The world of $\Delta\iota o\phi\alpha\nu\tau o\varsigma$

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Celebrating Pi Day 2025
IIT Roorkee


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Диофант из Александрии
(Diophantus of Alexandria, Διοφαντος ο Αλεξανδρευς)
(гг. рождения и смерти неизвестны, вероятно, 200/214 - 284/298 гг.)

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'Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
after attaining half the measure of his father's life chill fate took him.
After consoling his fate by the science of numbers for four years, he ended his life.'

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This puzzle implies that Diophantus's age $x = 84$ is a solution of the equation

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4.$$

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The amateur mathematician Pierre de Fermat had, in his copy of Bachet's translation of Diophantus's *Arithmetica*, made a famous marginal note which came to be known as Fermat's last theorem.

DIOPHANTI ALEXANDRINI ARITHMETICORVM LIBRI SEX.

ET DE NVMERIS MVLTANGVLIS LIBER VNVS.

Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.

AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V. C.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS

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unknown.

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Therefore, the problem as to which integers are sums of three integer cubes becomes very interesting.

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They used 'Charity Engine', a world-wide computer that harnessed idle, unused computing power from over 500000 home PCs to create a crowd-sourced platform.

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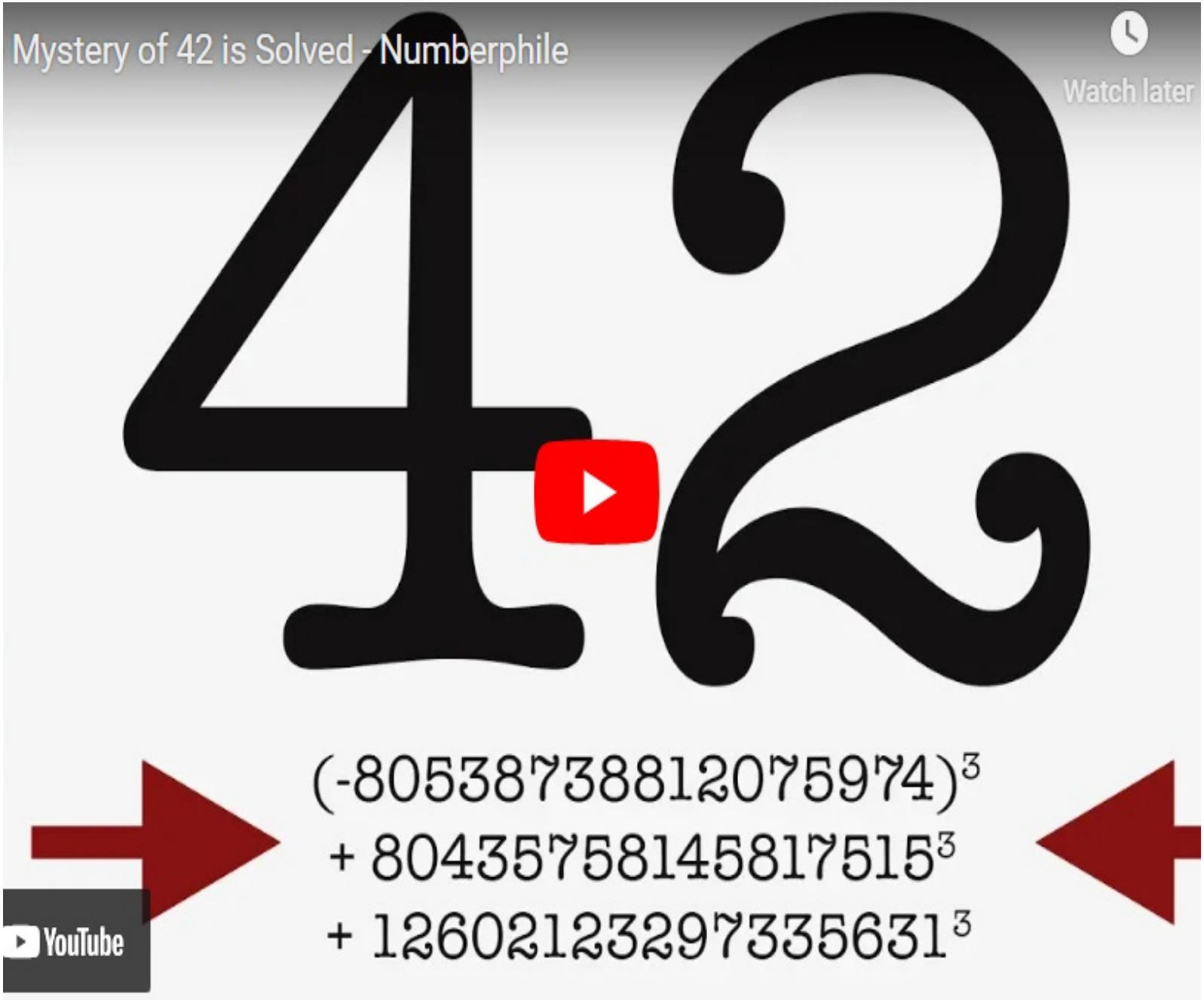
‘Deep Thought’ built by its creators to give the answer to the “Ultimate Question of Life, the Universe, and Everything”.
After eons of calculations, the answer was given simply as “42”.
Deep Thought was then instructed to design the Earth supercomputer to determine what the Question actually is!


