# Klassen Theorie

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*"The way I first visualized a K-group was as a group of classes of objects of an abelian (or more generally, additive) category, such as coherent sheaves on an algebraic variety, or vector bundles, etc. I would presumably have called this group C(X) (X being a variety or any other kind of space), C the initial letter of class, but my past in functional analysis may have prevented this, as C(X) designates also the space of continuous functions on X (when X is a topological space). Thus, I reverted to K instead of C, since my mother tongue is German, Class = Klasse (in German), and the sounds corresponding to C and K are the same"* .......................**Alexander Grothendieck**.

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# Introduction

In 1957, Grothendieck (handwritten notes) formulated the "*Grothendieck-Riemann-Roch theorem*". In 1958, there was an exposition of this by Borel & Serre. In 1959, Bott proved his periodicity theorems on stable homotopy groups. His proof involved Riemannian geometry and Morse theory. In 1961, Atiyah & Hirzebruch realized that Bott periodicity is related to a new K-theory (topologial version) analogous to Grothendieck's algebraic K-theory. They developed the topological theory which followed a similar pattern. A new feature was that they could define derived functors $K^{-n}$ and Bott periodicity simplified to the statements

$$K_{\mathbf{C}}^{-n} \cong K_{\mathbf{C}}^{-n-2} \ , \ K_{\mathbf{R}}^{-n} \cong K_{\mathbf{R}}^{-n-8}.$$

This K-theory is an extraordinary cohomology theory (satisfies all the axioms of cohomology theories save the dimension axiom).

It was observed by Swan and others that the category of vector bundles over a compact Hausdorff space $X$ is equivalent to the category of finitely generated projective modules over the ring of continuous functions on $X$. Thus began the study of K-theory of rings.

In these lectures, we will prove complex Bott periodicity (BP) and derive some applications. The applications of BP include Brouwer fixed point theorem, Adams's theorem that the only spheres $S^{n-1}$ which are $H$-spaces (hence, the values of $n$ for which the sphere $S^{n-1}$ has trivial tangent bundle or, equivalently, for which $\mathbf{R}^n$ has a bilinear multiplication with respect to which it forms a division algebra) correspond to $n = 1, 2, 4, 8$.

The real and complex Bott periodicity theorems have the following consequence on the homotopy groups of the infinite unitary, orthogonal and symplectic groups $U, O, Sp$ respectively.

$$\pi_{i+2}(U) \cong \pi(U) \ , \ \pi_{i+8}(O) \cong \pi_i(O) \ , \ \pi_{i+8}(Sp) \cong \pi_i(Sp).$$

$$\pi_0(U) = 0, \pi_1(U) \cong \mathbf{Z};$$

$$\pi_0(O) \cong \mathbf{Z}/2\mathbf{Z}, \pi_1(O) \cong \mathbf{Z}/2\mathbf{Z}, \pi_2(O) = 0, \pi_3(O) \cong \mathbf{Z},$$

$$\pi_4(O) = \pi_5(O) = \pi_6(O) = 0, \pi_7(O) \cong \mathbf{Z};$$

$$\pi_0(Sp) = \pi_1(Sp) = \pi_2(Sp) = 0, \pi_3(Sp) \cong \mathbf{Z},$$

$$\pi_4(Sp) \cong \mathbf{Z}/2\mathbf{Z}, \pi_5(Sp) \cong \mathbf{Z}/2\mathbf{Z}, \pi_6(Sp) = 0, \pi_7(Sp) \cong \mathbf{Z}.$$

# 1 Recollection of facts on vector bundles

• $F$ will denote either $\mathbf{R}$ or $\mathbf{C}$.
• We have inclusions

$$F^1 \subset F^2 \subset \cdots \subset F^n \subset \cdots$$

where $F^n = \{x \in F^{n+1} : x_{n+1} = 0\}$.
• $F^\infty := \bigcup_{n\geq 1} F^n = \{(x_1, x_2, x_3, \cdots) : x_r = 0$ for all but finitely many $r\}$; it has the weak/direct limit topology - a function $f : F^\infty \to Y$ is continuous if and only if, $f|F^n$ is continuous.
• $Vect^n(B)$ denotes isomorphism classes of vector bundle of rank $n$ over $B$.

• *For a compact Hausdorff space $B$, every vector bundle $E \to B$ has a complement; that is, a corresponding vector bundle $E' \to B$ such that $E \oplus E'$ is a trivial bundle.*

• *If $p : E \to B$ is a vector bundle, and $f_0, f_1 : A \to B$ are homotopic maps from a paracompact space, then the induced vector bundles $f_0^*(B), f_1^*(B)$ over $A$ are isomorphic.*
Consequently,
• *If $\theta : A \to B$ is a homotopy equivalence of paracompact spaces, then the induced vector bundle map $\theta^* : Vect^n(B) \to Vect^n(A)$ is a bijection.*

• *For $n \leq N$ (including infinite $N$), the space*

$$E_n(F^N) := \{(V, v) \in G_n(F^N) \times F^N : v \in V\}$$

*is a vector bundle over $G_n(F^N)$ via the projection map $E_n(F^N) \to G_n(F^N)$; $(V, v) \mapsto V$. In the case when $N$ is infinite, we write only $G_n, E_n$.*
Consequently,
• *If $B$ is paracompact, then every vector bundle of rank $n$ over $B$ can be obtained as the pull-back $f^*(E_n)$ where $f : B \to G_n$. Moreover, homotopic maps $f_0, f_1 : B \to G_n$ gives us isomorphic vector bundles and vice versa.*

# 2 Clutching construction

• *Clutching construction*
Let $X = X_1 \cup X_2$ be a compact, Hausdorff space where $X_1, X_2$ are closed

subspaces and, suppose $A = X_1 \cap X_2$. Given vector bundles $p_i : E_i \to X_i$ $(i = 1, 2)$ and an isomorphism

$$\phi : E_1|A \overset{\cong}{\to} E_2|A$$

there is a vector bundle $p : E_1 \cup_\phi E_2 \to X$ defined as follows:
• the space $E_1 \cup_\phi E_2 = (E_1 \sqcup E_2)/\sim$ where $v \sim \phi(v)$ for all $v \in E_1|A$; and
• $p$ is induced by $E_1 \to X_1 \hookrightarrow X$ and $E_2 \to X_2 \hookrightarrow X$.
This is called a clutching construction.

**Proof.**

We need to show that the bundle is locally trivial. Now,

$$(E_1 \cup_\phi E_2)|(X \setminus A) \cong E_1|(X \setminus A) \sqcup E_2|(X \setminus A)$$

which implies local triviality holds around points of $X \setminus A$. So, let $a \in A$. Consider neighbourhoods $U_1, U_2$ of $a$ in $X_1, X_2$ respectively, such that

$$h_i : E_i|U_i \overset{\cong}{\to} U_i \times F^n.$$

We will change $h_2$ so as to coincide with $h_1$ near $a$. Firstly, let $V \subset A \cap U_1 \cap U_2$ be a neighbourhood of $a$. Over the closure $\overline{V}$, we have

$$\overline{V} \times F^n \overset{h_2^{-1}}{\to} E_2|\overline{V} \overset{\phi^{-1}}{\to} E_1|\overline{V} \overset{h_1}{\to} \overline{V} \times F^n.$$

The composite map is of the form $(x, v) \mapsto (x, g(x)v)$ for some continuous map $g : \overline{V} \to GL_n(F)$.
Viewing $GL_n(F) \subset F^{n^2}$, an open subset, and applying the Tietze extension theorem, we get an extension of $g$ to a map $\bar{g}$ from $X_2$ to $F^{n^2}$. There exists a neighbourhood $W$ of $\overline{V}$ in $X_2$ such that the restriction of $\bar{g}$ to $W$ takes values in the open subset $GL_n(F)$. Choose small neighbourhoods $V_1 \subset U_1$ and $V_2 \subset U_2 \cap W$ such that $V_1 \cap A = V_2 \cap A = V$. Then, consider the trivializations

$$E_1|V_1 \overset{h_1}{\to} V_1 \times F^n;$$

$$E_2|V_2 \overset{h_2}{\to} V_2 \times F^n.$$

Compose the second one with the isomorphism

$$V_2 \times F^n \to V_2 \times F^n; (x, v) \mapsto (x, \bar{g}(x)v).$$

This matches with the first trivialization under $\phi$ to give a trivialization

$$(E_1 \cup_\phi E_2)|V \overset{\cong}{\to} V \times F^n.$$

• *In the case when $E_1, E_2$ above are trivial vector bundles of rank $n$, then giving an isomorphism between $E_1|A$ and $E_2|A$ is equivalent to giving a map from $A$ to $GL_n(F)$ - this is called a clutching function.*

• *With $X, X_1, X_2, A, E_1, E_2$ as above, let $\phi_t : E_1|A \overset{\cong}{\to} E_2|A$ be isomorphisms for $t \in [0,1]$. Then, $E_1 \cup_{\phi_0} E_2 \cong E_2 \cup_{\phi_1} E_2$.*
Consequently,
*Clutching constructions associated to homotopic maps from $A$ to $GL_n(F)$ are isomorphic.*
**Proof.**
The isomorphisms

$$\Phi : (E_1 \times [0,1])|(A \times [0,1]) \overset{\cong}{\to} E_2 \times [0,1])|(A \times [0,1]);$$

$$(v,t) \mapsto (\phi_t(v), t)$$

and the clutching constructions for $E_i \times [0,1] \to X_i \times [0,1]$ give rise to a vector bundle

$$(E_1 \times [0,1]) \cup_\Phi (E_2 \times [0,1]) \to X \times [0,1]$$

such that $E_1 \cup_{\phi_t} E_2$ is its pullback under the map

$$i_t : X \to X \times [0,1]; x \mapsto (x,t).$$

In particular, since $i_0, i_1$ are homotopic, the proof is complete.

**Key proposition on clutching constructions.**
(i) $(E_1 \cup_\phi E_2)|X_i = E_i$ *($i = 1,2$)*;
(ii) *If $E \to X$ is a vector bundle, and $E|X_i = E_i$ ($i = 1,2$), then $E$ is isomorphic to $E_1 \cup_{id} E_2$, where id is the identity isomorphism from $E|A$ to itself*;
(iii) *If $\beta_i : E_i \to E_i'$ ($i = 1,2$) are isomorphisms and $\phi : E_1|A \to E_2|A, \phi' : E_1'|A \to E_2'|A$ are isomorphisms satisfying $\phi' \circ \beta_1 = \beta_2 \circ \phi$, then $(E_1 \cup_\phi E_2) \cong (E_1' \cup_{\phi'} E_2')$;*

(iv) *Given clutching data* $(E_1, E_2, \phi)$ *and* $(E_1', E_2', \phi')$*, we have:*
(a) $(E_1 \cup_\phi E_2) \oplus (E_1' \cup_{\phi'} E_2') \cong ((E_1 \oplus E_1') \cup_{\phi\oplus\phi'} (E_2 \oplus E_2'))$;
(b) $(E_1 \cup_\phi E_2) \otimes (E_1' \cup_{\phi'} E_2') \cong ((E_1 \otimes E_1') \cup_{\phi\otimes\phi'} (E_2 \otimes E_2'))$;
(c) $(E_1 \cup_\phi E_2)^* \cong E_1^* \cup_{(\phi^*)^{-1}} E_2^*$.

**Special case when $E_i$ are trivial**

*The assertions (a),(b),(c) above reduce in the case of trivial vector bundles* $E_i$ *(write* $\epsilon_i^r$ *for the rank* $r$ *trivial bundle over* $X_i$*) to the following statements: Let* $\phi : A \to GL_m(F), \psi : A \to GL_n(F)$*. Then,*
(a') $(\epsilon_1^m \cup_\phi \epsilon_2^m) \oplus (\epsilon_1^n \cup_\psi \epsilon_2^n) \cong (\epsilon_1^{m+n} \cup_{\phi\oplus\psi} \epsilon_2^{m+n})$; *where*

$$\phi \oplus \psi : A \to GL_{m+n}(F);$$

$$a \mapsto \begin{pmatrix} \phi(a) & 0 \\ 0 & \psi(a) \end{pmatrix}.$$

(b') *Identifying* $F^m \otimes F^n$ *with* $F^{mn}$*, we have* $(\epsilon_1^m \cup_\phi \epsilon_2^m) \otimes (\epsilon_1^n \cup_\psi \epsilon_2^n) \cong (\epsilon_1^{mn} \cup_{\phi\otimes\psi} \epsilon_2^{mn})$; *where* $\phi \otimes \psi : A \to GL_{mn}(F)$ *is the map that sends* $a$ *to the matrix of the map*

$$F^{mn} \cong F^m \otimes F^n \overset{\phi(a)\otimes\psi(b)}{\to} F^m \otimes F^n \cong F^{mn}.$$

