Parabola Volume 59, Issue 1 (2022)

A potpourri of pretty identities involving Catalan, Fibonacci and trigonometric numbers

Enoch Suleiman[1] and **B. Sury**[2]

1 Introduction

Apart from the binomial coefficients which are ubiquitous in many counting problems, the Catalan and Fibonacci sequences seem to be appear almost as frequently. The Fibonacci numbers are defined by the linear recursion $F_{n+2} = F_{n+1} + F_n$ beginning with $F_0 = 0$ and $F_1 = 1$. For the labyrinth of avatars of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, the reader is encouraged to see Richard Stanley's lovely book [3] for more than two hundred interpretations. Two well-known interpretations of the Catalan numbers are as Dyck paths (lattice paths) going in right and upwards steps from $(0,0)$ to (n,n) and never going above the line $y = x$, and as the number of ways to connect $2n$ points on a circle via non-intersecting lines.

In this note, we start by obtaining some identities for sums involving the Catalan sequence. In addition, we discuss the beautiful binomial transform which does not seem to be as well known as it should be. These simple methods allow us to obtain several pretty identities involving Fibonacci numbers, Catalan numbers and trigonometric sums. As an appetizer, we list some of the identities proved in this article:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}C_{k+1} = \begin{cases} 0 & n \text{ is odd} \\ C_{n/2} & n \text{ is even} \end{cases};$$

$$\sum_{j\geq 1}\frac{1}{C_j} = 1 + \frac{4\pi}{9\sqrt{3}};$$

$$(-1)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{n+k+1};$$

$$\sin\frac{2\pi}{n} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}\sin\frac{2\pi(k+1)}{n}\cos^{n-k}\frac{2\pi}{n}.$$

[1] Enoch Suleiman teaches at Federal University Gashua, Gashua, Yobe State, Nigeria.
[2] B. Sury is a Professor of Mathematics at the Indian Statistical Institute in Bangalore, India.

2 Linear recursion for Catalan numbers

An identity for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ was discovered by Touchard in 1928; it asserts that

$$C_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} 2^{n-1-2k} C_k\,. \tag{B}$$

Note that this recursion involves the "bottom half" of the sequence $C_0, \ldots, C_{n-1}$. More generally, let us consider for each integer $a \geq 2$ the *generalised Catalan numbers*

$$C_a(n) := \frac{1}{(a-1)n+1}\binom{an}{n}\,.$$

It is remarkable that these numbers satisfy a *linear* recursion (also observed in [5]). In particular for any $a \geq 2$, the numbers $C_a(n)$ can be defined recursively by the value $C_a(0) = 1$ and, for all $n \geq 1$, the linear recursion

$$C_a(n) = \sum_{k=1}^{\lfloor \frac{(a-1)n+1}{a} \rfloor} (-1)^{k-1} \binom{(a-1)(n-k)+1}{k} C_a(n-k)\,. \tag{A}$$

In particular, the case $a = 2$ gives the usual Catalan numbers C_n by the linear recursion

$$C_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k}\,. \tag{A′}$$

The linear recursion (A) can be proved by using the *forward difference operator* Δ which we briefly recall now. Define a new function Δf for any function f on $\mathbb{R}$ by

$$(\Delta f)(x) := f(x+1) - f(x)\,,$$

and successively define $\Delta^{k+1} f = \Delta(\Delta^k f)$ for each $k \geq 1$. It is easily proved by induction on n that

$$(\Delta^n f)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+n-k)\,.$$

We note that if f is a polynomial of degree d, then Δf is also a polynomial and has degree $d-1$. In particular, $\Delta^N f$ is the zero function 0 when $N > d$, so $(\Delta^N f)(0) = 0$.

To see how the recursion follows from the difference operators, first rewrite (A) as

$$\sum_{k=0}^{\lfloor \frac{(a-1)n+1}{a} \rfloor} (-1)^{k-1} \binom{(a-1)(n-k)+1}{k} C_a(n-k) = 0\,.$$

In other words, using $C_a(n-k) = \frac{1}{(a-1)(n-k)+1}\binom{a(n-k)}{n-k}$, we are claiming that

$$\sum_{k \geq 0} (-1)^{k-1} \binom{(a-1)(n-k)+1}{k} \frac{1}{(a-1)(n-k)+1} \binom{a(n-k)}{n-k} = 0\,.$$