Mystery of 42 is Solved - Numberphile
Watch later
42
$(-80538738812075974)^3$
$+ 80435758145817515^3$
$+ 12602123297335631^3$
YouTube


569936821221962380720^3
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$(-569936821113563493509)^3$
+
$(-472715493453327032)^3$
=
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Challenge for the clairvoyant

"It occurs to me that these sorts of questions would be excellent challenge questions to pose to any psychics who claim to be in contact with super-intelligent aliens, since the solutions are already expected to be produced by computer search in a few years but would be instantly verifiable evidence of some extraordinary computational or intellectual resource if produced sooner".

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However, what may not be so well-known is that it has infinitely many rational solutions. For instance, if u, v is a solution, then so is $U = u(u^3 - 3458)/(1729 - 2u^3)$ and $V = v(u^3 + 1729)/(1729 - 2u^3)$.

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Sylvester predicted that:
primes $p \equiv 2, 5 \pmod 9$ are not sums of two rational cubes,
primes $p \equiv 4, 7, 8 \pmod 9$ are sums of two rational cubes.
In contrast, primes $p \equiv 1 \pmod 9$ may or may not be sums of two rational cubes.

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Here is a routine-looking question? Is the infinite series $\sum \frac{1}{n^3 \sin^2(n)}$ convergent?

The convergence of the series depends on the behavior of the sequence $n|\sin(n)|$ as $n \to \infty$ and, this is mysterious. It depends on something unknown as yet - how well can π be approximated by rational numbers?

The *irrationality measure* $\mu(\alpha)$ of an irrational number α is the infimum of all $a > 0$ such that the inequality $|\alpha - p/q| < 1/q^a$ has only finitely many solutions p, q.

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The Dirichlet box principle implies that $\mu(\alpha) \geq 2$ and generically (that is, almost all) α have $\mu(\alpha) = 2$.

For a specific number, it is difficult to find μ; for instance, $\mu(e) = 2$ but, the constant $\mu(\pi)$ is still unknown.

One knows $\mu(\pi) < 8$ but not much more is known.

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Here is the shocker - the series $\sum \frac{1}{n^3 \sin^2(n)}$ diverges if $\mu(\pi) > 5/2$ and converges if, $\mu(\pi) < 5/2$. So, take your pick!

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Hilbert's famous 10th problem asserted: *Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

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Theorem (Davis-Putnam-Robinson-Matiyasevich) There is no general algorithm that, given any Diophantine equation, decides whether it has solutions in positive integers or not. In more technical terms, the notions of 'effectively enumerable' sets and 'Diophantine' sets coincide.

a to z

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The set of positive values of the following polynomial in 26 variables when the variables take positive integer values, equals the set of prime numbers!

$$
\begin{aligned}
(k+2)\{\quad 1 \quad & -[wz+h+j-q]^2 \\
& -[(gk+2g+k+1)(h+j)+h-z]^2 \\
& -[2n+p+q+z-e]^2 \\
& -\left[16(k+1)^3(k+2)(n+1)^2+1-f^2\right]^2 \\
& -\left[e^3(e+2)(a+1)^2+1-o^2\right]^2 \\
& -\left[(a^2-1)y^2+1-x^2\right]^2 \\
& -\left[16r^2y^4(a^2-1)+1-u^2\right]^2-[n+l+v-y]^2 \\
& -\left[\left((a+u^2(u^2-a))^2-1\right)(n+4dy)^2+1-(x+cu)^2\right]^2 \\
& -\left[(a^2-1)l^2+1-m^2\right]^2 \\
& -\left[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x\right]^2 \\
& -\left[z+pl(a-p)+t(2ap-p^2-1)-pm\right]^2 \\
& -[ai+k+1-l-i]^2 \\
& -\left[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m\right]^2\ \}
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which require very different techniques; we will say more about them in a while but first, let us discuss a more elementary problem.