(c') $(\epsilon_1^m \cup_\phi \epsilon_2^m)^* \cong \epsilon_1^m \cup_{(\phi^t)^{-1}} \epsilon_2^m$ *where*

$$(\phi^t)^{-1} : A \to GL_{mn}(F);$$

$$a \mapsto (\phi(a)^t)^{-1}.$$

**Important example.**
We consider the canonical line bundle $\gamma^1$ on $X = \mathbf{C}P^1$ and its dual $H = (\gamma^1)^*$. An alternative way of looking at $\mathbf{C}P^1$ is to identify it with $\mathbf{C} \cup \{\infty\} = S^2$ by means of the homeomorphism $[z_1 : z_2] \mapsto z_1/z_2$. Consider the subspaces

$$D_0 = \{z \in S^2 : |z| \leq 1\};$$

$$D_\infty = \{z \in S^2 : |z| \geq 1\}.$$

Both $D_0, D_\infty$ are homeomorphic to the closed unit disc which is contractible and their intersection is the equatorial great circle $S^1 = \{z \in S^2 : |z| = 1\}$. We have the trivializations of $\gamma^1$ over $D_0$ and $D_\infty$ as:

$$\gamma^1|D_0 \overset{h_0}{\to} D_0 \times \mathbf{C};$$

$$(z, \lambda(z,1)) \mapsto (z, \lambda),$$

$$\gamma^1|D_\infty \overset{h_\infty}{\to} D_\infty \times \mathbf{C};$$

$$(z, \lambda(1, z^{-1})) \mapsto (z, \lambda).$$

The composite $h_\infty h_0^{-1}|S^1$ of the two isomorphisms $h_0^{-1}|S^1 : S^1 \times \mathbf{C} \to \gamma^1|S^1$ and $h_\infty|S^1 : S^1 \times \mathbf{C} \to \gamma^1|S^1$ is

$$(z, \lambda) \mapsto (z, \lambda z).$$

Calling the trivial line bundles $D_0 \times \mathbf{C}$ and $D_\infty \times \mathbf{C}$ as $\epsilon_0$ and $\epsilon_\infty$ respectively, we have

$$\gamma^1 \cong \epsilon_0 \cup_\phi \epsilon_\infty$$

where $\phi : S^1 \to GL_1(\mathbf{C})$ is the inclusion map $z \mapsto z$.
Let us apply the observation (c') above to obtain

$$H \cong \epsilon_0 \cup_\psi \epsilon_\infty$$

where $\psi : S^1 \to GL_1(\mathbf{C})$ is the map $z \mapsto z^{-1}$.
Again, applying observation (a'), we have the direct sum

$$H \oplus H \cong \epsilon_0 \cup_{\psi_\oplus} \epsilon_\infty$$

where $\psi_\oplus : S^1 \to GL_2(\mathbf{C})$ is

$$z \mapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

Applying observation (b'), we have the tensor product

$$H \otimes H \cong \epsilon_0 \cup_{\psi_\otimes} \epsilon_\infty$$

where $\psi_\otimes : S^1 \to GL_1(\mathbf{C})$ is the map $z \mapsto z^{-2}$. Hence, denoting by $\epsilon^1$, the trivial line bundle over $D_0 \cup D_\infty$, we have

$$(H \otimes H) \oplus \epsilon^1 \cong \epsilon_0 \cup_{\psi_{\otimes,1}} \epsilon_\infty$$

where the clutching function $\psi_{\otimes,1} : S^1 \to GL_2(\mathbf{C}$ is

$$z \mapsto \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

### 2.1 $H \oplus H \cong (H \otimes H) \oplus \epsilon^1$

We use the fact that $GL_2(\mathbf{C})$ is path-connected to deduce the above isomorphism. Indeed, consider a path $\alpha$ from the identity matrix to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, the map

$$S^1 \times [0,1] \to GL_2(\mathbf{C});$$
$$(z,t) \mapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \alpha(t) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \alpha(t)$$

is a homotopy from $\psi_{\oplus}$ to $\psi_{\otimes,1}$. This induces and isomorphism of vector bundles

$$H \oplus H \cong (H \otimes H) \oplus \epsilon^1.$$

• Note that we have used the connectedness of $GL_2(\mathbf{C})$. As $GL_2(\mathbf{R})$ has two connected components, the above proof does not carry over to reals.

## 3 Cone and suspension

For a topological space $X$, the unreduced cone over $X$ is defined as

$$CX = (X \times [0,1])/(X \times \{1\}).$$

One may identify $X$ naturally as he subspace $X \times \{0\}$ of $CX$.
• *$CX$ is contractible.*
Indeed, the map $CX \times [0,1] \to CX$ defined as

$$([(x,s)],t) \mapsto [(x, s+t-st)]$$

shows that $CX$ deformation retracts to the vertex of the cone.

The suspension of $X$ is the quotient space $SX = CX/(X \times \{0\})$. One usually identifies $X$ with the subspace $X \times \{1/2\}$ of $SX$. The subspaces
$C_+X$ = image of $X \times [1/2, 1]$ in $SX$, and
$C_-X$ = image of $X \times [0, 1/2]$ in $SX$
are homeomorphic to $CX$. Their union is $SX$ and the intersection is $X$.
Note that $CS^{n-1}$ is homeomorphic to $D^n = \{x \in \mathbf{R}^{n+1} : ||x|| \leq 1\}$ via $[x,t] \mapsto (1-t)x$, and $SS^n$ is homeomorphic to $S^{n+1}$.

As $C_+X, C_-X$ are contractible, vector bundles over $SX$ restrict to trivial bundles over $C_+X$ and $C_-X$. So, we may apply the clutching construction to $SX$ when $X$ is compact. Firstly, we look at the space $[X, GL_n(F)]$ of homotopy classes of maps from $X$ to $GL_n(F)$. On this space, we consider the orbits under the action of the group $\pi_0 GL_n(F) \times \pi_0 GL_n(F)$. The action is

$$([a],[b])(\phi) = [a\phi b^{-1}].$$

The group $\pi_0 GL_n(F) \times \pi_0 GL_n(F)$ is trivial if $F = \mathbf{C}$ and is $\mathbf{Z}/2 \times \mathbf{Z}/2$ if $F = \mathbf{R}$. With these notations, we have:

**Theorem.** *Let $X$ be compact and Hausdorff. Then, the clutching construction provides a bijection*

$$\Phi : [X, GL_n(F)]/(\pi_0 GL_n(F) \times \pi_0 GL_n(F)) \stackrel{\cong}{\to} Vect_F^n(SX);$$

$$[\phi] \mapsto [\epsilon_+^n \cup_\phi \epsilon_-^n].$$

Here, $\epsilon_+^n, \epsilon_-^n$ are the trivial vector bundles of rank $n$ over $C_+X$ and $C_-X$ respectively.

**Proof.**

Let us construct a map

$$\Psi : Vect_F^n(SX) \to [X, GL_n(F)]/(\pi_0 GL_n(F) \times \pi_0 GL_n(F))$$

which would be the inverse of $\Phi$. To construct $\Psi$, start with any $[E] \in Vect_F^n(SX)$. Note that $C_+$ and $C_-$ are contractible which gives trivializations

$$h_+ : E|C_+ \stackrel{\cong}{\to} \epsilon_+^n;$$

$$h_- : E|C_- \stackrel{\cong}{\to} \epsilon_-^n.$$

The composite

$$h_- h_+^{-1} : \epsilon_+^n|X \to E|X \to \epsilon_-^n|X$$

is of the form

$$(x,v) \mapsto (x, \phi(h_+,h_-)(x)v)$$

where $\phi(h_+,h_-) : X \to GL_n(F)$. Now, we may define

$$\Psi([E]) = [\phi(h_+,h_-)].$$

To show this definition is independent of the trivializations $h_\pm$, let $h'_\pm$ be other choices of trivializations. Then, $h'_\pm = d_\pm \circ h_\pm$ for some automorphisms

$$d_\pm : \epsilon^n_\pm \xrightarrow{\cong} \epsilon^n_\pm.$$

Corresponding to $d_\pm$, we have maps

$$\delta_\pm : C_\pm \to GL_n(F).$$

Observe that for each $x \in X$, we have

$$\phi(h'_+, h'_-)(x) = \delta_-(x)\phi(h_+, h_-)\delta_+(x)^{-1}.$$

Since $C_\pm X$ are contractible, the maps $\delta_\pm$ are homotopic to a constant; suppose they map onto the elements $g_\pm \in GL_n(F)$. Hence, $\phi(h'_+, h'_-)$ is homotopic to the map

$$X \to GL_n(F); x \mapsto g_-\phi(h_+, h_-)g_+^{-1}.$$

Hence, we have independence of $[\phi(h_+, h_-)]$ with respect to choices of $h_\pm$. Let us now show that if $E$ is replaced by an isomorphic $E'$, then this class above remains the same. Indeed, for an isomorphism

$$\alpha : E' \to E$$

the maps $h_\pm \circ \alpha$ are trivializations of $E'|C_\pm X$ such that

$$\phi(h_+ \circ \alpha, h_- \circ \alpha) = \phi(h_+, h_-).$$

Hence, $\Psi$ is well-defined.
Now, $\Psi\Phi([\phi]) = \Psi([\epsilon^n_+ \cup_\phi \epsilon^n_-]) = [\phi]$ for all $\phi$. Conversely, by construction, for any $E$, the class $[\phi] := \Psi([E])$ satisfies

$$E \cong \epsilon^n_+ \cup_\phi \epsilon^n_-.$$

So, $\Phi\Psi([E]) = \Phi([\phi]) = [\epsilon^n_+ \cup_\phi \epsilon^n_-] = [E]$.

**Important variation.**
One can define a weaker notion of vector bundles where the fibres may not have the same dimension. Many properties of vector bundles in the earlier sense carry over. Henceforth, we will consider the new notion. Note that if $X$

is connected, then the new notion reduces to the old notion. Let $Vect_F(X)$ denote the isomorphism classes of vector bundles over a paracompact space $X$. Under the direct sum operation, $Vect_F(X)$ is a commutative monoid. For instance, if $X$ is a point, $Vect_F(X)$ can be identified via the dimension function with the monoid $\mathbf{Z}_{\geq 0}$ under addition. To get a group, one needs negatives of the positive integers. This process can be imitated for a general commutaive monoid. We will carry this out for $Vect_F(X)$ for a general paracompact space in the next section.

# 4 Defining K-group as Grothendieck group

Given a commutative monoid $(M, +)$, define an equivalence relation on $M \times M$ by:

$$(x, y) \sim (x', y') \Leftrightarrow \exists m \in M \textit{ such that } x + y' + m = y + x' + m.$$

The equivalence classes $(M \times M)/\sim$ form the Grothendieck group $Gr(M)$ of $M$; the group operation

$$[(x, y)] + [(x', y')] = [(x + x', y + y']$$

is well-defined. One may think of the class of $(x, y)$ as the formal difference $x - y$. Further, the map $x \mapsto ([(x, 0)]$ gives a (not necessarily 1-1) monoid homomorphism from $M$ to $Gr(M)$. One writes informally $[m]$ instead of $[(m, 0)]$.

**Lemma.** *$Gr(M)$ has the universal property with respect to monoid homomorphisms from $M$ to abelian groups.*

**Definition.** For a compact Hausdorff space $X$, one defines $K_F(X)$ to be $Gr(Vect_F(X))$. For a vector bundle $E \to X$, we write $[E]$ for the class of $(E, 0)$ in $K_F(X)$.

**Lemma.** $[(E_1, E_2)] = [E_1] - [E_2]$ *in $K_F(X)$. In particular, every elsment of $K_F(X)$ can be written as $[E] - [\epsilon^n]$ for some vector bundle $E \to X$ and some $n \geq 0$.*

**Proof.** The first statement clearly implies that every element of $K_F(X)$ is expressible as $[E_1] - [E_2]$. Now, since $X$ is compact, every vector bundle has a complement; that is, there exists a vector bundle $E_2'$ such that $E_2 \oplus E_2' = \epsilon^n$ for some $n$. Hence, $[E_1] - [E_2] = [E_1 + E_2'] - [\epsilon^n]$.

**Definition.** Two vector bundles $E_1, E_2$ are stably isomorphic if there exists $n \geq 0$ such that $E_1 \oplus \epsilon^n \cong E_2 \oplus \epsilon^n$. We write $E_1 \cong_s E_2$

**Lemma.** *$E_1 \oplus E_2' \cong_s E_2 \oplus E_1'$ if and only if, $[E_1] - [E_1'] = [E_2] - [E_2']$ in $K_F(X)$.*

**Proof.** Now, in $K_F(X)$,

$$[E_1] - [E_1'] = [E_2] - [E_2']$$

$$\Leftrightarrow [(E_1, E_1')] = [(E_2, E_2')]$$

$$\Leftrightarrow E_1 \oplus E_2' \oplus E \cong E_1' \oplus E_2 \oplus E$$

for some $E$. This happens if and only if

$$E_1 \oplus E_2' \oplus \epsilon^n \cong E_1' \oplus E_2 \oplus \epsilon^n$$

because $E$ has a complement. That is, it happens if, and only if, $E_1 \oplus E_2' \cong_s E_1' \oplus E_2$.

**Remark.** In other words, one may think of $K_F(X)$ as classes of vector bundles over $X$ where two bundles are identified if they are stably isomorphic.

**Lemma.** *For a compact Hausdorff space $X$, $K_F(X)$ becomes a commutative ring under the multiplication*

$$[(E_1, E_2)][(E_1', E_2')] := [((E_1 \otimes E_1') \oplus (E_2 \otimes E_2'), (E_1 \otimes E_2') \oplus (E_1' \otimes E_2))].$$

*In particular,*

$$[E_1][E_1'] = [E_1 \otimes E_1'].$$

**Definition.** For a continuous map $f : X \to Y$ of compact, Hausdorff spaces, we have a ring homomorphism

$$f^* : K_F(Y) \to K_F(X); [E] \mapsto [f^*E].$$

This map $f^*$ is called the induced map.