Expanding the binomial coefficients

$$\binom{(a-1)(n-k)+1}{k} = \frac{\big((a-1)(n-k)+1\big)!}{k!\big((a-1)n-ak+1\big)!}$$

and

$$\binom{a(n-k)}{n-k} = \frac{\big(a(n-k)\big)!}{(n-k)!\big((a-1)(n-k)\big)!}$$

and multiplying and dividing by $n!$, the above equation becomes simply

$$-\frac{1}{n}\sum_{k\geq 0}(-1)^k\binom{n}{k}\binom{a(n-k)}{n-1} = 0\,.$$

The function $f(x) = ax(ax-1)\cdots(ax-n+2)$ is a polynomial of degree $n-1 < n$. Therefore,

$$(\Delta^n f)(x) = \sum_{k\geq 0}(-1)^k\binom{n}{k}f(x+n-k) = 0\,.$$

This gives

$$(\Delta^n f)(0) = \sum_{k\geq 0}(-1)^k\binom{n}{k}\binom{a(n-k)}{n-1} = 0\,.$$

This is the asserted recursion (A). □

Note that the expression (A′) involves the "top half" of the number $C_0,\ldots,C_n$. By combining it with the identity (B) which involves the "bottom half" of those numbers, we immediately obtain a new identity. We state it separately for odd and even n for the sake of clarity:

Lemma 1.

$$(2n+1)C_{2n} = \sum_{r=0}^{n-1}\binom{2n}{2r}2^{2n-2r}C_r + (1-(-1)^n)C_n - \sum_{k=1}^{n-1}(-1)^{n+k}\binom{n+k+1}{n-k+1}C_{n+k} \quad \text{(C)}$$

$$2nC_{2n-1} = \sum_{r=0}^{n-1}\binom{2n-1}{2r}2^{2n-1-2r}C_r + \sum_{k=0}^{n-2}(-1)^{n+k}\binom{n+k+1}{n-k}C_{n+k}\,. \quad \text{(D)}$$

Proof. Consider (A′) and (B) respectively, for an even integer $2n$:

$$C_{2n} = \sum_{k=1}^{n}(-1)^{k-1}\binom{2n-k+1}{k}C_{2n-k}\,;$$

$$C_{2n} = \sum_{k=0}^{n-1}\binom{2n-1}{2k}2^{2n-1-2k}C_k\,.$$

Equating the right hand sides and writing on one side the term $2nC_{2n-1}$ corresponding to the largest suffix $2n-1$, we have

$$2nC_{2n-1} = \sum_{r=0}^{n-1} \binom{2n-1}{2r} 2^{2n-1-2r} C_r + \sum_{k=0}^{n-2} (-1)^{n+k} \binom{n+k+1}{n-k} C_{n+k}\,.$$

This is the identity (D). Similarly, taking $2n+1$ in place of n in the identities (A′) and (B), we obtain (C). □

3 Binomial transform

Let $a_0, a_1, \ldots$ be a sequence of numbers and define a sequence of numbers $b_0, b_1, \ldots$ by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

Then, for all $n \geq 0$,

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k\,. \tag{E}$$

This identity is the *inverse binomial transform* and is extremely useful.

To prove (E), first use binomial expansions as follows:

$$\begin{aligned} x^n = ((1+x)-1)^n = \sum_{k=0}^{n} \binom{n}{k} (1+x)^k (-1)^{n-k} &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{r=0}^{n} \binom{k}{r} x^r \\ &= \sum_{r=0}^{n} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{r} \right) x^r\,. \end{aligned}$$

By equating the coefficients of the powers of x on both sides of this equation, we obtain

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{r} = \begin{cases} 0\,, & r < n\,; \\ 1\,, & r = n\,. \end{cases}$$

Therefore,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k = \sum_{k=0}^{n} \sum_{r=0}^{k} \binom{k}{r} a_r (-1)^{n-k} \binom{n}{k} = \sum_{r=0}^{n} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{r} \right) a_r = a_n\,.$$

Thus, the inverse of the binomial transform is established. □

More identities involving Catalan numbers can be deduced from the earlier ones using the binomial transform. Let us apply this transform to the identity due to Touchard.