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For instance, if he has 36 fruits, he can do this because $6^2 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.

Think of a fruit-seller arranging her fruits in a triangular pattern in the morning and in a square pattern in the evening.

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Which other squares are so expressible as 'triangular numbers'?

If $n^2 = 1 + 2 + \cdots + k = k(k+1)/2$, then

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Thus, $(2k+1, 2n)$ is a solution of the equation $x^2 - 2y^2 = 1$.

Being aesthetically-minded fruit-sellers, all of us of course want to know what the solutions of $x^2 - 2y^2 = 1$ are !

Here is another instance when the above equation occurs and involves Ramanujan:

December 22, 1887 to April 26, 1920

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Ramanujan thought for a moment and replied "Take down the solution" and dictated a continued fraction saying that it contained the solution!

Evidently, Ramanujan wanted to have some fun instead of directly giving the answer! So, what is behind this?

If the house number is r, then we have

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Multiplying by 8 and adding 1, we have $8r^2 + 1 = (2n+1)^2$, the very same equation encountered by the fruit-seller!

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Can it take the value 1? Can it take the value -1? How many solutions are there?
This rich area of mathematics is popularly (and erroneously!) known as the theory of the Pell equations.
Interestingly, it turns out that there are infinitely many pairs m, n for which $m^2 - dn^2 = 1$ and essentially, they are all generated from a single pair.

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What is more - they gave an algorithm (the so-called Chakravala or cyclic method) which produces all the solutions.

Brahmagupta lived during 598-670 AD

Bhaskaracharya lived from 1114 to 1185 AD.

In 1657, Fermat, writing to his friend Frenicle, he posed "to the English mathematicians and all others" the problem of finding a solution of $x^2 - Ny^2 = 1$ "pour ne vous donner pas trop de peine" like $N = 61,109$.

Fermat : August 17, 1601 to January 12, 1665

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The 20th century great André Weil's comment on this was: "What would have been Fermat's astonishment if some missionary, just back from India, had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier ?"

Andre Weil : May 6, 1906 to August 6, 1998

Indeed, in 1150 A.D., Bhaskara II gave the explicit solutions

$$1766319049^2 - 61(226153980)^2 = 1$$

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Brahmagupta had remarked, "Kurvannaavatsaraad ganakah" - meaning (approximately), "*a person who is able to solve these within a year is truly a mathematician*"' !

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In view of the above understanding of mathematical history, now the equations can better be referred to as the Brahmagupta-Pell equations.

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Why?

6 is easy from the usual $3, 4, 5$ triangle.

To see that 5 is a congruent number, consider the right-angled triangle with sides $3/2, 20/3, 41/6$.

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How did we guess this? More importantly, how do we decide if a given number is a congruent number?

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How did we guess this? More importantly, how do we decide if a given number is a congruent number?
This will be done by relating it to another problem!

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The congruent number problem and the above question are equivalent!

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Then $x = w/2$ is such that $(v-u)^2/4, w^2/4, (u+v)^2/4$ form an arithmetic progression.

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Then $x = w/2$ is such that $(v-u)^2/4, w^2/4, (u+v)^2/4$ form an arithmetic progression.
Conversely, if $x^2 - d = y^2, x^2, x^2 + d = z^2$ are three rational squares in arithmetic progression, then:
$z - y, z + y$ are the legs of a right angled triangle with rational legs, area $(z^2 - y^2)/2 = d$ and rational hypotenuse $2x$ because $2(y^2 + z^2) = 4x^2$.

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The fact that $1, 2$ are not congruent numbers is essentially equivalent to Fermat's last theorem for the exponent 4(!)
Indeed, if $a^2 + b^2 = c^2$, $\frac{1}{2}ab = 1$ for some rational numbers a, b, c then $x = c/2, y = |a^2 - b^2|/4$ are rational numbers satisfying $y^2 = x^4 - 1$.
Similarly, if $a^2 + b^2 = c^2$, $\frac{1}{2}ab = 2$ for rational numbers a, b, c, then $x = a/2, y = ac/4$ are rational numbers satisfying $y^2 = x^4 + 1$.

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Similarly, $y^2 = x^4 + 1$ for rational x, y gives a right-angled triangle with sides $2x, 2/x, 2y/x$ and area 2.