**Lemma.** *If $f, g : X \to Y$ are homotopic, then $f^* = g^*$ on $K_F(Y)$.*

**Proof.** This is already true at the level of vector bundles.

**Proposition.** *Let $X, Y$ be compact, Hausdorff spaces and let $X \sqcup Y$ denote their disjoint union. If $i, j$ are the inclusions of $X, Y$ in $X \sqcup Y$ respectively, there is a ring isomorphism*

$$K_F(X \sqcup Y) \stackrel{(i^*,j^*)}{\rightarrow} K_F(X) \times K_F(Y).$$

**Proof.** This is an isomorphism of semirings at the level of vector bundles.

# 5 Pairs and pointed spaces

Instead of just $X$, one often considers pairs $(X, A)$, where $A$ is a fixed subspace of $X$. For pairs $(X, A), (Y, B)$ one denotes $(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y)$. Then, one has the following natural notions:
(i) one writes $f : (X, A) \to (Y, B)$, a continuous map of pairs, if $f : X \to Y$ is continuous and $f(A) \subset B$;
(ii) given $f, g$ maps of pairs as above, a homotopy from $f$ to $g$ as maps of pairs, is a continuous map $h : X \times [0, 1] \to Y$ such that $h(-, 0) = f$, $h(-, 1) = g$, $h(A \times [0, 1]) \subset B$.
When $A, B$ are points, pairs and maps of pairs etc. are called pointed spaces and maps of pointed spaces.
In particular, a base-point preserving homotopy from $f : (X, x_0) \to (Y, y_0)$ to $g : (X, x_0) \to (Y, y_0)$ is a usual homotopy $h$ from $f$ to $g$ such that $h(x_0, t) = y_0$ for all $t \in [0, 1]$.

**Definition.** If $X$ is a space, then $X_+$ is the pointed space $(X \sqcup pt, pt)$; here, a disjoint base point is added.

**Lemma.** *A map $f : (X, A) \to (Y, B)$ induces a map of pointed spaces $(X/A, A/A)$ to $(Y/B, B/B)$. Here, we allow $A$ or $B$ to be empty also; so, $(X, \emptyset)$ is to be interpreted as $X_+$.*

# 6 Reduced and relative K-theories

**Definition.** Given a (compact, Hausdorff) pointed space $(X, x_0)$, the reduced K-group is defined as

$$\widetilde{K_F(X)} := Ker(K_F(X) \stackrel{i^*}{\rightarrow} K_F(\{x_0\})$$

where $i : \{x_0\} \to X$ is the inclusion map. Note that $K_F(pt) \cong \mathbf{Z}$. The reduced K-group is also a ring but has no unity.

Since the constant map $r : X \to \{x_0\}$ satisfies $i^*r^* = id_{K_{pt}}$, we have:

**Splitting lemma.** *For a pointed space $(X, x_0)$, we have an isomorphism*

$$K_F(X) \cong \widetilde{K_F}(X) \oplus \mathbf{Z}.$$

**Definition.** Let $X$ be compact, Hausdorff and $A$ be a closed subspace. Then, the relative K-group $K_F(X, A)$ is defined to be $\widetilde{K_F}(X/A)$.

**Excision theorem.** *Let $(X, A)$ be a compact pair, and let $U \subset A$ be open. Then, the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism*

$$K_F(X, A) \stackrel{\cong}{\to} K_F(X - U, A - U).$$

**Proof.** This is simply from the homeomorphism from $(X - U)/(A - U)$ to $X/A$.

**Definition.** Two vector bundles $E_1, E_2$ over a compact, Hausdorff spaces are *stably equivalent* (written $E_1 \sim_s E_2$) if, there exist $m, n \geq 0$ such thar $E_1 \oplus \epsilon^m \cong E_2 \oplus \epsilon^n$. This is an equivalence relation and the stable equivalence class of $E$ is written as $[E]_s$.

**Proposition.** *If $(X, x_0)$ is a compact, $T_2$ pointed space, then the map*

$$\phi : Vect_F(X)/\sim_s \to \widetilde{K_F}(X);$$

$$[E]_s \mapsto [E] - [\epsilon^{dim(E_{x_0})}]$$

*is an isomorphism of abelian groups.*
**Proof.**
$\phi$ is surjective by the lemma on the bottom of page 10 and is injective by the lemma on top of page 11.

# 7 Some constructions on pointed spaces

**Reduced cone.**
For a pointed space $(X, x_0)$, the (base-pointed) quotient $(X \times [0,1])/(X \times$

$\{1\}) \cup (\{x_o\} \times [0,1])$ is called the reduced cone on $X$ at $x_0$, and is denoted by $C(X, x_0)$. As usual, $X$ can be identified with a subspace of the reduced cone via $x \mapsto [(x, 0)]$. We sometimes use the notation $CX$ for the reduced cone also.

**Lemma.** *Given $(X, x_0)$, the base-point of $C(X, x_0)$ (which is the subspace $X \times \{1\}$ collapsed to a point) is a deformation retract of the reduced cone $C(X, x_0)$.*
**Proof.** The map $C(X, x_0) \times [0,1] \to C(X, x_0)$ given by

$$([x, s], t) \mapsto [x, (1-t)s + t]$$

is such a deformation retraction.

If $X$ is any compact space, the reduced cone made with the pointed space $X_+$ is homeomorphic to the unreduced cone $CX$.

**Reduced suspension.**
For a pointed space $(X, x_0)$, the (base-pointed) quotient space $C(X, x_0)/(X \times \{0\})$ is called the reduced suspension of $X$ at $x_0$ and is denoted by $\Sigma(X, x_0)$. Equivalently, $\Sigma(X, x_0)$ is the quotient of the unreduced suspension by the subspace $\{x_0\} \times [0, 1]$. One often thinks of $X$ as the "equatorial" subspace $X \times \{1/2\}$ of $\Sigma(X, x_0)$. The reason we use $\Sigma$ instead of $S$ is we will be talking about iterated suspensions and do not want that to be confused with the spheres.

**The wedge sum.**
For pointed spaces $(X, x_0), (Y, y_0)$, their wedge sum $(X, x_0) \vee (Y, y_0)$ is their categorical co-product in the category of pointed spaces and pointed maps. In other words, there are "inclusion maps" $i_X, i_Y$ of $(X, x_0)$ and of $(Y, y_0)$ in the wedge sum so that, for any $(Z, z_0)$, and any

$$f : (X, x_0) \to (Z, z_0), g : Y, y_0) \to (Z, z_0)$$

there is a unique map $(f, g) : (X, x_0) \vee (Y, y_0) \to (Z, z_0)$ satisfying $(f, g) \circ i_X = f, (f, g) \circ i_Y = g$.
One may construct the wedge sum up to a unique isomorphism as

$$(X, x_0) \vee (Y, y_0) = \{(x, y) \in X \times Y : x = x_0 \ \ or \ \ y = y_0\}.$$

**Smash product.**
For pointed spaces $(X, x_0), (Y, y_0)$, the smash product is the (base-pointed)

quotient space $(X, x_0) \wedge (Y, y_0) := (X \times Y)/((X, x_0) \vee (Y, y_0))$. Sometimes, we simply write $X \vee Y$ and $X \wedge Y$ without specifically mentioning the base points.

**Examples.**

• (i) $C(X, x_0) = (X, x_0) \wedge ([0, 1], 1)$.
• (ii) $(X, x_0) \wedge (S^0, 1) \cong (X, x_0)$.
$\Sigma(X, x_0) \cong (X, x_0) \wedge (S^1, 1)$.
Indeed, we may view $S^1$ as $[0, 1]/\delta[0, 1]$, both sides amount to the quotient $X \times [0, 1]/(X \times \{0, 1\}) \cup (\{x_0\} \times [0, 1])$.
*More generally,* $\Sigma^n(X, x_0) \cong X \wedge S^n$ *for all* $n$.
This is because $S(S^{n-1})$ is homeomorphic to $S^n$ and the wedge product is associative (see the lemma below!).
• $X_+ \wedge Y_+ \cong (X \times Y)_+$.
• *A pointed map* $X \wedge [0, 1]_+ \to Y$ *is just a pointed homotopy* $X \times [0, 1] \to Y$.

**Lemma.** *The wedge operation is associative on compact Hausdorff pointed spaces.*
**Proof.**
Consider $a : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z); (x \wedge y) \wedge z \mapsto x \wedge (y \wedge z)$.
We will show this is a homeomorphism. The map fits into the commutative diagram

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow & & \searrow & \\ (X \wedge Y) \wedge Z & & \stackrel{a}{\to} & & X \wedge (Y \wedge Z) \end{array}$$

where the first southwest-going arrow $\pi_1$ is the composite

$$X \times Y \times Z \stackrel{quotient \times id}{\to} (X \wedge Y) \times Z \stackrel{quotient}{\to} (X \wedge Y) \wedge Z$$

and the second southeast-going arrow $\pi_2$ is the composite

$$X \times Y \times Z \stackrel{id \times quotient}{\to} X \times (Y \wedge Z) \stackrel{quotient}{\to} X \wedge (Y \wedge Z).$$

As $\pi_1, \pi_2$ are continuous, surjections from a compact space to a Hausdorff space, they are open. The fibres of $\pi_1, \pi_2$ agree which gives that $a$ is a homeomorphism.

**Remark.** The smash product is associative also when the spaces are locally compact but may not be so in general. For instance,

$$(\mathbf{N} \wedge \mathbf{Q}) \wedge \mathbf{Q} \ncong \mathbf{N} \wedge (\mathbf{Q} \wedge \mathbf{Q}).$$

# 8 A short exact sequence

We will show that the reduced $K$-groups of wedge sums are the direct products of the reduced $K$-groups of the respective factors. We prove a more general result which is:

**Proposition.** *Let $X$ be a compact Hausdorff space with a base point, and let $A$ be a closed subspace containing the base point. Then, the inclusion map $i : A \to X$ and the quotient map $q : X \to X/A$ induce an exact sequence*

$$\widetilde{K_F}(X/A) \stackrel{q^*}{\to} \widetilde{K_F}(X) \stackrel{i^*}{\to} \widetilde{K_F}(A).$$

The proof of this uses the following key fact about vector bundles.

**Key lemma.** *Let $X$ be a compact Hausdorff space and let $A$ be a closed subspace. Let $p : E \to X$ be a vector bundle. Then,*
(i) *any trivialization*

$$h : E|A \stackrel{\cong}{\to} A \times F^n$$

*defines a vector bundle $E/h \to X/A$ satisfying the property that $E$ is isomorphic to the pull-back $q^*(E/h)$ where $q : X \to X/A$ is the quotient map.*
(ii) *If trivializations $h_0, h_1 : E|A \stackrel{\cong}{\to} A \times F^n$ are homotopic through trivializations $h_t$ of $E|A$ ($t \in [0,1]$), then $E/h_0 \cong E/h_1$.*
(iii) *For a trivialization $h_0 : E|A \stackrel{\cong}{\to} A \times F^n$, and $c \in GL_n(F)$, consider the trivialization $h_1 := (id \times c) \circ h_0$. Then, $E/h_0 \cong E/h_1$.*
The basic idea of proof of (i) is as follows. Consider the quotient space of $E$ where we identify points $v_1, v_2$ in the restriction $E|A$ if, and only if, $h(v_1)$ and $h(v_2)$ have the same second co-ordinate. This quotient space can be shown to be the vector bundle $E/h$ sought.

**Proof of proposition.**
Since $q \circ i$ maps to a point and the reduced K-group of a point is 0, we have that $i^*q^* = 0$.

Let us prove the opposite inclusion $ker(i^*) \subset Im(q^*)$. Let $x \in ker(I^*)$. Write $x$ as $[E] - [\epsilon^n]$ for some $n \geq 0$ and some vector bundle $E$ over $X$. As $E$ restricts trivially over $A$, we have $[E|A] = [\epsilon^n] \in K_F(A)$. In other words,

$$E|A \oplus \epsilon^m \cong \epsilon^{m+n}$$

for some $m \geq 0$. Fix an isomorphism

$$h : (E \oplus \epsilon^m)|A = (E|A) \oplus \epsilon^m \overset{\cong}{\to} \epsilon^{m+n}.$$

Therefore, $[(E \oplus \epsilon^m)/h] - [\epsilon^{m+n}] \in \widetilde{K_F}(X/A)$ is a class whose image under $q^*$ to the class $[E \oplus \epsilon^m] - [\epsilon^{m+n}] = [E] - [\epsilon^n] = x \in \widetilde{K_F}(X)$.

**Corollary.** *Let $X, Y$ be compact Hausdorff base-pointed spaces. Denote by $i_X, i_Y$ the inclusions of $X, Y$ respectively in $X \vee Y$. Then, we have an isomorphism*

$$\widetilde{K_F}(X \vee Y) \overset{(i_X^*, i_Y^*)}{\to} \widetilde{K_F}(X) \times \widetilde{K_F}(Y).$$

**Proof.**
Let $q_X, q_Y$ be the quotient maps from $X \vee Y$ to $X, Y$ respectively and $i_X, i_Y$ inclusions of $X, Y$ in $X \vee Y$. The above proposition shows that

$$\widetilde{K}(Y) \overset{q_Y^*}{\to} \widetilde{K}(X \vee Y) \overset{i_X^*}{\to} \widetilde{K}(X)$$

is an exact sequence. The maps $i_X^*, i_Y^*$ give retractions of $q_X^*, q_Y^*$ while the latter give sections of the former maps. In other words, we have a short exact sequence

$$0 \to \widetilde{K}(Y) \overset{q_Y^*}{\to} \widetilde{K}(X \vee Y) \overset{i_X^*}{\to} \widetilde{K}(X) \to 0$$

with $i_Y^*$ giving a retraction of $q_Y^*$. This gives the direct sum asserted.

