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B. Sury

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Lower Bound for the Least Common Multiple

B. Sury

Abstract. It is well known that the prime number theorem can be phrased as the statement that the least common multiple (lcm) of the first n natural numbers is asymptotic to the exponential of n. Suitable weaker bounds of this lcm already suffice to deduce certain striking properties of primes such as the existence of a prime between n and $2n$ for sufficiently large n. In this note we prove in an elementary manner that the lcm of the first n natural numbers is bigger than 2^n when n is bigger than 6.

It is well known that the prime number theorem can be phrased in terms of the least common multiple (abbreviated lcm) of numbers. Indeed, denoting $\log \operatorname{lcm}(1, 2, \ldots, n)$ by $\psi(n)$, the prime number theorem can be rephrased as the statement $\lim_{n\to\infty} \frac{\psi(n)}{n} = 1$. If we do not use the (deep) fact that the above limit exists, we can still prove estimates (for large enough n) such as

$$2^n < \operatorname{lcm}(1, 2, \ldots, n) < 4^n.$$

In fact, such estimates were obtained by Chebyshev from which facts such as Bertrand's postulate are deduced. From the lower bound above, one can deduce that $\pi(n) \geq \frac{n}{\log(n)} \log(2)$, where $\pi(x)$ is the prime counting function (see [**4**, p. 12]). In fact, Mohan Nair obtained a slick proof of the lower bound using a certain beta integral (loc. cit.). Our purpose is to give a different proof by combining the two observations below. We prove in an elementary manner that the lcm of the first n natural numbers is bigger than 2^n when n is bigger than 6. We must mention the following two references that were brought to the author's attention by the referees. A paper by Hanson [**2**] gives an elementary proof of the upper bound 3^n. Further, the book by Crandall and Pomerance (see [**1**, Exercise 1.28, pp. 55–56]) gives the steps for an elementary proof of the lower bound $\prod_{p \leq x} > 2^x$ for $x \geq 31$. (As a matter of fact, the steps allow us to deduce the result for $x \geq 2^{12}$, and the gap from 31 to 2^{12} has to be filled by direct calculation.)

Observation I. $\operatorname{lcm}(1, 2, \ldots, n) = \operatorname{lcm}\left(2\binom{n}{2}, 3\binom{n}{3}, \ldots, n\binom{n}{n}\right).$

Observation II. $\lceil n/2 \rceil \binom{n}{\lceil n/2 \rceil} > 2^n$ *if* $n > 6$, *where* $\lceil x \rceil$ *denotes the smallest integer not less than* x.

The two observations evidently yield the lower bound $\operatorname{lcm}(1, 2, \ldots, n) > 2^n$ for $n > 6$.

Proof of Observation II. This follows easily by induction on $n > 6$. For $n = 7$, we have $4\binom{7}{4} = 140 > 2^7$. We assert

$$(n+1)\binom{2n+2}{n+1} = 2(n+1)\binom{2n+1}{n+1},$$

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$$(n+2)\binom{2n+3}{n+2} > 4(n+1)\binom{2n+1}{n+1}.$$

These are easy to see and imply that the assumption $(n+1)\binom{2n+1}{n+1} > 2^{2n+1}$ leads to the conclusion $\lceil n/2 \rceil \binom{n}{\lceil n/2 \rceil} > 2^n$ for $n > 6$. Hence, this observation follows. ■

Now we prove Observation I which is slightly harder. Nair gives a proof using a beta integral (see [**4**, p. 12]). We make use of another little observation that interprets the power of a prime dividing the lcm being considered:

For a natural number n, if p^a is the highest power of a prime p dividing lcm$(1, 2, \ldots, n)$*, then $p^a \leq n < p^{a+1}$. In other words, $a+1$ is the number of digits of n when written in base p.*

To prove this, consider any prime p dividing lcm$(1, 2, \ldots, n)$; then $p \leq n$. If a is the largest integer so that $p^a \leq n$, then p^a evidently divides lcm$(1, 2, \ldots, p^a, \ldots, n)$. As the power of p dividing lcm$(1, 2, \ldots, n)$ is the maximum of the powers of p dividing the numbers $1, 2, \ldots, n$, it follows that p^{a+1} does not divide lcm$(1, 2, \ldots, n)$ as $n < p^{a+1}$. Thus, $p^a \leq n < p^{a+1}$ clearly implies that the number of digits of n written in base p is $a+1$.

Proof of Observation I. First, it is evident that left-hand side is at most equal to the right-hand side because each of $2, 3, \ldots, n$ divides the numbers on the right-hand side whose least common multiple is being considered.

To prove the other inequality, we will prove that the power of p dividing $r\binom{n}{r}$ for any $0 < r < n$ is less than the number $a+1$ of digits of n in base p (and, hence, is at most a). This will imply our assertion. We use the Kummer–Legendre formula asserting that the power of p dividing a binomial coefficient $\binom{n}{r}$ $(0 < r < n)$ is the number of carry-overs while adding r and $n-r$ written in base p (see [**3**, pp. 229–233]).

Write

$$r = * * \cdots * 0 \cdots 0$$

in base p where there are precisely $u \geq 0$ zeros at the end.

Next, observe that if $n = r + (n-r)$ in base p and n has $a+1$ digits in base p, then at most a of those digits incorporate a carry, since the top digit does not incorporate a carry. As r ends in precisely u zeros in base p, those u places do not propagate carries, and the first digit of n that includes a carry from earlier places is place $u+1$ or later. Thus, the number of carries is at most $a-u$. So, the power of p in $r\binom{n}{r}$ is at most $u + (a-u) = a$, and the proof is complete. ■

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Statistics & Mathematics Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore 560059, India.
surybang@gmail.com

Primes in (0, *n*] vs. (*n*, *m*]

The prime number theorem gives us an approximate distribution of primes, but what can be said about the relative number of primes in the intervals $(0, n]$ and $(n, m]$, for natural numbers $m > n$? The case $m = 2n \gg 0$ was known clasically, but recent advances allow for a sharper result.

Suppose that $m > n \geq 5393$. These nice bounds for the prime counting function π were proved in [1]:

$$\frac{n}{\log n - 1} < \pi(n) \leq \pi(m) < \frac{m}{\log m - 1.112}.$$

Theorem 1. *Suppose that* $m > n \geq 5393$. *Suppose further that* $\frac{m}{n} > e^{0.112} \approx 1.12$. *Then*

$$\frac{\pi(n)}{n} = \frac{\pi(n) - \pi(0)}{n - 0} > \frac{\pi(m) - \pi(n)}{m - n}.$$

Proof. $\frac{m}{n}\pi(n) - \pi(m) > \frac{m}{n}\frac{n}{\log n - 1} - \frac{m}{\log m - 1.112} = m\left(\frac{1}{\log n - 1} - \frac{1}{\log m - 1.112}\right)$. This is positive, since $\frac{m}{n} > e^{0.112}$ and thus $\log n - 1 < \log m - 1.112$. We now rearrange $\frac{m}{n}\pi(n) - \pi(m) > 0$ into the desired statement. ■

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