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Here is a (rather unusual!) way of using the above fact that 1 is not a congruent number to show that $\sqrt{2}$ is irrational! Indeed, consider the right-angled triangle with legs $\sqrt{2}, \sqrt{2}$ and hypotenuse 2. If $\sqrt{2}$ were rational, this triangle would exhibit 1 as a congruent number!

Though it is an ancient problem to determine which natural numbers are congruent, it is only in late 20th century that substantial results were obtained and progress has been made which is likely to lead to its complete solution.

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It is easy to show that:
d is a congruent number if, and only if, the elliptic curve $E_d : y^2 = x^3 - d^2x$ has a solution with $y \neq 0$.

In fact, $a^2+b^2=c^2, \frac{1}{2}ab=d$ implies $bd/(c-a), 2d^2/(c-a)$ is a rational solution of $y^2=x^3-d^2x$.

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In a nutshell, here is the reason we got this elliptic curve.
The real solutions of the equation $a^2 + b^2 = c^2$ defines a surface in 3-space and so do the real solutions of $\frac{1}{2}ab = d$.
The intersection of these two surfaces is a curve whose equation in suitable co-ordinates is the above curve!

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Therefore, if d is a congruent number, there are infinitely many rational-sided right-angled triangles with area d(!)

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Further, assuming the truth of a famous, deep, open conjecture known as the *weak Birch & Swinnerton-Dyer conjecture*, it has been shown that this is a complete characterization of congruent numbers!

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No solutions are known.

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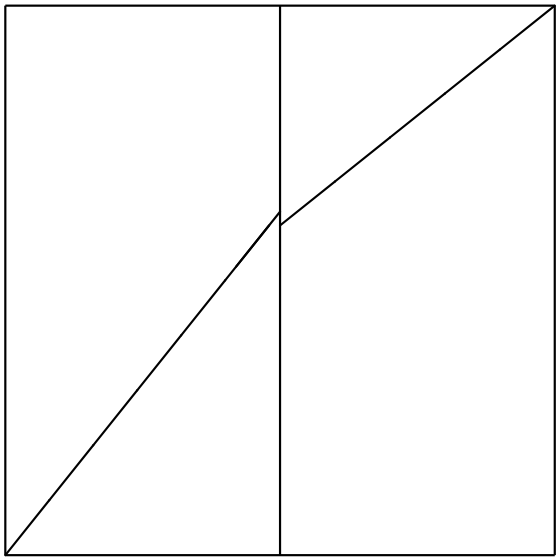

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It is an easy exercise to prove that such a 'rational' walk is impossible because 1 is not a congruent number!

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For instance, these are easy to observe when $r = 2, 3$.

However, for $r > 3$, the proof needs deeper properties of prime numbers, such as:

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By the way, the product of any four consecutive integers is one less than a perfect square:

$$n(n+1)(n+2)(n+3) = (n^2+3n+1)^2 - 1.$$

Which natural numbers have all their digits to be 1 *with respect to two different bases?*

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Equivalently, solve

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}$$

in natural numbers $x, y > 1; m, n > 2$.

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However, it is still unknown whether there are only finitely many solutions in x, y, m, n. In fact, no other solutions are known.

For any *fixed bases* x, y, it was proved only as recently as in 2002 that the number of solutions for m, n is at the most 2.

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Are there other solutions to the equation

$$x(x+d_1) \cdots (x+(m-1)d_1) = y(y+d_2) \cdots (y+(n-1)d_2)$$

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It is only in 1999 that using ideas from algebraic geometry, it was proved that if m, n, d_1, d_2 are fixed, then the equation has only finitely many solutions in integers apart from some exceptions which occur when $m = 2, n = 4$.

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For each $c \in \mathbb{Q}$, the number of (x, y, m, n) satisfying

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with $y \geq x+m$, $\min(m, n) \geq 3$, is finite.