The basic idea is now to create a long exact sequence from the short exact sequence above. Leading up to that, we first prove:

**Proposition.** *Let $X$ be compact, Hausdorff and let $A$ be a closed subspace which is contractible. Then, the quotient map $q : X \to X/A$ induces a bijection $q^* : Vect_F(X/A) \to Vect_F(X)$. Therefore, $q$ induces an isomorphism from $K_F(X/A)$ to $K_F(X)$. If $X$ has a base point which is contained in $A$, then $q$ induces an isomorphism from $\widetilde{K_F}(X/A)$ to $\widetilde{K_F}(X)$.*

**Proof.**
The first assertion implies the others and we prove the first assertion now. Let $E$ be a vector bundle over $X$. As $A$ is contractible, $E$ restricts to a trivial bundle over it. So, there exists an isomorphism

$$h : E|A \xrightarrow{\cong} A \times F^n.$$

We obtain a vector bundle $E/h$ over $X/A$. We claim that $E/h$ is (up to isomorphism) independent of the choice of $h$. Suppose $h'$ is another trivialization of $E|A$. Then, the composite $h'h^{-1} : A \times F^n \xrightarrow{\cong} A \times F^n$ is an isomorphism. In other words, there exists a continuous map $g : A \to GL_n(F)$ such that $h'h^{-1}(a, v) = (a, g(a)v)$. As $A$ is path-connected (being contractible), the image $g(A)$ is path-connected. By replacing $h'$ with $(id_A \times \gamma) \circ h' : E|A \to A \times F^n \to A \times F^n$ for some $\gamma \in GL_n(F)$ if necessary, we may assume that $g(A)$ is in the identity component. Then, we may obtain a homotopy from $g$ to the identity and hence obtain a homotopy from $h' = (h'h^{-1})h$ to $(id)h = h$ through trivializations of $E|A$. Hence, the above lemma on vector bundles shows $E/h \cong E/h'$. Thus, the map

$$[E] \mapsto [E/h]$$

is well-defined as a map from $Vect_F(X)$ to $Vect_F(X/A)$.
This is an inverse map for $q^*$. Indeed, for any vector bundle $E \to X$, we have by the above lemma, $E \cong q^*(E/h)$ and, for any vector bundle $\mathcal{E} \to X/A$, we have $(q^*\mathcal{E})/h \cong \mathcal{E}$ for the obvious trivialization $h$ of $q^*\mathcal{E}|A$.
The proof is complete.

# 9 A long exact sequence

Let $A$ be a closed subspace of a compact Hausdorff pointed space where $A$ contains the base point. Write $i$ for the inclusion of $A$ into $X$. Consider the following maps:
• the inclusion $j(i) : X \hookrightarrow X \cup CA$ (note $(X \cup CA)/X \cong \Sigma A$);
• the map collapsing $CA$, $\pi(i) : X \cup CA \to X/A$.
• The map $q(i) : X \to X/A$ that collapses $A$ is the composite $\pi(i) \circ j(i)$.
As $CA$ is contractible, we apply the above proposition to the quotient map $\pi(i)$ to obtain

$$\pi(i)^* : \widetilde{K_F}(X/A) \xrightarrow{\cong} \widetilde{K_F}(X \cup CA).$$

The functor $\widetilde{K_F}$ applied to $A \stackrel{i}{\to} X \stackrel{j(i)}{\to} X \cup CA$ gives an exact sequence. This is seen from the above statement and the proposition since $\pi(i) \circ j(i) = q(i)$.

**Observation.** *The functor $\widetilde{K_F}$ applied to the sequence*

$$A \stackrel{i}{\to} X \stackrel{j(i)}{\to} X \cup CA \stackrel{q(j(i))}{\to} \Sigma A \stackrel{\Sigma i}{\to} \Sigma X$$

*is exact.*
To see why, we consider the diagram

$$\begin{array}{ccccccccc} A & \stackrel{i}{\to} & X & \stackrel{j(i)}{\to} & (X \cup CA) & \stackrel{j^2(i)}{\to} & (X \cup CA) \cup CX & \stackrel{qj^2(i)}{\to} & \Sigma X \\ & & & & & \searrow & \downarrow & & \downarrow \\ & & & & & & \Sigma A & \stackrel{\Sigma i}{\to} & \Sigma X \end{array}$$

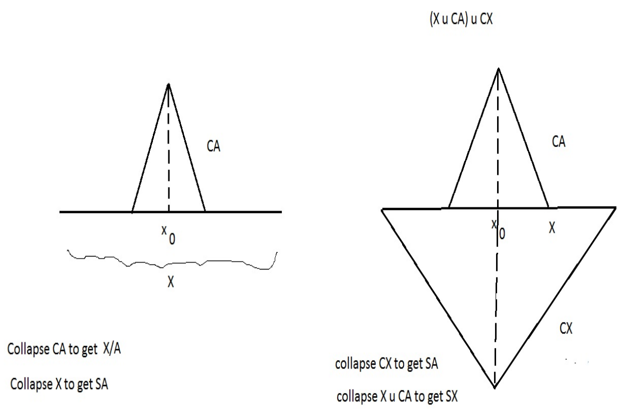
where the south-east pointing arrow is $qj(i)$, the vertical arrow in the middle is $\pi j(i)$ and the rightmost vertical arrow is the map $r : [x, t] \mapsto [x, 1 - t]$. The triangle commutes. The square also commutes up to pointed homotopy, if we note that

$$h : ((X \cup CA) \cup CX) \times [0, 1] \to \Sigma X;$$

$$([a, s], t) \mapsto [i(a), 1 - (1 - s)t] \quad on \quad CA;$$

$$([x, s], t) \mapsto [x, (1 - s)(1 - t)] \quad on \quad CX$$

is a pointed homotopy from $r \circ qj^2(i)$ to $(\Sigma i) \circ \pi j(i)$. On applying $\widetilde{K_F}$, the top row is exact as we saw earlier. Finally, since $\pi j(i)$ and $r$ induce isomorphisms on reduced K-groups, the observation is finally observed!




We wish to prove:

**Theorem.** *The following sequence is exact:*

$$\widetilde{K_F}(A) \overset{i^*}{\leftarrow} \widetilde{K_F}(X) \overset{q^*}{\leftarrow} \widetilde{K_F}(X/A) \overset{\delta}{\leftarrow}$$

$$\widetilde{K_F}(\Sigma A) \overset{(\Sigma i)^*}{\leftarrow} \widetilde{K_F}(\Sigma X) \overset{(\Sigma q)^*}{\leftarrow} \widetilde{K_F}(\Sigma X/A) \overset{\delta}{\leftarrow}$$

$$\widetilde{K_F}(\Sigma^2 A) \overset{(\Sigma^2 i)^*}{\leftarrow} \widetilde{K_F}(\Sigma^2 X) \overset{(\Sigma^2 q)^*}{\leftarrow} \widetilde{K_F}(\Sigma^2 X/A) \overset{\delta}{\leftarrow}$$

$$\dots\dots\dots$$

Here, the coboundary map $\delta$ has the following description as a composition. The isomorphism $\widetilde{K_F}(\Sigma A) \to \widetilde{K_F}(X \cup CA)$ is $(q(j(i)))^*$ coming from collapsing $X$ in $X \cup CA$. The second map is the inverse of the isomorphism $(\pi(i))^*$ from $\widetilde{K_F}(X/A)$ to $\widetilde{K_F}(X \cup CA)$ which comes by collapsing the contractible space $CA$ in $X \cup CA$.
The proof of the theorem is based on the above observations and the following lemma.

**Lemma.** *Applying the functor $\widetilde{K_F}$ to the sequence*

$$\Sigma A \overset{\Sigma i}{\to} \Sigma X \overset{\Sigma j(i)}{\to} \Sigma(X \cup CA) \overset{\Sigma q(j(i))}{\to} \Sigma^2 A \overset{\Sigma^2 i}{\to} \Sigma^2 X$$

*we get an exact sequence.*

**Proof.**

Let us define the map $\alpha : \Sigma X \cup C\Sigma A \to \Sigma(X \cup CA)$ by:
$\alpha = id$ on $\Sigma X$, $\alpha([a,t,s]) = [a,s,t]$ for $[a,t,s] \in C\Sigma A$.
Here, $s,t$ can be thought of as the cone and suspension co-ordinates. The map $\alpha$ is clearly an isomorphism.
Consider the map $\beta : \Sigma^2 A \to \Sigma^2 A$ which is the isomorphism $[a,t,s] \mapsto [a,s,t]$.
Finally, we have an isomorphism $\gamma : \Sigma^2 X \to \Sigma^2 X$ given by $[x,t,s] \mapsto [x,s,t]$.
These isomorphisms fit into the following commutative diagram:

$$\begin{array}{ccccccccc}
\Sigma A & \stackrel{\Sigma i}{\to} & \Sigma X & \stackrel{j\Sigma i}{\to} & \Sigma X \cup C\Sigma A & \stackrel{qj\Sigma i}{\to} & \Sigma^2 A & \stackrel{\Sigma^2 i}{\to} & \Sigma^2 X \\
 & & & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & & & \Sigma(X \cup CA) & \stackrel{\Sigma q(j(i))}{\to} & \Sigma^2 A & \stackrel{\Sigma^2 i}{\to} & \Sigma^2 X
\end{array}$$

where the vertical arrows from left to right are $\alpha, \beta, \gamma$ and the southeast pointing arrow is $\Sigma j(i)$. The observation above shows that applying $\widetilde{K_F}$ to the top row keeps it exact. Hence, applying $\widetilde{K_F}$ to the sequence in the statement of the lemma also takes it to an exact sequence of reduced K-groups.

**Proof of theorem.**

We saw that applying $\widetilde{K_F}$ to the top row of the following commutative diagram gives an exact sequence of reduced K-groups:

$$\begin{array}{ccccccccccccccc}
A & \stackrel{i}{\to} & X & \stackrel{j(i)}{\to} & X \cup CA & \stackrel{q(j(i))}{\to} & \Sigma A & \stackrel{\Sigma i}{\to} & \Sigma X & \stackrel{\Sigma j(i)}{\to} & \Sigma(X \cup CA) & \stackrel{\Sigma qji}{\to} & \Sigma^2 A & \cdots \\
 & & & \searrow & \downarrow & & & & & \searrow & \downarrow & & & \\
 & & & & X/A & & & & & & \Sigma(X/A) & & &
\end{array}$$

where the two southeast arrows $q$ and $\Sigma q$ are induced by the quotient map from $X$ to $X/A$ and the vertical arrows $\pi(i)$ and $\Sigma\pi(i)$ are induced by collapsing $A$.

Now, we note that for every positive integer $n$, the map $\Sigma^n \pi(i)$ fits in the commutative diagram

$$\begin{array}{ccc} \Sigma^n(X \cup CA) & \stackrel{\cong}{\leftarrow} & \Sigma^n X \cup C\Sigma^n A \\ \downarrow & & \downarrow \\ \Sigma^n(X/A) & \stackrel{\cong}{\leftarrow} & \Sigma^n X/\Sigma^n A \end{array}$$

where the horizontal isomorphisms above and below are respectively $\alpha$ and $\beta$, where
$\alpha = id$ on $\Sigma^n X$ and
$\alpha([a, t_1, \cdots, t_n, s]) = [a, s, t_1, \cdots, t_n]$ on $C\Sigma^n A$ and
$\beta([x, t_1, \cdots, t_n]) = [[x], t_1, \cdots t_n]]$.
The vertical maps are $\Sigma^n \pi(i)$ and $\pi(\Sigma^n i)$ on the left and right respectively. Since $\pi(\Sigma^n i)$ induces an isomorphism on the reduced K-groups, so does $\Sigma^n \pi(i)$. Thus, the proof of the theorem is complete.

# 10 Negative indices

The long exact sequence we obtained prompts us to define K-groups with negative indices for compact Hausdorff spaces.
For a pointed space $X$, let us define $\widetilde{K_F}^{-n}(X) := \widetilde{K_F}(\Sigma^n X)$.
The reason to define negative indices is the convention that the coboundary map goes to higher indices.
In case $X$ is not pointed, one defines the LHS by the corresponding LHS for $X_+$.
If $(X, A)$ is a pair with $A$ closed and containing the base point of $X$, one defines $\widetilde{K_F}^{-n}(X, A) := \widetilde{K_F}^{-n}(X/A)$.

The long exact sequence of the above theorem can be written as

$$\widetilde{K_F}^0(A) \stackrel{i^*}{\leftarrow} \widetilde{K_F}^0(X) \stackrel{q^*}{\leftarrow} \widetilde{K_F}^0(X/A) \stackrel{\delta}{\leftarrow} \widetilde{K_F}^{-1}(A) \stackrel{i^*}{\leftarrow} \widetilde{K_F}^{-1}(X) \stackrel{q^*}{\leftarrow} \widetilde{K_F}^{-1}(X/A) \stackrel{\delta}{\leftarrow} \cdots$$

Instead of the quotient map $q : X \to X/A$, if we look at the map of pairs $j : (X, \emptyset) \to (X, A)$, the long exact sequence we get is

$$\widetilde{K_F}^0(A) \stackrel{i^*}{\leftarrow} \widetilde{K_F}^0(X) \stackrel{j^*}{\leftarrow} \widetilde{K_F}^0(X, A) \stackrel{\delta}{\leftarrow} \widetilde{K_F}^{-1}(A) \stackrel{i^*}{\leftarrow} \widetilde{K_F}^{-1}(X) \stackrel{j^*}{\leftarrow} \widetilde{K_F}^{-1}(X, A) \stackrel{\delta}{\leftarrow} \cdots$$

In this manner, we get a $\mathbf{Z}_{\leq 0}$-graded abelian group $\widetilde{K_F}^{\leq 0}(X) = \oplus_{n \geq 0} \widetilde{K_F}^{-n}(X)$.