Defining $a_n = \frac{1}{2^n} C_{\frac{n}{2}}$ if n is even, and $a_n = 0$ if n is odd, and defining $b_n = \frac{1}{2^n} C_{n+1}$, Touchard's identity

$$C_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} 2^{n-1-2k} C_k$$

can be expressed as

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \,.$$

Therefore, the inverse binomial transform yields our next result:

Lemma 2.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} C_{k+1} = \begin{cases} 0 & n \textit{ is odd;} \\ C_{\frac{n}{2}} & n \textit{ is even.} \end{cases} \tag{1}$$

4 Polynomial identities and Fibonacci numbers

Akin to the binomial transform, we have the following result whose proof we leave as a challenging exercise - it is similar to the proof of the inverse binomial transform:

Lemma 3. *Let $a_0, a_1, \dots$ and $b_0, b_1, \dots$ be sequences of complex numbers related by the identity*

$$b_n = \sum_{k \geq 0} \binom{n-k}{k} a_k$$

for all $n \geq 0$. Then

$$a_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} b_{n+k} \,.$$

Using this transform and the polynomial identity

$$\sum_{k \geq 0} (-1)^k \binom{n-k}{k} (XY)^k (X+Y)^{n-2k} = X^n + X^{n-1}Y + \cdots + XY^{n-1} + Y^n \,,$$

which follows easily by induction, as noted in [4], we obtain the next result.

Lemma 4.

$$(XY)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (X^{n+k} + X^{n+k-1}Y + \cdots + Y^{n+k})(X+Y)^{n-k} \,.$$

Proof. Set $a_n = (-XY/(X+Y)^2)^n$ and $b_n = \frac{X^n + X^{n-1}Y + \cdots + Y^n}{(X+Y)^n}$ in Lemma 3. □

Here are some consequences of Lemma 4.

Corollary 5.

$$(-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{n+k+1}$$
$$1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+k+1)2^{n-k}$$
$$\sin\frac{2\pi}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} \sin\frac{2\pi(k+1)}{n} \left(\cos\frac{2\pi}{n}\right)^{n-k}.$$

Proof. It is well known that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ where α, β are the roots of $x^2 - x - 1 = 0$ and must satisfy $\alpha\beta = -1$ and $\alpha + \beta = 1$. Substituting $X = \alpha, Y = \beta$ in Lemma 4, we immediately obtain the first identity.
For the second identity, substitute $X = Y = 1$ in Lemma 4.
Finally, the substitution $X = e^{\frac{2\pi i}{n}}, Y = e^{-\frac{2\pi i}{n}}$ yields the last identity. □

5 Harmonic Sums

We give an identity involving harmonic numbers

$$H_k = \sum_{r=1}^{k} \frac{1}{r} = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$$

that follows by an application of the binomial transform:

Lemma 6.
$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{H_k}{k}. \tag{F}$$

Proof. By the binomial transform, the identity (F) it follows from the identity

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{1}{ij} = \frac{1}{n^2}.$$

This identity was posed as Problem 11164 in the American Mathematical Monthly in 2005 by Dias-Barrero citation details More generally, W. Chu and Q.L. Yan citation details proved that

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \le i_1 \le \cdots \le i_r \le k} \frac{1}{i_1 i_2 \cdots i_r} = \frac{1}{n^r}.$$

The last identity is the specialization of the following formal identity of rational functions where we let $x = 1$ and take $n - 1$ instead of n:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} \sum_{0 \le i_1 \le \cdots \le i_r \le k} \frac{1}{(x+i_1)\cdots(x+i_r)} = \frac{x}{(x+n)^{r+1}}. \tag{♠}$$

Chu proved the rational function identity; as we are unable to provide a reference where this proof is given, we give a proof below for the sake of completeness.

Denote by L the left hand side of (♠). By the relation

$$\binom{n}{k}\binom{x+k}{k}^{-1} = \binom{x+n}{n-k}\binom{x+n}{n}^{-1},$$

we have

$$\binom{x+n}{n}L = \sum_{0\le i_1\le\cdots\le i_r\le n} \frac{1}{(x+i_1)\cdots(x+i_r)} \sum_{k=i_r}^{n}(-1)^k\binom{x+n}{n-k}.$$

Let us now use the well-known binomial identity

$$\sum_{k=i_r}^{n}(-1)^k\binom{x+n}{n-k} = (-1)^{i_r}\binom{x+n}{n-i_r}\frac{x+i_r}{x+n} \tag{G}$$

to get

$$\binom{x+n}{n}L = \frac{1}{x+n}\sum_{0\le i_1\le\cdots\le i_{r-1}\le n} \frac{1}{(x+i_1)\cdots(x+i_{r-1})} \sum_{i_r=i_{r-1}}^{n}(-1)^{i_r}\binom{x+n}{n-i_r}.$$

We use (G) once again on the rightmost sum, obtaining

$$\sum_{i_r=i_{r-1}}^{n}(-1)^{i_r}\binom{x+n}{n-i_r} = (-1)^{i_{r-1}}\binom{x+n}{n-i_{r-1}}\frac{x+i_{r-1}}{x+n}.$$

