangement of the atoms.

1. Let us find the number of benzene rings with cl substituted in the place of H.
The symmetry group of the benzene ring is D_6 (i.e., the symmetries of a regular hexagon).
Now, $z(D_6) = \frac{1}{12}[s_1^6 + 4s_2^3 + 2s_3^2 + 3s_1^2 s_2^2 + 2s_6]$. Here, $D = \{1, 2, 3, 4, 5, 6\}$ $R = \{H, cl\}$.
Let content $(H) = 0$, content $(cl) = 1$. So $c(x) = 1 + x$.

$$\begin{aligned}\sum \Phi(x) &= \frac{1}{12}[(1+x)^6 + 4(1+x^2)^3 + 2(1+x^3)^2 + 3(1+x)^2(1+x^2)^2 \\ &= 1 + x + 3x^2 + 3x^3 + 3x^4 + x^5 + x^6\end{aligned}$$

Therefore there are 13 chemical compounds obtained in this manner (see the figure).

2. Similarly, there are two isomers of the octahedral molecule $PtBr_4Cl_2$ with Pt at the centre and Br and cl at the vertices. This is proved by using the cycle index in example 6, of the vertices of a regular octahedron.

3. To find the number of (simple, undirected) graphs upto isomorphism on a set of n vertices.
Here D is the set of $\binom{n}{2}$ pairs of vertices. If there is an edge between a pair of vertices, let it have content 1, if it has no edge, then let it have content 0.

So, a configuration is a graph, and its content is the number of edges.
Then, $c(x) = 1 + x$.
Now, two labelled graphs are isomorphic if there exists a bijection between the vertices that preserves adjacency (therefore, two graphs are equivalent if if they are isomorphic). Now, the group G of symmetries of the set of pairs of vertices is called $S_n^{(2)}$.

$$z(S_4^{(2)}) = \frac{1}{24}[s_1^6 + 9s_1^2s_2^2 + 8s_3^2 + 6s_2s_4]$$

Counting series for such graphs

$$\begin{aligned} &= \frac{1}{24}[(1+x)^6 + 9(1+x)^2(1+x^2)^2 + 8(1+x^3)^2 + 6(1+x^2)(1+x^4)] \\ &= 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6. \end{aligned}$$

Therefore total number of unlabelled graphs on 4 vertices = 11. These are shown in the figure.

Suggested Reading:

Combinatorics: V. Krishnamurthy.
Book review in Amer.Math.Monthly: Russell Merris.

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