# 11 External product

We want to view the above graded abelian group as a graded commutative ring.

**Definition.** For compact, Hausdorff spaces $X, Y$, one defines the *external product* to be the operation

$$K_F(X) \otimes K_F(Y) \xrightarrow{*} K_F(X \times Y)$$

where $a * b = \pi_X^*(a)\pi_Y^*(b)$.

External product has the naturality property; that is, if $f : X \to X', g : Y \to Y'$, then $f^*(a) * g^*(b) = (f, g)^*(a, b)$.

The basic result we wish to prove is that when $F = \mathbf{C}$, the external product gives an isomorphism

$$K_F(X) \otimes K_F(S^2) \to K_F(X \times S^2).$$

We need to develop some preliminary tools.

We will define a reduced version of the external product which will go from product of the reduced K-groups of $X$ and $Y$ to the reduced K-group of their smash product. Let $(X, x_0), (Y, y_0)$ be pointed spaces. There is a commutative diagram:

$$\begin{array}{ccc} K_F(X) \otimes K_F(Y) & \xrightarrow{*} & K_F(X \times Y) \\ \downarrow & & \downarrow \\ K_F(x_0) \otimes K_F(y_0) & \xrightarrow{*} & K_F((x_0, y_0)) \end{array}$$

where the vertical maps are induced by the inclusion maps of the base points. It follows that if $a \in ker(i_{x_0}^*) = \widetilde{K_F}(X), b \in ker(i_{y_0}^*) = \widetilde{K_F}(Y)$, then by the naturality of $*$,

$$a * b \in ker(i_{(x_0,y_0)}^*) = \widetilde{K_F}(X \times Y).$$

### Reduced external product

Consider the map from the wedge sum $X \vee Y \hookrightarrow X \times Y$ and the quotient map $q$ from $X \times Y$ to $X \wedge Y$. They induce an exact sequence

$$\widetilde{K_F}(\Sigma(X \times Y)) \to \widetilde{K_F}(\Sigma(X \vee Y)) \to \widetilde{K_F}(X \wedge Y) \xrightarrow{q^*} \widetilde{K_F}(X \times Y) \to \widetilde{K_F}(X \vee Y).$$

Further, the second group $\widetilde{K_F}(\Sigma(X \vee Y))$ is isomorphic to

$$\widetilde{K_F}(\Sigma X \vee \Sigma Y) \cong \widetilde{K_F}(\Sigma X) \oplus \widetilde{K_F}(\Sigma Y)$$

and the last group $\widetilde{K_F}(X \vee Y) \cong \widetilde{K_F}(X) \oplus \widetilde{K_F}(Y)$.
The first and the last map in the above exact sequence have evident splittings $(\Sigma\pi_X)^* + (\Sigma\pi_Y)^*$ and $\pi_X^* + \pi_Y^*$ respectively. Hence, we have a split short exact sequence

$$0 \to \widetilde{K_F}(X \wedge Y) \stackrel{q^*}{\to} \widetilde{K_F}(X \times Y) \stackrel{(i_X^*, i_Y^*)}{\to} \widetilde{K_F}(X) \oplus \widetilde{K_F}(Y) \to 0$$

in which $i_X, i_Y$ are the inclusions $x \mapsto (x, y_0)$ and $y \mapsto (x_0, y)$ of $X, Y$ respectively in $X \times Y$.

Note the consequence of the split exact sequence:

$$\widetilde{K}_F(X \times Y) \stackrel{q^* + \pi_X^* + \pi_Y^*}{\leftarrow} \widetilde{K}_F(X \wedge Y) \oplus \widetilde{K}_F(X) \oplus \widetilde{K}_F(Y).$$

Now, we show that the external product has a reduced form. Indeed, let $(a, b) \in \widetilde{K_F}(X) \times \widetilde{K_F}(Y)$. Then, from the definition

$$a * b = \pi_X^*(a)\pi_Y^*(b),$$

we have that $i_X^*\pi_Y^*(b) = 0 = i_Y^*\pi_X^*(a)$. Therefore,

$$a * b \in ker((i_X^*, i_Y^*)) = Im(q^*).$$

As $q^*$ is 1-1, the pull-back of $a * b$ by $q^*$ is a unique element of $\widetilde{K_F}(X \wedge Y)$.

**Lemma.** *A commutative diagram involving the reduced and the unreduced external products is:*

$$\begin{array}{ccc} K_F(X) \otimes K_F(Y) & \stackrel{\cong}{\leftarrow} & (\widetilde{K_F}(X) \otimes \widetilde{K_F}(Y)) \oplus \widetilde{K_F}(X) \oplus \widetilde{K_F}(Y) \oplus \mathbf{Z} \\ \downarrow & & \downarrow \\ K_F(X \times Y) & \stackrel{\cong}{\leftarrow} & (\widetilde{K_F}(X \wedge Y) \oplus \widetilde{K_F}(X) \oplus \widetilde{K_F}(Y) \oplus \mathbf{Z} \end{array}$$

*In writing the above diagram, we have used the fact that* $K_F(X) \cong \widetilde{K_F}(X) \oplus \mathbf{Z}$. *Also, the vertical arrows are the external product on the left and the* *reduced external product along with three identity maps, on the right. Finally, the bottom row isomorphism is due to the split short exact sequence observed above.*

**Definition.** If $X, Y$ are pointed compact, Hausdorff spaces, then for all $m, n \geq 0$, define the reduced external product on negative K-groups:

$$\widetilde{K_F}^{-n}(X) \otimes \widetilde{K_F}^{-m}(Y) \xrightarrow{*} \widetilde{K_F}^{-n-m}(X \wedge Y)$$

as the composite

$$\widetilde{K_F}(X \wedge S^n) \otimes \widetilde{K_F}(Y \wedge S^m) \to \widetilde{K_F}(X \wedge S^n \wedge Y \wedge S^m)$$

$$\xrightarrow{(1 \wedge T \wedge 1)^*} \widetilde{K_F}(X \wedge Y \wedge S^n \wedge S^m) \xrightarrow{\cong} \widetilde{K_F}(X \wedge Y \wedge S^{n+m})$$

where

$$T : Y \wedge S^n \to S^n \wedge Y; (y \wedge t) \mapsto (t \wedge y),$$

a homeomorphism.

In case $X, Y$ are not pointed spaces, one considers the pointed spaces $X_+, Y_+$ and apply the above construction to obtain the *unreduced* external product

$$K_F^{-n}(X) \otimes K_F^{-m}(Y) \xrightarrow{*} K_F^{-n-m}(X \times Y).$$

# 12 Bott periodicity - complex case

Throughout this section, we will take $F = \mathbf{C}$. For simplicity of notation, we will write $K(X)$ etc. instead of $K_{\mathbf{C}}(X)$ etc.

**Complex Bott periodicity theorem.**
*Let $X$ be a compact, Hausdorff space. Then, the external product*

$$K(X) \otimes K(S^2) \to K(X \times S^2)$$

*is an isomorphism and $K(S^2)$ is the free abelian group of rank 2 generated by the class of the trivial line bundle and the class of the dual $[H]$ of the canonical bundle.*

This is a periodicity result because it has the (equivalent) version

$$K^{-n}(X) \cong K^{-n-2}(X).$$

The real version is more complicated and has $S^8$ in place of $S^2$.
First, we note equivalent versions of the complex Bott periodicity and prove one of these versions.

**Proposition.** *The following statements are equivalent (in the statements BP2 to BP5, $X$ is pointed):*

$$(BP1) \quad K(X) \otimes K(S^2) \to K(X \times S^2)$$

*given by the external product is an isomorphism and $K(S^2)$ is the free abelian group generated by* $1$ *and* $[H]$.

$$(BP2) \quad \widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(X \wedge S^2)$$

*given by the reduced external product is an isomorphism and $\widetilde{K}(S^2)$ is the infinite cyclic group generated by the "Bott element" $b = [H] - 1$.*

$$(BP3) \quad \widetilde{K}(X) \xrightarrow{\beta} \widetilde{K}(X \wedge S^2)$$

*given by multiplication with the Bott element, is an isomorphism.*

$$(BP4) \quad \widetilde{K}(X) \to \widetilde{K}^{-2}(X)$$

*given by multiplication with the Bott element $b \in \widetilde{K}(S^2) = \widetilde{K}^{-2}(S^0)$ is an isomorphism.*

$$(BP5) \quad \widetilde{K}^{-n}(X) \to \widetilde{K}^{-n-2}(X)$$

*given by multiplication with the Bott element is an isomorphism, for all $n \geq 0$.*

**Proof.**

The above lemma connecting reduced and unreduced external products shows that (BP1) and (BP2) are equivalent. Further, (BP2) obviously implies (BP3). If we take $X = S^0$, then (BP3) implies $\widetilde{K}(S^2)$ is generated by the Bott element. Also, (BP2) and (BP4) are evidently equivalent. Further, (BP3) implies (BP5) by the definition of the reduced external product and (BP4) is a special case of (BP5).

We will prove the version (BP3) using clutching functions which are Laurent polynomials. In fact, we will explicitly produce an inverse to the mapping $\beta$ in (BP3). The following proposition will be used to induce a map on the reduced K-groups and provide the sought inverse of $\beta$.

**Reduction Proposition.** *There exists a map $\alpha : K(X \times S^2) \to K(X)$ satisfying:*
(i) *$\alpha$ is natural in $X$,*
(ii) *$\alpha$ is $K(X)$-linear (that is, a homomorphism of $K(X)$-modules, where $K(X)$ acts on $K(X \times S^2)$ by $a.p = \pi_X^*(a)p$ where $\pi_X : X \times S^2 \to X$, the projection, and*
(iii) *$X = pt$, the Bott class $b := [H] - 1 \in K(S^2)$ is in the kernel of $\alpha$.*

# 13 Towards the proof of periodicity

## 13.1 Reduced version of $\alpha$

From the above $\alpha$, we construct a reduced version of $\alpha$ from $\widetilde{K}(X \times S^2)$ to $\widetilde{K}(X)$ as follows.
Let $i : \{x_0\} \hookrightarrow X$ and $j : \{x_0\} \times S^2 \hookrightarrow X \times S^2$ be the inclusions defined by the base point $x_0$. One has the following diagram with exact rows:

$$\begin{array}{ccccccccc} & & \widetilde{K}(X \times S^2/\{x_0\} \times S^2) & \to & K(X \times S^2) & \stackrel{\jmath^*}{\to} & K(\{x_0\} \times S^2) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{K}(X) & \to & K(X) & \stackrel{j^*}{\to} & K(x_0) & \to & 0 \end{array}$$

where the middle and right vertical arrows are $\alpha$ and the left vertical arrow is defined as a consequence because the right square above commutes by naturality of $\alpha$. If we compose the left vertical map with the map $\widetilde{K}(X \wedge S^2) \to \widetilde{K}(X \times S^2/\{x_0\} \times S^2)$ induced by the quotient map, we finally have

$$\alpha : \widetilde{K}(X \wedge S^2) \to \widetilde{K}(X).$$

By construction, $\alpha$ is natural and the diagram

$$\begin{array}{ccc} \widetilde{K}(X \wedge S^2) & \stackrel{\alpha}{\to} & \widetilde{K}(X) \\ \downarrow & & \downarrow \\ K(X \times S^2) & \stackrel{\alpha}{\to} & K(X) \end{array}$$

is commutative. So, we use the same notation $\alpha$.

## 13.2 $\alpha \circ \beta = id$

We are left with showing that $\alpha$ and $\beta$ are inverse maps of each other. In this subsection, we will observe $\alpha \circ \beta = id_{K(X)}$.

**Lemma.** *$\alpha \circ \beta = id$ on $\widetilde{K}(X)$.*
**Proof.**
We look at the commutative diagram

$$\begin{array}{ccccc} \widetilde{K}(X) & \stackrel{\beta}{\to} & \widetilde{K}(X \wedge S^2) & \stackrel{\alpha}{\to} & \widetilde{K}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{K}(X) & \stackrel{\beta}{\to} & \widetilde{K}(X \times S^2) & \stackrel{\alpha}{\to} & \widetilde{K}(X) \end{array}$$

By the definition of $\beta$, we have $\beta(1) = \pi_{S^2}^*(b)$; here, $\pi_{S^2}$ is the second projection from $X \times S^2$. By naturality and the property that $\alpha(b) = 1$ when $X = \{pt\}$, we obtain $\alpha(\pi_{S^2}^*(b)) = 1$. As both $\alpha, \beta$ in the bottom row are $K(X)$-linear, this means $\alpha \circ \beta = id$ on bottom row. Therefore, this is true on the top row as well.

## 13.3 $\beta \circ \alpha = id$

The reverse equality $\beta \circ \alpha = id$ is more complicated to prove. This is done in two steps as follows.

**Lemma.** *Consider the transpose map $T : S^2 \wedge S^2 \to S^2 \wedge S^2; (x \wedge y) \mapsto (y \wedge x)$. Then, we have the commutative diagram*

$$\begin{array}{ccccc} \widetilde{K}(X \wedge S^2) & \stackrel{\beta}{\to} & \widetilde{K}(X \wedge S^2 \wedge S^2) & \stackrel{(id \wedge T)^*}{\to} & \widetilde{K}(X \wedge S^2 \wedge S^2) \\ \downarrow & & & & \downarrow \\ \widetilde{K}(X) & & \stackrel{\beta}{\to} & & \widetilde{K}(X \wedge S^2) \end{array}$$

**Lemma.** *The map $\widetilde{K}(X \wedge S^2 \wedge S^2) \stackrel{(id \wedge T)^*}{\to} \widetilde{K}(X \wedge S^2 \wedge S^2)$ is the identity map.*
**Proof.**
We view $S^2 \wedge S^2$ which is homeomorphic to $S^4$, as an one-point compactification of $\mathbf{R}^4$. Then, the transpose map $T$ above is the one induced by the linear map

$$\mathbf{R}^4 \to \mathbf{R}^4; (a, b, c, d) \mapsto (c, d, a, b).$$

This clearly has determinant 1 which means there is a connected path in $GL_4(\mathbf{R})$ which connects it to the identity map. In other words, one has homotopy between $T$ and $id_{S^2\wedge S^2}$ and hence a homotopy from $id_X \wedge T$ to $id_{X\wedge S^2\wedge S^2}$.