We thereby obtain

$$\binom{x+n}{n}L = \frac{1}{(x+n)^2}\sum_{0\le i_1\le\cdots\le i_{r-1}\le n} \frac{1}{(x+i_1)\cdots(x+i_{r-2})} \sum_{i_{r-1}=i_{r-2}}^{n}(-1)^{i_{r-1}}\binom{x+n}{n-i_{r-1}}.$$

Repeating this a total of r times, we finally obtain

$$\binom{x+n}{n}L = \frac{1}{(x+n)^r}\binom{x+n}{n}\frac{x}{x+n}$$

which immediately gives the asserted identity

$$L = \frac{x}{(x+n)^{r+1}}.$$

Thus, the proof is complete. □

6 An identity of Lehmer

Here are some more identities involving the Catalan numbers and the Fibonacci numbers, where the starting point is the following generating series, due to D.H. Lehmer citation details, that is valid when for $|z| < 1$:

$$\sum_{j\geq 1} \frac{(4z^2)^j}{\binom{2j}{j}} = \frac{z^2}{1-z^2} + \frac{z\arcsin(z)}{(1-z^2)^{3/2}}\,.$$

By multiplying by $4z^2$ and writing $t = 4z^2$, we have

$$\sum_{j\geq 1} \frac{t^{j+1}}{\binom{2j}{j}} = \frac{t^2}{4-t} + \frac{t^{3/2}\arcsin(\sqrt{t}/2)}{2(1-t/4)^{3/2}}$$

for $0 < t < 4$. The series on the left converges uniformly and we can compute its derivative by differentiating term by term. If we do this, and substitute $t = 1$, then we obtain an identity where the left-hand side is the sum of the reciprocals of the Catalan numbers $C_j = \frac{1}{j+1}\binom{2j}{j}$ for $j \geq 1$. Similarly, we may substitute $t = 2, 3, \frac{1}{2}$ etc. after differentiating, giving the identities below. The identities that follow from the generating function of Lehmer are as follows:

Lemma 7.

$$\sum_{j\geq 1} \frac{t^j}{C_j} = \frac{10t-t^2}{(4-t)^2} + \frac{6\sqrt{t}\arcsin(\sqrt{t}/2)}{(4-t)^{3/2}} + \frac{6t^{3/2}\arcsin(\sqrt{t}/2)}{(4-t)^{5/2}} \quad \textit{for } 0 < t < 4\,;$$

$$\sum_{j\geq 1} \frac{1}{C_j} = 1 + \frac{4\pi}{9\sqrt{3}}\,;$$

$$\sum_{j\geq 1} \frac{1}{jC_j} = \frac{1}{3} + \frac{5\pi}{9\sqrt{3}}\,;$$

$$\sum_{j\geq 1} \frac{1}{(j+1)C_j} = \frac{1}{3} + \frac{2\pi}{9\sqrt{3}}\,;$$

$$\sum_{j\geq 1} \frac{2^j}{C_j} = 4 + \frac{3\pi}{2}\,;$$

$$\sum_{j\geq 1} \frac{3^j}{C_j} = 21 + 8\sqrt{3}\pi\,;$$

$$\sum_{j\geq 1} \frac{1}{2^jC_j} = \frac{19}{49} + \frac{96}{49\sqrt{7}}\arcsin\frac{1}{\sqrt{2}}\,;$$

$$\sum_{j\geq 1} \frac{j}{C_j} = 2 + \frac{7\pi\sqrt{3}}{27}\,.$$

As a corollary, we may obtain expressions for $\sum_{j\geq 1} \frac{F_j}{C_j}$ by using the first identity above with $t = \alpha = \frac{1+\sqrt{5}}{2}$ and $t = -\beta = \frac{\sqrt{5}-1}{2}$, and by recalling that $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$.

For more identities involving sums where the binomial coefficients appear in the denominator, please see [6].

More general infinite series involving the Catalan numbers in the denominator can also be expressed as finite sums as follows. These were first discovered by C. Elsner [1] but let us give a simpler, one-sentence proof. Consider the two infinite series

$$\sum_{n\geq 0} \frac{1}{(2n+1)(2n+3)\cdots(2n+2k+1)\binom{2n}{n}}$$

and

$$\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)(2n+3)\cdots(2n+2k+1)\binom{2n}{n}}$$

as finite sums. We use the notation $k!!$ for $1 \times 3 \times 5 \times \cdots \times (2k-1)$.