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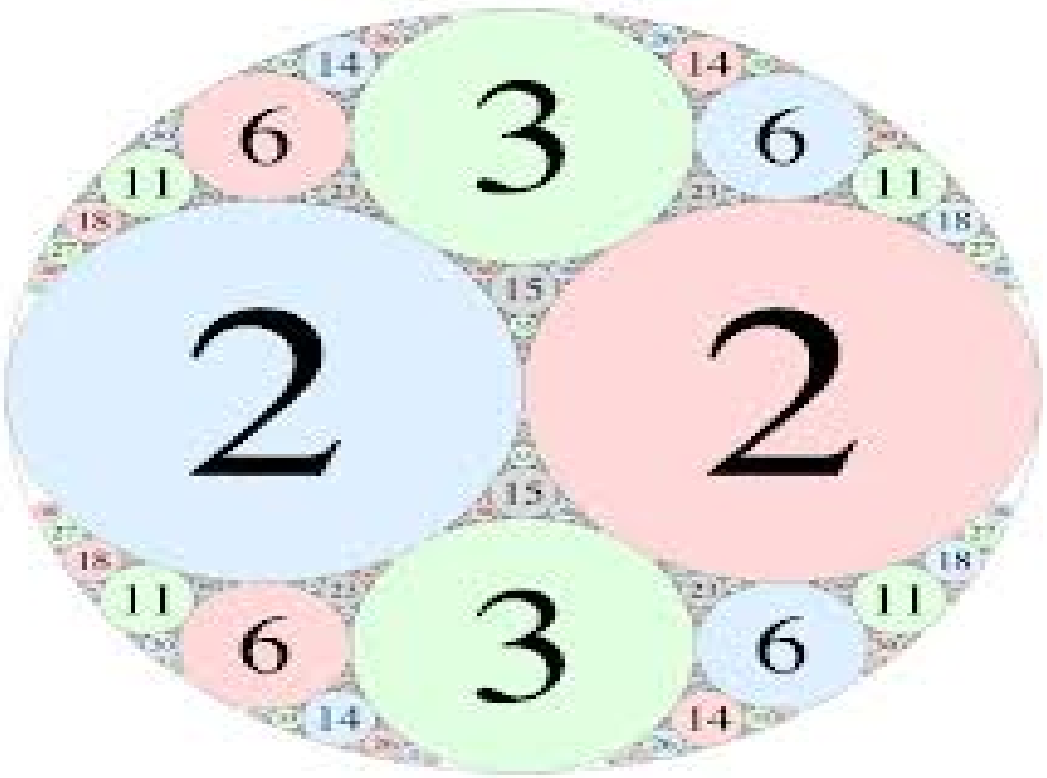
This is unsettled as yet.

Apollonian circle packing

Apollonius from 200 BC discovered something beautiful.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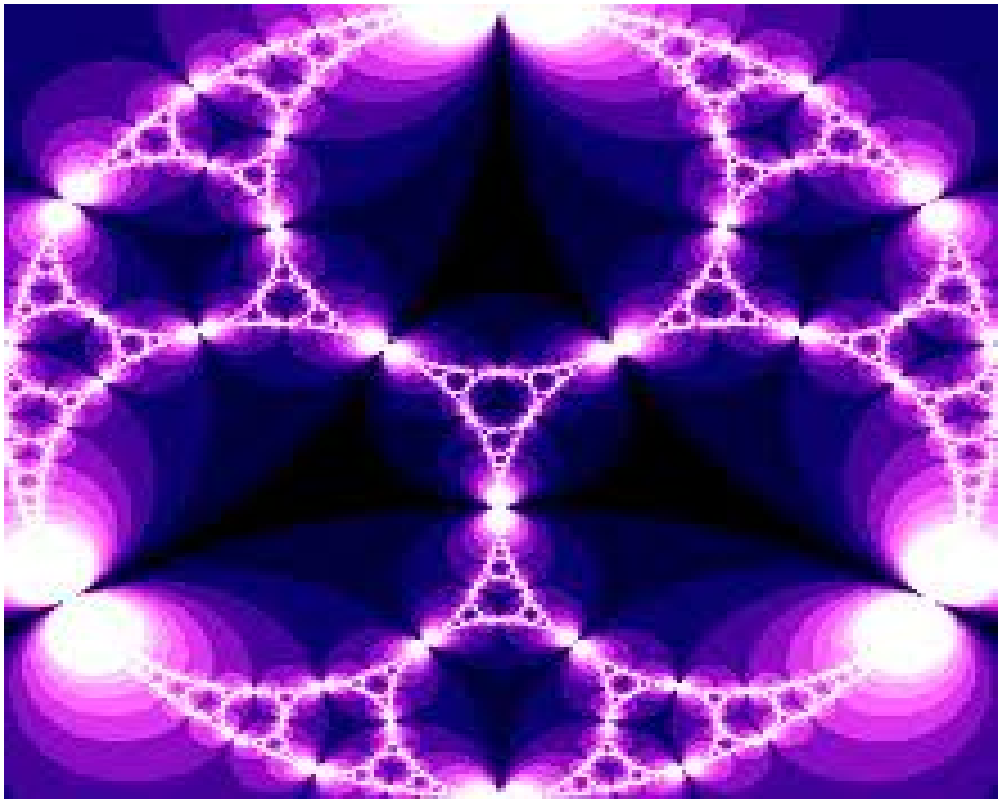