**Corollary.** *$\beta \circ \alpha = id$ on $\widetilde{K}(X \wedge S^2)$.*
**Proof.**
$\beta \circ \alpha = \alpha \circ (id_X \wedge T)^* \circ \beta = \alpha \circ \beta = id.$

## 13.4 Constructing $\alpha$ via clutching

Thus, we have proved that BP holds modulo the construction of $\alpha$. In what follows, we will construct $\alpha$ using clutching constructions. The idea of constructing $\alpha$ as in the reduction proposition is by considering clutching functions in increasing generality.

Recall the notations $D_0 = \{z \in S^2 : |z| \leq 1\}$ and $D_\infty = \{z \in S^2 : |z| \geq 1\}$. Both $D_0, D_\infty$ are homeomorphic to the closed unit disc which is contractible and their intersection is the equatorial great circle $S^1 = \{z \in S^2 : |z| = 1\}$). Let $X$ be a pointed space, and $\pi_0 : X \times D_0 \to X$, $\pi_\infty : X \times D_\infty \to X$, $\pi : X \times S^1 \to X$ be the natural projections. Write $s : X \to X \times S^2$; $x \mapsto (x, 1)$.

**Proposition.** *Let $E \to X \times S^2$ be a complex vector bundle, and $\mathcal{E} = s^*(E)$. Then, there is an isomorphism of vector bundles over $X \times S^1$:*

$$u : \pi^*\mathcal{E} \to \pi^*\mathcal{E}$$

*satisfying:*
(i) *$u|X = id$ - therefore, we may identify $\mathcal{E}$ with the clutching construction $\pi_0^*\mathcal{E} \cup_u \pi_\infty^*\mathcal{E})$ restricted to $X$;*
(ii) *there is an isomorphism of $E$ with $\pi_0^*\mathcal{E} \cup_u \pi_\infty^*\mathcal{E})$ that is identity on $X \subset X \times S^2$.*
*Further, the two properties (i),(ii) determine $u$ uniquely up to homotopy (through vector bundle isomorphisms).*
**Proof.**
Since the composite maps

$$X \times D_0 \overset{\pi_0}{\to} X \hookrightarrow X \times D_0$$

and

$$X \times D_\infty \stackrel{\pi_\infty}{\to} X \hookrightarrow X \times D_\infty$$

are homotopic to the respective identity maps, we have isomorphisms

$$h_0 : E|X \times D_0 \stackrel{\cong}{\to} \pi_0^*(\mathcal{E});$$

$$h_\infty : E|X \times D_\infty \stackrel{\cong}{\to} \pi_\infty^*(\mathcal{E}).$$

For a suitable automorphism $\alpha \in Aut(\mathcal{E})$, we may compose $h_0$ with $\alpha \times id_{D_0}$ to assume $h_0$ is actually the identity on $X$. Similarly, we may assume $h_\infty$ is also identity on $X$. With this choice, define

$$u : \pi_0^*(\mathcal{E}) \stackrel{h_0^{-1}}{\to} E|X \times S^1 \stackrel{h_\infty}{\to} \pi_\infty^*(\mathcal{E}).$$

Clearly, (i) and (ii) are satisfied. Let us show uniqueness now.
Suppose $u'$ is another such isomorphism.
The isomorphisms $E \cong \pi_0^*\mathcal{E} \cup_u \pi_\infty^*\mathcal{E})$ induce isomorphisms

$$h_0' : E|X \times D_0 \stackrel{\cong}{\to} \pi_0^*(\mathcal{E});$$

$$h_\infty' : E|X \times D_\infty \stackrel{\cong}{\to} \pi_\infty^*(\mathcal{E})$$

which restrict to the identity on $X$ and such that $u'$ agrees with $h_\infty'(h_0')^{-1}$ on $\pi^*(\mathcal{E})$.
Now, the difference $h_0 - h_0'$ gives an automorphism of $\pi_0^*(\mathcal{E}) = \mathcal{E} \times D_0$ which is identity on $X$. All such automorphisms are of the form

$$\mathcal{E} \times D_0 \to \mathcal{E} \times D_0;$$

$$(v, z) \mapsto (g(v, z), z)$$

where $g(-, z)$ is a vector bundle automorphism for each $z \in D_0$ and $g(-, 1) = id$. Using a deformation retraction $H$ from $D_0$ to 1, we obtain a homotopy

$$\mathcal{E} \times D_0 \times [0, 1] \to \mathcal{E} \times D_0;$$

$$(v, z, t) \mapsto (g(v, H(z, t)), z)$$

from $h_0 - h_0'$ to $id$; hence, we get a homotopy from $h_0$ to $h_0'$ through vector bundle isomorphisms. In the same manner, we have a homotopy between $h_\infty$

and $h'_{infty}$ through vector bundle automorphisms on $\pi_\infty^*(\mathcal{E})$. Therefore, we end up with a homotopy from $u$ to $u'$ through vector bundle isomorphisms.

**Laurent polynomial clutching functions**

**Definition.** Given a vector bundle $p : E \to X$, an automorphism of $\pi^{|ast}(E)$ amounts to a continuous family of automorphisms $u(x,z) : \mathcal{E}_x \to \mathcal{E}_x$ (for $x \in X, z \in S^1$). It is called a Laurent polynomial clutching function if we can write

$$u(x,z) = \sum_{|k| \leq n} a_k(x) z^k$$

for some endomorphisms $a_k$ of $E$.

Further, it is called a polynomial clutching function of the sums above are over $k \geq 0$; it is said to be linear if the sum is linear in $z$. In other words, there are endomorphisms $a, b$ of $E$ such that for each $x \in X, z \in S^1$, $a(x)z + b(x)$ is an isomorphism of $p^{-1}(x)$ sending $v$ to $za(x)(v) + b(x)(v)$ (here, $z$ is multiplication by $z \in S^1 \subset \mathbf{C}$).

*Often, we will simply write $E$ instead of $s^*(E)$ when the context is clear (and $X$ is identified via $s$ with $X \times \{1\}$ in $X \times S^2$. In particular, for $E$ over $X \times S^2$ with a clutching function $u$, the map $u$ is written as an automorphism of $\pi^*(E)$ where $\pi : X \times S^1$ to $X$.*

**Lemma.** *Every clutching function can be approximated by Laurent polynomial clutching functions. In particular, for any clutching function $u : \pi^*\mathcal{E} \stackrel{\cong}{\to} \pi^*\mathcal{E}$, there is a Laurent polynomial clutching function homotopic to it. Further, two Laurent polynomial clutching functions which are homotopic through clutching functions are homotopic through Laurent polynomial clutching functions.*

**Proof.**

Given a clutching function $u : \pi^*(E) \stackrel{\cong}{\to} \pi^*(E)$, consider

$$a_k(x) := \frac{1}{2\pi} \int_0^{2\pi} u(x, e^{i\theta}) e^{-ik\theta} d\theta.$$

The Cesaro sums

$$u_n(x,z) := \frac{1}{n+1} \sum_{0 \leq k \leq n} s_k(x,z)$$

where

$$s_k(x,z) := \sum_{|j|\le k} a_j(x) z^j$$

are clutching functions which converge uniformly in $x$ and $z$ (by Fejer's theorem) to $u(x,z)$. This proves the first assertion.
To show the second one, for a sufficiently close Laurent polynomial clutching function $\ell$ to $u$, the map

$$t \mapsto tu + (1-t)\ell$$

is a homotopy through clutching functions, from $\ell$ to $u$.
Finally, for the final assertion, observe that a homotopy from $\ell_0$ to $\ell_1$ (two Laurent polynomial clutching functions) through clutching functions, gives and automorphism of $\pi^*(E)\times[0,1]$. We approximate this automorphism by a Laurent polynomial clutching function and obtain a homotopy $\ell'_t$ through Laurent polynomial clutching functions. For a sufficiently close approximation of the automorphism, one can combine the homotopy $\ell'_t$ with a linear homotopy from $\ell_0$ to $\ell'_0$ and a linear homotopy from $\ell'_1$ to $\ell_1$ to get a homotopy from $\ell_0$ to $\ell_1$ which is entirely through Laurent polynomial clutching functions.

**Linear clutching functions case.**
Let $E \to X \times S^2$ be a vector bundle, and consider the restriction to $X \times \{1\}$ which is identified with $X$. Writing $\pi : X \times S^1 \to X$ for the projection, look at a clutching function $p : \pi^*(E) \xrightarrow{\cong} \pi^*(E)$ which is linear; that is, $p(x,z) = a(x)z + b(x)$ say.

**Lemma.** *The operator $Q_p : E \to E$ defined by*

$$Q_p(x) = \frac{1}{2\pi i}\int_{|z|=1} (a(x)z + b(x))^{-1} a(x) dz$$

*is a projection operator. In particular, we have a decomposition*

$$E = Im(Q_p) \oplus Ker(Q_p).$$

**Proof.**
Note that for $z \neq w$, one has

$$\frac{(az+b)^{-1}}{w-z} + \frac{(aw+b)^{-1}}{z-w} = (az+b)^{-1} a (aw+b)^{-1}.$$

Since, for all $x \in X$, the operator $a(x)z + b(x)$ is invertible for all $z \in S^1$, there exists $\epsilon > 0$ such that it remains invertible for all $1 - \epsilon < |z| < 1 + \epsilon$. The Cauchy's integral theorem implies that the equality

$$Q_p(x) = \frac{1}{2\pi i} \int_{|z|=r} (a(x)z + b(x))^{-1} a(x) dz$$

holds for all $r \in (1 - \epsilon, 1 + \epsilon)$. Take $r, R$ such that

$$1 - \epsilon < r < R < 1 + \epsilon.$$

Then,

$$Q_p^2 = \frac{1}{(2\pi i)^2} (\int_{|z|=R} (az + b)^{-1} a dz)(\int_{|w|=r} (aw + b)^{-1} a dw)$$

$$= \frac{1}{(2\pi i)^2} \int_{|w|=r} \int_{|z|=R} \frac{(az + b)^{-1}}{w - z} a dz dw + \frac{1}{(2\pi i)^2} \int_{|w|=r} \int_{|z|=R} \frac{(aw + b)^{-1}}{z - w} a dz dw$$

$$= \frac{1}{(2\pi i)^2} \int_{|w|=r} \int_{|z|=R} \frac{(az + b)^{-1}}{w - z} a dz dw + \frac{1}{2\pi i} \int_{|w|=r} (aw + b)^{-1} a dw$$

$$= \frac{1}{2\pi i} \int_{|z|=R} 0 dz + Q_p = Q_p$$

where we have simply used Fubini's theorem and Cauchy's integral formula.

**Remarks.**
We observe that a projection operator on a vector bundle has locally constant rank function. Thus, the image and kernel are vector bundles. One denotes the two vector bundles $Im(Q_p)$ and $Ker(Q_p)$ by $(E, p)_+$ and $(E, p)_-$ respectively. We observe:

**Proposition.**
(i) $(E, 1)_+ = \epsilon_X^0$ *and* $(E, z)_+ = E$.
(ii) $(E, p_0)_+ \cong (E, p_1)_+$ *if* $p_0, p_1$ *are homotopic through linear clutching functions.*
(iii) $(E_1 \oplus E_2, p_1 \oplus p_2) \cong (E_1, p_1)_+ \oplus (E_2, p_2)_+$.
(iv) $(E' \otimes E, id \otimes p)_+ \cong E' \otimes (E, p)_+$ *for every vector bundle* $E' \to X$.

**Proof.**
(i) follows by Cauchy's integral formula. Indeed,

$$Q_1 = \frac{1}{2\pi i} \int_{|z|=1} Id_E dx = 0$$

and

$$Q_z = \frac{1}{2\pi i} \int_{|z|=1} \frac{Id_E}{z} dz = Id_E.$$

For (ii), note that a homotopy between $p_0, p_1$ through linear clutching functions is a function $q : E \times [0,1] \to X \times [0,1]$ such that the ends (at $t = 0$ and $t = 1$) of the vector bundle $(E \times [0,1], q)_+ \to X \times [0,1]$ are $(E, p_0)_+$ and $(E, p_1)_+$. Parts (iii), (iv) are clear from the construction.

**Remarks.**
We remark that we need a homotopy through *linear* clutching functions so that the the image and kernel of the operator $Q_{p_0}$ are carried to the corresponding spaces for the operator $Q_{p_1}$.

**From linear to polynomial clutching functions.**
Given a polynomial clutching function of degree $n$, we will associate a linear clutching function on the $(n+1)$-fold direct sum of $E$ as follows.