Lemma 8.

$$\sum_{n\geq 0} \frac{1}{(2n+1)(2n+3)\cdots(2n+2k+1)\binom{2n}{n}} = \frac{4}{k!!}\left((-3)^k \frac{\pi}{6\sqrt{3}} + \sum_{s<k} \frac{(-\frac{1}{3})^{s-k+1}}{2s+1}\right);$$

$$\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)(2n+3)\cdots(2n+2k+1)\binom{2n}{n}} = \frac{4}{k!!}\left(\frac{5^k}{\sqrt{5}}\log(\frac{1+\sqrt{5}}{2}) - \sum_{s<k} \frac{(1/5)^{s-k+1}}{2s+1}\right).$$

Proof. One may write

$$\begin{aligned}
\sum_{n\geq 0} \frac{1}{(2n+1)(2n+3)\cdots(2n+2k+1)\binom{2n}{n}} &= \sum_{n\geq 0} \frac{1}{4^n k!!}\int_0^{\pi/2} (\sin t)^{2k}(\cos t)^{2n+1} dt \\
&= \frac{4}{k!!}\int_0^1 \frac{x^{2k}}{3+x^2} dx \\
&= \frac{4}{3k!!}\sum_{r\geq 0}\int_0^1 x^{2k}(-x^2/3)^r dx \\
&= \frac{4}{3k!!}\sum_{r\geq 0} \frac{(-\frac{1}{3})^r}{2k+2r+1} \\
&= \frac{4}{3k!!}\sum_{s\geq k} \frac{(-3)^{k-s}}{2s+1} \\
&= \frac{4}{k!!}\left((-3)^k \frac{\pi}{6\sqrt{3}} + \sum_{s<k} \frac{(-3)^{k-s-1}}{2s+1}\right).
\end{aligned}$$

In the last step, we have used the arctan series to conclude that

$$\sum_{s\geq 0} \frac{(-3)^{-s}}{2s+1} = \sqrt{3}\,\frac{\pi}{6}.$$

Note that the summation and the integral above have been interchanged using uniform convergence.

For the other series, the proof is similarly given as follows.

$$\begin{aligned}\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)(2n+3)\cdots(2n+2k+1)\binom{2n}{n}} &= \sum_{n\geq 0}\frac{1}{(-4)^n k!!}\int_0^{\pi/2}(\sin t)^{2k}(\cos t)^{2n+1}dt \\ &= \frac{4}{k!!}\int_0^1 \frac{x^{2k}}{5-x^2}dx \\ &= \frac{4}{5k!!}\sum_{s\geq k}\frac{(\frac{1}{5})^{s-k}}{2s+1} \\ &= \frac{4}{k!!}\left(\frac{5^k}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right)-\sum_{s<k}\frac{5^{k-s-1}}{2s+1}\right)\end{aligned}$$

Here, we used the expansions of $\log(1\pm t)$ to conclude that

$$\sum_{s\geq 0}\frac{5^{-s}}{2s+1}=\frac{1}{\sqrt{5}}\log\frac{\sqrt{5}+1}{2}.$$

□

7 Some questions for further investigation

In view of the results we have obtained, some natural questions arise.

Question 1
Do the numbers $C_a(n)$ admit an identity generalizing Touchard's identity (B)?

Question 2
Is there a counting proof for the identity (A) for $C_a(n)$, such as that for (B) in [2]?

Question 3
Is there a proof by counting for the new identities (C) and (D) above?

Question 4
Are there analogues of the above lemmas for generalized Catalan numbers?

Acknowledgment

It is a pleasure to thank Professor Thomas Britz for suggestions to make the article easier and more enjoyable to read.

References

[1] C. Elsner, On recurrence formulae for sums involving binomial coefficients, *The Fibonacci Quarterly* **43** (2005), 31–45.

[2] W. Shapiro, A short proof of an identity of Touchard's concerning Catalan numbers, *Journal of Combinatorial Theory, Ser. A* **20** (1976), 375–376.

[3] R.P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[4] B. Sury, A curious polynomial identity, *Nieuw Archief voor Wiskunde (IV)* **11** (1993), 93–96.

[5] B. Sury, Generalized Catalan numbers: linear recursion and divisibility, *Journal of Integer Sequences* **12** (2009), Article 09.7.5.

[6] B. Sury, Tianming Wang and Feng-Zhen Zhao, Identities involving reciprocals of binomial coefficients, *Journal of Integer Sequences* **7** (2004), Article 04.2.8.