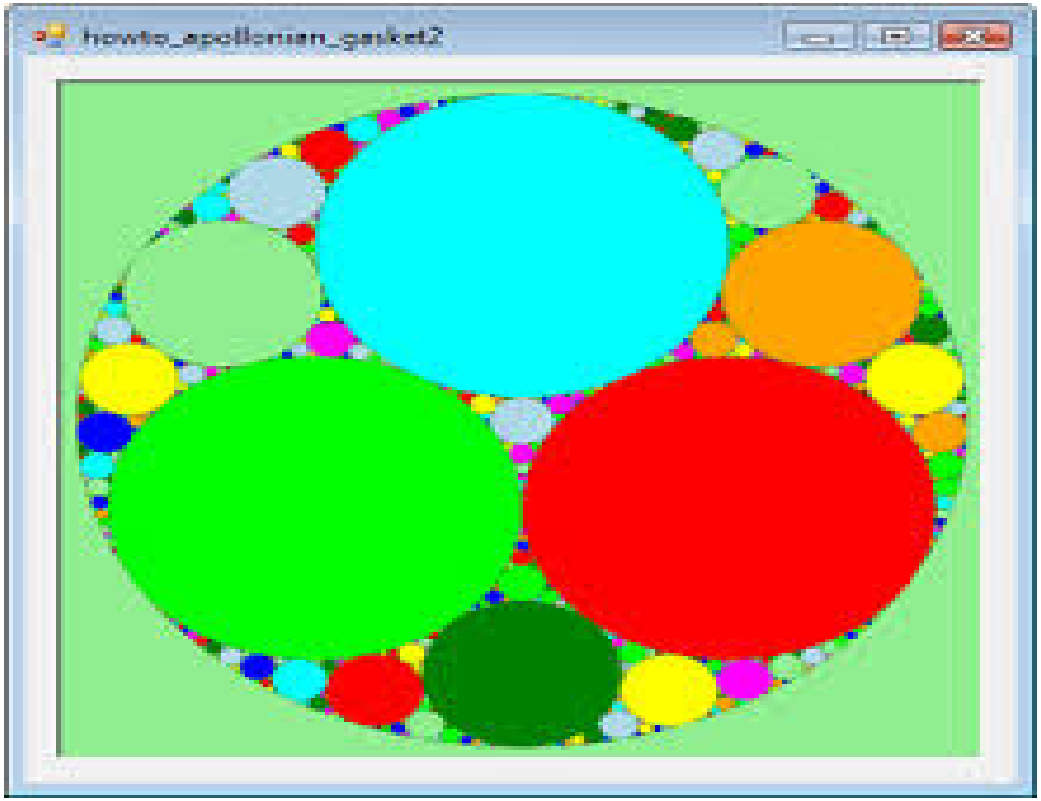
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$$\left(\sum_{i=1}^{4} \frac{1}{r_i} \right)^2 = 2 \sum_{i=1}^{4} \frac{1}{r_i^2}.$$

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$$\left(\sum_{i=1}^{4} \frac{1}{r_i}\right)^2 = 2\sum_{i=1}^{4} \frac{1}{r_i^2}.$$

Here, the circles are supposed to have no common interior point which means by convention that the outermost circle's exterior is the interior and the interior is the exterior and the radius is negative.

In terms of the curvature, which is the reciprocal of the radius, the equation becomes

$$(C_1 + C_2 + C_3 + C_4)^2 = 2(C_1^2 + C_2^2 + C_3^2 + C_4^2).$$

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Thus, if we are given 3 of the circles and they have integer curvatures,t he fourth must also have integral curvature because of the equation!