Let $p(x,z) = \sum_{k-0}^{n} a_k(x) z^k : \pi^*(E) \stackrel{\cong}{\to} \pi^*(E)$. Consider the vector bundle $L^n(E) := \oplus^{n+1} E \to X$ and the linear clutching function

$$L^n(p) : \pi^*(L^n(E)) \stackrel{\cong}{\to} \pi^*(L^n(E))$$

defined by

$$L^n(p) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -z & 1 \end{pmatrix}$$

Note that

$$L^n(p) = \begin{pmatrix} 1 & g_1 & g_2 & \cdots & g_n \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix}$$

where the polynomials $g_i(z)$'s are defined by $g_1(z) = \frac{p(x,z)-a_0}{z}, g_{r+1}(z) = \frac{g_r(z)-g_r(0)}{z} = \frac{p-a_0-a_1z-\cdots-a_rz^r}{z^{r+1}}$. Therefore, $L^n(p)$ is actually an automorphism. We denote the vector bundles $(L^n(E), L^n(p))_+$ and $(L^n(E), L^n(p))_-$ over $X$ by $L^n(E,p)_+$ and $L^n(E,p)_-$ respectively. We have:

**Proposition.**
(i) *If $p$ is a polynomial clutching function of degree $n$, then $L^{n+1}(E,p)_+ \cong L^n(E,p)_+$, and $L^{n+1}(E,zp)_+ \cong L^n(E,p)_+ \oplus E$.*
(ii) *$L^n(E,p_0)_+ \cong L^n(E,p_1)_+$ if $p_0, p_1$ are polynomial clutching functions homotopic through polynomial clutching functions of degree $\leq n$.*
(iii) *$L^n(E_1 \oplus E_2, p_1 \oplus p_2)_+ \cong L^n(E_1,p_1)_+ \oplus L^n(E_2,p_2)_+$ where $p_1, p_2$ are polynomial clutching functions of degree at most $n$.*
(iv) *$L^n(E' \otimes E, id \otimes p)_+ \cong E' \otimes L^n(E,p)_+$.*
**Proof.**
(i) As $t$ varies in $[0,1]$, consider $\begin{pmatrix} & L^n(p) & & 0 \\ 0 & \cdots & -tz & 1 \end{pmatrix}$ gives a homotopy from $L^n(p) \oplus 1$ to $L^{n+1}(p)$ through linear clutching functions. Now, if $g : [0,1] \to GL_2(\mathbf{C})$ connects the identity matrix to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, a homotopy from $L^{n+1}(zp)$ to $z \oplus L^n(p)$ is given by the product

$$\begin{pmatrix} g(t) & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & a_0 & a_1 & \cdots & a_n \\ -z & 1-t & & & \\ & -z & 1 & & \\ & & \ddots & & \\ & & & -z & 1 \end{pmatrix}.$$

All the assertions now follow from the results proved for linear clutching functions above. Indeed, a homotopy from $p_0$ to $p_1$ via polynomial clutching functions $p_t$ of degree $\leq n$ induces a linear homotopy $L^n(p_t)$ from $L^n(p_0)$ to $L^n(p_1)$.

**From polynomial to Laurent polynomial clutching functions**
If $\ell$ is a Laurent polynomial clutching function, then for large enough $n$, $z^n\ell$ is a polynomial clutching function of degree $\leq 2n$. One defines

$$\alpha(E,\ell) := nE - L^{2n}(E, z^n\ell)_+ \in K(X).$$

By the proposition (i) above, the above definition is independent of $n$ large enough. Further $\alpha(E,\ell_0) \cong \alpha(E,\ell_1)$ if $\ell_0, \ell_1$ are Laurent polynomial clutching functions which are homotopic through Laurent polynomial clutching functions. By the approximation theorem on clutching functions proved earlier, for an arbitrary clutching function $u$, one defines $\alpha(E,u) := \alpha(E,\ell) \in K(X)$ for a Laurent polynomial clutching function $\ell$ homotopic to $u$ through clutching functions. We have the properties:

**Proposition.**
(i) $\alpha(E,u_0) = \alpha(E,u_1)$ *for clutching functions* $u_0, u_1$ *which are homotopic through clutching functions.*
(ii) $\alpha(E_1 \oplus E_2, u_1 \oplus u_2) = \alpha(E_1,u_1) + \alpha(E_2,u_2) \in K(X)$.
(iii) $\alpha(E' \otimes E, id \otimes u) = E'\alpha(E,u) \in K(X)$.
**Proof.**
(i) is evident from the very definition. The other parts follow from the previous proposition immediately.

**Proving $\alpha$ satisfies the reduction proposition**

Recall that any vector bundle over $X \times S^2$ arises by a clutching function $u$; that is, $u : \pi^*(E) \xrightarrow{\cong} \pi^*(E)$. One defined $\alpha(E) := \alpha(E,u)$ . This has the properties:

**Lemma.**
(i) $\alpha(H) = 1 \in K(pt)$.
(ii) $\alpha(\epsilon^1_{S^2}) = 0 \in K(pt)$.
**Proof.**
For (i), recall from the example of $S^2$ discussed earlier, that for $X = pt$, one may take the clutching function $u = z^{-1}$. Then,

$$\alpha(H) = \alpha(H_1, z^{-1}) = H_1 - L^2(H_1, 1)_+ = 1 - (L^2(H_1), L^2(1))_+ \in K(pt).$$

Here $L^2(1) = \begin{pmatrix} 1 & 0 & 0 \\ -z & 1 & 0 \\ 0 & -z & 1 \end{pmatrix} \cong I_3$ where the isomorphism is through linear clutching functions. But, we saw during the discussion on linear clutching functions that

$$(L^2(H_1), L^2(1))_+ = 0 \in K(pt).$$

This proves (i).
For (ii), observe that we may choose the identity as the clutching function on $\pi^*(\epsilon^1_{pt})$. Thus,

$$\alpha(\epsilon^1_{S^2}) = \alpha(\epsilon^1_{pt}, 1) = -L^0(\epsilon^1_{pt}, 1)_+ = -(\epsilon^1_{pt}, 1)_+ = \epsilon^0_{pt} = 0 \in K_{pt}.$$

Hence (ii) follows.

The proof of BP is complete.

# 14 Applications of Bott periodicity

## 14.1 Positive K-groups

For a pointed compact Hausdorff space $X$, we have the reduced K-groups $\widetilde{K}^{-n}(X)$ for $n \geq 0$. As Bott periodicity produces isomorphisms of the above with $\widetilde{K}^{-n-2}(X)$, we may define, for $n > 0$,

$$\widetilde{K}^{2n-1}(X) = \widetilde{K}^{-1}(X);$$

$$\widetilde{K}^{2n}(X) = \widetilde{K}^{0}(X).$$

With these notations, we have $\widetilde{K}^n(X) = \widetilde{K}^{n-2}(X)$ for all integers $n$.
For non-pointed spaces $X$, one may look at $X_+$ and define $K^n(X) = \widetilde{K}^n(X_+)$.
Similarly, for a compact pair $(X, A)$, define $K^n(X, A) = \widetilde{K}^n(X/A)$.
Thus, the $\mathbf{Z}$-graded group $\widetilde{K}^*(X) := \oplus_{n \in \mathbf{Z}} \widetilde{K}^n(X)$ can be viewed, in view of Bott periodicity, as the $\mathbf{Z}/2$-graded group

$$\widetilde{K}^*(X) = \widetilde{K}^0(X) \oplus \widetilde{K}^1(X).$$

Similarly, the unreduced $K^*(X)$ and $K^*(X, A)$ are $\mathbf{Z}/2$-graded.

**Proposition.** *The positive K-groups satisfy the following properties similar to reduced cohomology groups. Thus, topological K-theory of compact spaces gives an extraordinary cohomology theory (all axioms other than the dimension axiom).*

• **Homotopy invariance:**
*For a pointed map $f : X \to Y$, the induced map $\widetilde{K}^n(Y) \to \widetilde{K}^n(X)$ depends only on the pointed homotopy class of $f$.*

• **Suspension:**
*For all $n$, there is a natural isomorphism $\widetilde{K}^n(X) \cong \widetilde{K}^{n+1}(\Sigma X)$.*

• **Exactness:**
*For a closed subspace $A$ containing the base point of $X$, the sequence $\widetilde{K}^n(X/A) \to \widetilde{K}^n(X) \to \widetilde{K}(A)$ is exact.*

• **Additivity:**
*The inclusions $i_X : X \hookrightarrow X \vee Y, i_Y : Y \hookrightarrow X \vee Y$ induce $\widetilde{K}^n(X \vee Y) \stackrel{\cong}{\to} \widetilde{K}^n(X) \times \widetilde{K}^n(Y)$.*

**Proof.**
Homotopy invariance of induced maps on $K^0$ (and, hence on $\widetilde{K}^0$) and the observation that when $f, g$ are homotopic, then so are $\Sigma f, \Sigma g$ imply the assertion of homotopy invariance on $\widetilde{K}^n$.
The assertion on suspension is obvious from the definition and Bott periodicity.
The claim of exactness follows from the corresponding statement for nonpositive K-groups proved earlier.
Finally, the additivity claim follows from the corresponding statement for $n = 0$ and the homeomorphism between $\Sigma(X \vee Y)$ and $\Sigma X \vee \Sigma Y$.

## 14.2 Spheres

**Proposition.**
(i) *For $n > 0$, $\widetilde{K}(S^{2n})$ is infinite cyclic with a generator $([H] - 1)^{*n}$. The product in this ring is trivial.*
(ii) *For any pointed $X$, the reduced external product $\widetilde{K}(X) \otimes \widetilde{K}(S^{2n}) \to \widetilde{K}(X \wedge S^{2n})$ is an isomorphism.*
(iii) *For a non-pointed space $X$, the external product*

$$K(X) \otimes K(S^{2n}) \to K(X \times S^{2n})$$

*gives an isomorphism.*
(iv) $K(S^{2n}) \cong \mathbf{Z}[T]/(T^2)$ *as rings.*

(v) *For all* $n > 0$, $K(S^{2n-1}) \cong \mathbf{Z}, \widetilde{K}(S^{2n-1}) = 0$.
(vi) $\widetilde{K}^1(S^{2n}) = 0, \widetilde{K}^1(S^{2n-1}) \cong \mathbf{Z}$ *for* $n > 0$.
**Proof.**
Repeatedly applying BP, the iterated external product map

$$\widetilde{K}(S^2) \otimes \cdots \otimes \widetilde{K}(S^2) \xrightarrow{*} \widetilde{K}(S^{2n})$$

is an isomorphism. This is actually an isomorphism of rings because the reduced external product also preserves product (being the restriction of external product which does by definition; that is, $(a * b)(a' * b') = (aa') * (bb')$). As $\widetilde{K}(S^2)$ is generated by $[H] - 1$ with the relation $([H] - 1)^2 = 0 \in \widetilde{K}(S^2)$, we have (i).
This also implies (iv) immediately.
For odd dimensional spheres, first look at $S^1$. Now, rank $n$ vector bundles on $S^1$ are in bijection with homotopy classes of maps from $S^0$ to $GL_n(\mathbf{C})$; the latter is a single point as $GL_n(\mathbf{C})$ is path-connected. Hence, all vector bundles over $S^1$ are trivial. Thus, $Vect_{\mathbf{C}}(S^1)$ can be identified with the monoid of natural numbers and so, $K(S^1) \cong \mathbf{Z}$. Hence, $\widetilde{K}(S^1) = (0)$ and using BP, we get $\widetilde{K}(S^{2n+1}) = (0)$ for all $n$. Thus, $K(S^{2n-1}) \cong \mathbf{Z}$ for all $n \geq 1$ which proves (v).
By the suspension isomorphism, $\widetilde{K}^n(X) \cong \widetilde{K}^{n+1}(\Sigma X)$. Therefore, we immediately obtain (vi).
Further, (i) and BP implies (ii).
Using the commutative diagram involving the reduced and unreduced products (a previous lemma), (iii) also follows.

The proposition implies that $\widetilde{K}(S^n)$ is isomorphic to $\mathbf{Z}$ for all $n \geq 0$ where the copy of $\mathbf{Z}$ occurs in degree 0 when $n$ is even and in degree 1 when $n$ is odd. From this, one may deduce:

**Brouwer fixed point theorem.**
*A continuous map* $f : D^n \to D^n$ *has a fixed point.*

Indeed, if $f$ did not have a fixed point, we would have a retraction of $D^n$ to $S^{n-1}$ but the corresponding reduced $k$-groups are 0 and non-zero. This is a contradiction.

We state another following result which can be proved using Bott periodicity.

*Let* $X$ *be a finite cell complex with* $n$ *cells. Then,* $K^*(X)$ *is an* $n$*-generated*

*abelian group. If all the cells have even dimensions, then $K^1(X) = 0$ and $K^0(X)$ is free abelian of rank equal to $n$.*
*In particular, $K^1(\mathbf{C}P^n) = 0$ and $K^0(\mathbf{C}P^n) \cong \mathbf{Z}^{n+1}$ as abelian groups.*

## 14.3 Adams operation and division algebras

As an application of BP, we prove that the only values of $n$ for which $\mathbf{R}^n$ has a bilinear multiplication for which it forms a division algebra, are $n = 1, 2, 4, 8$. This purely algebraic theorem has a beautiful proof due to Adams. The algebraic result goes hand in hand with a topological result which we state now.

