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A Polynomial Parent to a Fibonacci–Lucas Relation

The ubiquitous Fibonacci and Lucas sequences

$$\{F_n : n \geq 0\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$$

$$\{L_n : n \geq 0\} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots\}$$

are defined via the same recursion $a_{n+2} = a_{n+1} + a_n$ with different starting values. There are many known relations between them. The purpose here is to observe one more relation, which follows from a simple polynomial identity (thought of as a parent?).

Theorem.

$$2^{n+1} F_{n+1} = \sum_{i=0}^{n} 2^i L_i.$$

Here is the proof.

Note that for indeterminates u, v and any positive integer n, we have the polynomial identity

$$(2u)^{n+1} - (2v)^{n+1} = (u - v) \sum_{i=0}^{n} \left((2u)^i + (2v)^i\right) (u + v)^{n-i}.$$

The proof follows simply by summing the two geometric series

$$\sum_{i=0}^{n} (2u)^i (u + v)^{n-i}, \quad \sum_{i=0}^{n} (2v)^i (u + v)^{n-i};$$

these are, respectively, $\frac{(2u)^{n+1} - (u+v)^{n+1}}{u-v}$ and $\frac{(u+v)^{n+1} - (2v)^{n+1}}{u-v}$.

Put $u = 1/2, v = \sqrt{5}/2$; then $u + v, u - v$ are the two roots α, β of $t^2 - t - 1 = 0$. So, the Cauchy–Binet formulas $F_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ for the Fibonacci and Lucas numbers give the asserted identity:

$$F_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \sum_{i=0}^{n} \frac{2^i (\alpha^i + \beta^i)}{2^{n+1}} = \frac{\sum_{i=0}^{n} 2^i L_i}{2^{n+1}}.$$

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