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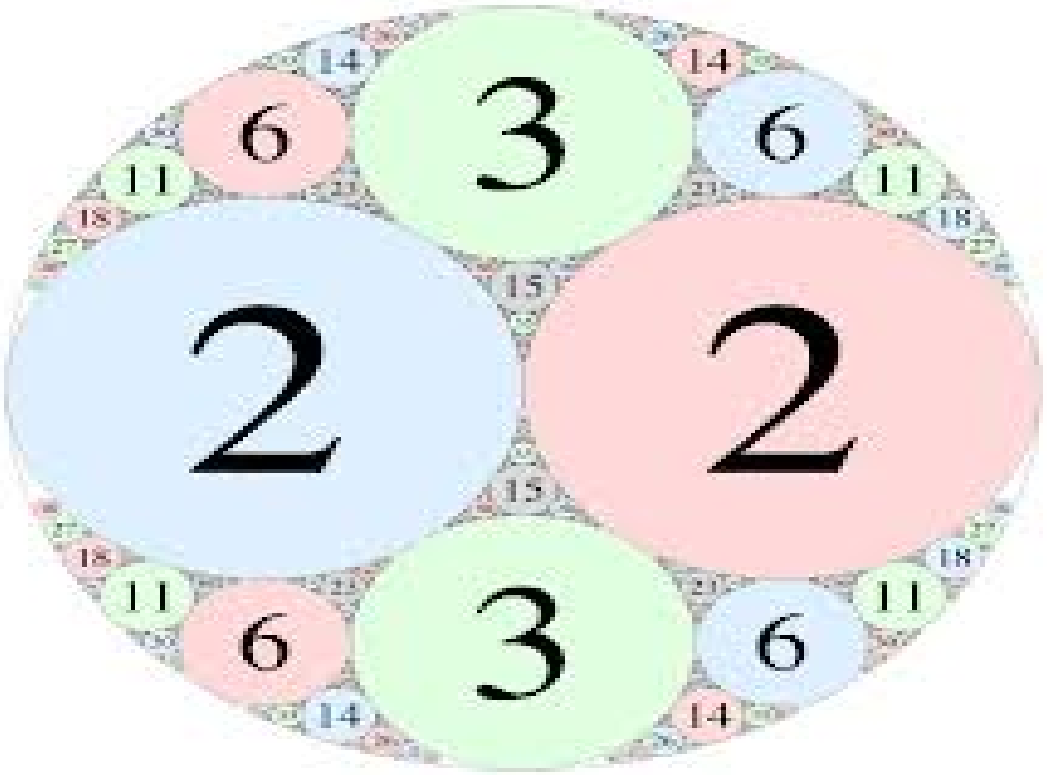

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Recently, very deep mathematical tools have been brought to bear on these problems and there is a veritable treasure for the eye as well as the brain awaiting you if you are interested!

Upping the ante

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The number of integral points on the n-dimensional octahedron $|x_1| + |x_2| + \cdots + |x_n| \leq r$ is given by the expression $p_n(r) = \sum_{i=0}^{n} 2^i \binom{n}{i} \binom{r}{i}$.

The question of whether two octahedra of different dimensions m, n can contain the same number of integral points becomes equivalent to the solvability of $p_m(x) = p_n(y)$ in integers x, y; this is proved now to have only finitely many solutions.

Modus operandi - Siegel's theorem

Theorem (Siegel, 1929). If $F \in \mathbb{Z}[X, Y]$ is absolutely irreducible and the curve $F = 0$ has genus > 0, then the number of integral points on the curve is finite. Further, the finiteness of the number of integer points holds good except when the (projective completion of the) curve defined by $F = 0$ has genus 0 and at most 2 points at infinity.

Let $f, g \in \mathbb{C}[X]$ be two polynomials such that the polynomial $f(X) - g(Y) \in \mathbb{C}[X, Y]$ in two variables is irreducible. Suppose the stationary points of f and g are simple. For each stationary point $a \in S_f$, define

$$r_a := |\{b \in S_g : f(a) = g(b)\}|.$$

Then, the genus g of the curve $f(X) = g(Y)$ is given by

$$2g = \sum_{a \in S_f} (\deg(g) - 2r_a) - \deg(f) + 2 - GCD(\deg(f), \deg(g)).$$

Given $F \in \mathbb{Q}[X, Y)$, one may homogenize this polynomial to a homogeneous polynomial of three variables X, Y, Z. Then, the points in the projective space corresponding to the solutions of $F(x, y, 0) = 0$ are called the points at infinity of the curve $F = 0$.

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Yuri Bilu and Robert Tichy found a novel way to use Siegel's theorem in a more effective manner.

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The following theorems address Erdös's conjecture and gives finiteness of integral solutions when the genus is > 0 and the finiteness of rational solutions when the genus is > 1.

Theorem 2.1. *Let m and n be positive integers with $m \leq n$ and let $\lambda \in \mathbb{C}^*$. If $X(X+1)\cdots(X+m-1) - \lambda Y(Y+1)\cdots(Y+n-1)$ is reducible in $\mathbb{C}[X,Y]$ then one of the following possibilities holds:*

1. *$m = n$, $\lambda = 1$, in which case $X - Y$ is a factor,*
2. *$m = n$ is odd, $\lambda = -1$, in which case $X + Y + m - 1$ is a factor,*
3. *$m = 2$, $n = 4$, $\lambda = \frac{1}{4}$, in which case we have*

$$4X(X+1) - Y(Y+1)(Y+2)(Y+3) = (2X - Y^2 - 3Y)(2X + 2 + 3Y + Y^2).$$

Theorem 2.2. *Consider the curve*

$$X(X+1)\cdots(X+m-1) = \lambda Y(Y+1)\cdots(Y+n-1)$$

with $n \geq m > 1$ *and* $\lambda \in \mathbb{C}^*$. *Suppose it is irreducible. Then its genus is zero in the following cases:*

1. $m = 2, n = 2,$
2. $m = 2, n = 3, \lambda = \pm 3\sqrt{3}/8,$
3. $m = 2, n = 4, \lambda = -4/9,$
4. $m = 2, n = 6, \lambda = (-10 \pm 7\sqrt{7})/576.$

The genus is one in the following cases:

1. $m = 2, n = 3, \lambda \neq \pm 3\sqrt{3}/8,$
2. $m = 2, n = 4, \lambda \neq -4/9,$
3. $m = 2, n = 5, \lambda = -1/4t, 3125t^4 - 47500t^2 + 82944 = 0,$
4. $m = 2, n = 6, \lambda = 16/225,$
5. $m = 2, n = 8, \lambda = -1/4t, t^3 + 567t^2 - 54432t - 4665600 = 0,$
6. $m = 3, n = 3,$
7. $m = 3, n = 4, \lambda = \pm 3\sqrt{3}/2,$
8. $m = n = 4, \lambda = -9/16, -16/9.$

In all other cases the genus is strictly bigger than one.

ABC

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For instance, for the Catalan equation $x^m - y^n = 1$, Robert Tijdeman's finiteness result was made effective by Langevin who obtained an upper bound for x, y, m, n that was of the order of $exp(exp(exp(exp(730))))$.

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A more general conjecture due to S S Pillai is still open; it asserts that the gaps in the sequence of perfect powers tends to infinity.

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It formalizes the observation that when two numbers A and B are divisible by large powers of small primes, then $A + B$ tends to be divisible by small powers of large primes.

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It formalizes the observation that when two numbers A and B are divisible by large powers of small primes, then $A + B$ tends to be divisible by small powers of large primes.

More precisely:

For any $\epsilon > 0$, there are only finitely many triples A, B, C of relatively prime integers satisfying $A + B = C$, and $max(A, B, C) > Rad(ABC)^{1+\epsilon}$, where $Rad(n)$ is the product of all distinct prime divisors of n.

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For instance, the ABC-conjecture implies Fermat's last theorem for sufficiently large exponents; in fact, it implies finiteness of the number of solutions of the generalized Fermat equation $Ax^r + By^s = Cz^t$.

I end with two remarkable results from a century back:

Ritt's first theorem. Let $f_1 \circ f_2 \circ \cdots f_r = g_1 \circ g_2 \circ \cdots g_s$ where $f_i, g_j \in \mathbf{C}[X]$ be nontrivial decompositions into indecomposables. Then, $r = s$ and the sets of degrees $\{deg(f_1), \cdots, deg(f_r)\} = \{deg(g_1), \cdots, deg(g_s)\}$.

Ritt's second theorem. let $f_1 \circ g_1 = f_2 \circ g_2$ be two proper decompositions over $\mathbf{C}$ where $deg(f_1) = deg(g_2)$ is relatively prime to $deg(g_1) = deg(f_2)$.
Then, either

$$f_1(X) = X^r P(X)^s = g_2(X) \ , \ g_1(X) = f_2(X) = X^s$$

or

$$f_1(X) = g_2(X) = D_m(X) \ , \ g_1(X) = f_2(X) = D_n(X)$$

where $D_n(X)$ is the Dickson polynomial of degree n defined by

$$D_n(X + 1/X) = X^n + 1/X^n.$$

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but somehow 3.14 is kosher.
But who cares? Let's celebrate any way!

Thank You For Listening!

Which numbers are popular in IIT?

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All love Pi, and know by sight e.

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- the best way to start is with High Tea!