**Parallelizability/division algebras theorem.**
(i) *If $\mathbf{R}^n$ admits the structure of a division algebra, then $n = 1, 2, 4$ or $8$.*
(ii) *If the sphere $S^{n-1}$ is parallelizable (that is, its tangent bundle is trivial), then $n = 1, 2, 4$ or $8$.*

**Definition.** An H-space $X$ is a space with a special element $e$and a ("multiplication") map $m : X \times X \to X$ which is continuous and has the properties that the maps $x \mapsto m(e, x)$ and $x \mapsto m(x, e)$ are homotopic to the identity map.
It can be shown that in this case, one can produce a map homotopic to $m$ which satisfies the stricter properties $m(e, x) = m(x, e) = x$ for all $x$. The main observation is:

**Proposition.** *If $\mathbf{R}^n$ is a division algebra, or if $S^{n-1}$ is parallelizable, then $S^{n-1}$ is an H-space.*
**Proof.** Suppose $\mathbf{R}^n$ is a division algebra (we may assume it has a unity). Now, with the above multiplication, we have an $H$-space structure on $S^{n-1}$:

$$S^{n-1} \times S^{n-1} \to S^{n-1};$$

$$(x, y) \mapsto \frac{xy}{||xy||}.$$

Now, assume $S^{n-1}$ is parallelizable. Choose linearly independent sections $s_1, \cdots, s_{n-1}$ of the tangent bundle of $S^{n-1}$. The vectors

$$s_1(e_1), s_2(e_1), \cdots, s_{n-1}(e_1)$$

give a basis of $T_{e_1}S^{n-1} = \left\langle \{e_2, \cdots, e_n\} \right\rangle \subset \mathbf{R}^n$. By replacing the $s_i$'s by linear combinations, we may assume

$$s_1(e_1) = e_2, s_2(e_1) = e_3, \cdots, s_{n-1}(e_1) = e_n.$$

Notice that $e_1$ along with the above $n-1$ vectors forms an orthonormal basis of $\mathbf{R}^n$. Therefore, by the Gram-Schmidt process (which does not affect the above basis as it is already orthonormal), we may also assume that for all $x \in S^{n-1}$, the vectors $x, s_1(x), s_2(x), \cdots, s_{n-1}(x)$ form an orthonormal basis for $\mathbf{R}^n$. For each $x \in S^{n-1}$, the matrix $g_x$ whose last column is the above vector, belongs to $SO(n)$ and provides an $H$-space structure on $S^{n-1}$ by

$$(x, y) \mapsto g_x(y).$$

**Lemma.** *$S^{2n}$ is never an H-space.*
**Proof.**
Suppose $\mu : S^{2n} \times S^{2n} \to S^{2n}$ is a multiplication making $S^{2n}$ is an H-space with identity element $e \in S^{2n}$. Let $i_1, i_2$ be the inclusions $x \mapsto (x, e)$ and $x \mapsto (e, x)$ respectively of $S^{2n}$ in the product. Write $i : pt \to S^{2n}$; $pt \mapsto e$. Consider the commutative diagram

$$\begin{array}{ccccc} & & K(S^{2n}) & \overset{*}{\leftarrow} & K(S^{2n} \otimes K(pt) \\ & \nearrow & \uparrow & & \uparrow \\ K(S^{2n}) & \overset{\mu^*}{\rightarrow} & K(S^{2n} \times S^{2n}) & \overset{*}{\leftarrow} & K(S^{2n}) \otimes K(S^{2n}) \\ & \searrow & \downarrow & & \downarrow \\ & & K(S^{2n}) & \overset{*}{\leftarrow} & K(pt) \otimes K(S^{2n}) \end{array}$$

The horizontal maps are isomorphisms. The vertical maps on the left are $i_1^*$ on top and $i_2^*$ at the bottom. Now, $K(S^{2n}) \cong \mathbf{Z}[x]/(x^2)$ and $i^*(x) = 0$ since $x \in \widetilde{K}(S^{2n})$. So, the left side of the above diagram consisting of the two triangles can be written as

$$\begin{array}{ccc} & & \mathbf{Z}[\alpha]/(\alpha^2) \\ & \nearrow & \uparrow \\ \mathbf{Z}[\gamma]/(\gamma^2) & \overset{\mu^*}{\rightarrow} & \mathbf{Z}[\alpha, \beta]/(\alpha^2, \beta^2) \\ & \searrow & \downarrow \\ & & \mathbf{Z}[\beta]/(\beta^2) \end{array}$$

where the top vertical arrow sends $\beta$ to 0 and the bottom one takes $\alpha$ to 0. Therefore, the horizontal arrow $\mu^*$ is given by

$$\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$$

for some integer $m$. Hence

$$0 = \mu^*(\gamma^2) = \mu^*(\gamma)^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0$$

which is a contradiction.

**Lemma.** *If $S^{2n-1}$ is an H-space, then there exists a map $f : S^{4n-1} \to S^{2n}$ which has Hopf invariant $\pm 1$.*
Here, the *Hopf invariant* of a map $f : S^{4n-1} \to S^{2n}$ is defined as follows.

Consider the space $C_f = S^{2n} \cup_f D^{4n}$ (mapping cone) obtained by means of the attaching function $f$. In other words, $C_f$ is obtained by attaching a $4n$-cell to the $2n$-sphere. As the even spheres $S^{2n}$ and $S^{4n}$ have trivial $\widetilde{K}^1$, the maps

$$S^{2n} \overset{i}{\hookrightarrow} C_f \overset{q}{\to} S^{4n}$$

give rise to the long exact sequence (which is actually short):

$$0 \to \widetilde{K}(S^{4n}) \overset{q^*}{\to} \widetilde{K}(C_f) \overset{i^*}{\to} \widetilde{K}(S^{2n}) \to 0.$$

As the groups on the left and right extremes are infinite cyclic, generated respectively by $(([H]-1)^*)^{2n}$ and by $(([H]-1)^*)^n$, the element $\alpha \in \widetilde{K}(C_f)$ which is the image of the generator $(([H]-1)^*)^{2n}$ and ANY element $\beta \in \widetilde{K}(C_f)$ mapping to the generator $(([H]-1)^*)^n$ are related by

$$\beta^2 = h\alpha$$

for some integer $h$. The integer $h$ is independent of the choice of $\beta$ and is called the Hopf invariant of $f$.

We reiterate the last statement:
*The integer $h$ is independent of the choice of $\beta$.*
Any other lift is of the form $\beta + m\alpha$. Now,

$$i^*(\alpha\beta) = i^*(\alpha)i^*(\beta) = 0.$$

This means $\alpha\beta \in ker(i^*) = Im(q^*)$ so that $\alpha\beta = d\alpha$ for some integer $d$. Then,

$$d^2\alpha = d\alpha\beta = \alpha\beta^2 = h\alpha^2 = 0$$

which gives $d^2 = 0$ as $\alpha$ is a generator of an infinite cyclic group. Thus, we have $d = 0$ and so $\alpha\beta = 0$. So,

$$(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta + m^2\alpha^2 = \beta^2.$$

**Proposition.** *If $S^{2n-1}$ is an $H$-space, then there exists a map $f$ with Hopf invariant $\pm 1$ from $S^{4n-1}$ to $S^{2n}$.*

**Proof.**

Let $\mu : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ be a multiplication map with an identity element $e$. We consider the $2n$-dimensional sphere $S^{2n}$ as a union of two discs $S^{2n} = D^{2n}_{+} \cup D^{2n}_{-}$ and the $(4n-1)$-dimensional sphere $S^{4n-1}$ as the boundary of the product of two discs $D^{2n}$:

$$S^{4n-1} = \delta(D^{2n} \times D^{2n}) = \delta D^{2n} \times D^{2n} \cup D^{2n} \times \delta D^{2n}.$$

We obtain a map

$$f : S^{4n-1} \to S^{2n}$$

by fitting together the two continuous maps

$$\delta D^{2n} \times D^{2n} \to D^{2n}_{+};$$

$$(x, y) \mapsto |y|\mu(x, y/|y|)$$

and

$$D^{2n} \times \delta D^{2n} \to D^{2n}_{-};$$

$$(x, y) \mapsto |x|\mu(x/|x|, y).$$

Note that the above maps are meaningful (and continuous) at $x = 0$ and $y = 0$ and agree with $\mu$ on $S^{2n-1} \times S^{2n-1}$.

Recall the mapping cone $C_f = S^{2n} \cup_f D^{4n}$. Then, we have a map

$$\Phi : (D^{4n}, S^{4n-1}) = (D^{2n} \times D^{2n}, \delta(D^{2n} \times D^{2n})) \to (C_f, S^{2n})$$

where the map from $D^{4n}$ to $C_f = S^{2n} \cup_f D^{4n}$ is the obvious map. Therefore, we have a commutative diagram

$$
\begin{array}{ccc}
\widetilde{K}(C_f) & \rightarrow & \widetilde{K}(S^{2n}) \\
\uparrow & & \uparrow \\
K(C_f, D^{2n}_+) & \rightarrow & K(S^{2n}, D^{2n}_+) \\
\downarrow & & | \\
K((D^{2n}, \delta D^{2n}) \times D^{2n}) & & | \\
\downarrow & & \downarrow \\
K((D^{2n}, \delta D^{2n}) \times \{e\}) & \stackrel{f|^*}{\leftarrow} & K(D^{2n}_-, \delta D^{2n}) \\
\uparrow & & \\
\widetilde{K}(S^{2n}) & &
\end{array}
$$

In this diagram, the top two vertical arrows are isomorphisms since the reduced K-group of $D^{2n}_+$ is trivial. The top horizontal arrow induced by the inclusion of $S^{2n}$ in $C_f$ takes $\beta$ to a generator. The long vertical arrow on the right is an isomorphism by excision. The second from top vertical arrow on the left is $\Phi^*$ and the vertical arrows below it are isomorphisms induced by inclusion of $\{e\}$ and the quotient map from $(D^{2n}, \delta D^{2n}) \times \{e\}$ onto $S^{2n}$. Finally, the horizontal arrow $f|^*$ is an isomorphism since the restricted function $f|$ gives a homeomorphism from $D^{2n} \times \{e\}$ to $D^{2n}_-$. A similar diagram holds with $+$ replaced by $-$.

To deduce that $\beta^2 = \pm\alpha$, we just need to look at the following commutative diagram and follow where elements are carried to:

$$
\begin{array}{ccc}
\widetilde{K}(C_f) \otimes \widetilde{K}(C_f) & \stackrel{prod}{\rightarrow} & \widetilde{K}(C_f) \\
\uparrow & & \uparrow \\
K(C_f, D^{2n}_+) \otimes K(C_f, D^{2n}_-) & \stackrel{prod}{\rightarrow} & K(C_f, S^{2n}) \\
\downarrow & & \uparrow \\
K((D^{2n}, \delta D^{2n}) \times D^{2n}) \otimes K(D^{2n} \times (D^{2n}, \delta D^{2n})) & \stackrel{prod}{\rightarrow} & K(D^{2n} \times D^{2n}, \delta(D^{2n} \times D^{2n})) \\
\downarrow & \nearrow & \uparrow \\
K((D^{2n}, \delta D^{2n}) \times \{e\}) \otimes K(\{e\} \times (D^{2n}, \delta D^{2n})) & & \widetilde{K}(S^{2n}) \\
\uparrow & \nearrow & \\
\widetilde{K}(S^{2n} \otimes \widetilde{K}(S^{2n}) & &
\end{array}
$$

The top horizontal arrow takes $\beta \otimes \beta$ obviously to $\beta^2$. The top right vertical arrow carries a generator to $\alpha \in \widetilde{K}(C_f)$. The second-from-top vertical arrows are $\Phi^* \otimes \Phi^*$ and $\Phi^*$. The north-east bound arrows are isomorphisms induced by the star maps. Other unspecified maps are clearly induced by inclusions or quotient maps. Since the previous diagram showed that $\beta \in \widetilde{K}(C_f)$ corresponded to a generator $g$ of $\widetilde{K}(S^{2n})$, following what happens to $g \otimes g \in \widetilde{K}(S^{2n}) \otimes \widetilde{K}(S^{2n})$ on the bottom left, we find that $\beta^2 = \pm\alpha$.

## 14.4 Proof of Adams's theorem.

On combining the above observations, the parallelizability/division algebras theorem is a consequence of the following beautiful theorem due to J.F.Adams. His second proof of the theorem in 1966 uses his construction of the so-called Adams operations - his first proof in 1960 used secondary cohomology operations.

**Adams's theorem.** *There is a map $f$ of Hopf invariant $1$ or $-1$ from $S^{4n-1}$ to $S^{2n}$ if, and only if, $n = 1, 2, 4$.*

**Definition.** *If $X$ is a compact, Hausdorff space, then a sequence $\psi^n (n \geq 1)$ of ring homomorphisms from $K(X)$ to itself satisfying the following properties, are called Adams operations:*
(i) $\psi^m \circ \psi^n = \psi^{mn}$;
(ii) $\psi^n f^* = f^* \psi^n \ \forall \ f : X \to Y$;
(iii) $\psi^n(L) = L^n$ *for any line bundle* $L$;
(iv) *for each prime $p$, $\psi^p(x) \equiv x^p$ mod $p$ for $x \in K(X)$.*

**Theorem.** *Adams operations exist.*

The proof will depend on a splitting principle and the following observations. If $E = \oplus_{i=1}^r L_i$ is a sum of line bundles $L_i$'s, then the property $\psi^n(E) = \sum_{i=1}^r L_i^n$ is required above and we could define $\psi^n$ in this manner. The idea of defining $\psi^n(E)$ for general $E$ comes from this itself because the sum on the RHS is expressible as a polynomial in the exterior powers of $E$ and this makes sense for general $E$ (even if $E$ is not a sum of line bundles) and we could define $\psi^n$ for general $E$ also. We will make it precise presently. First, we state the splitting principle.

**Splitting principle.** *For a compact, Hausdorff space $X$ and a complex vector bundle $p : E \to X$, there is a compact, Hausdorff space $Y$ and a map $f : Y \to X$ such that $f^* : K(X) \to K(Y)$ is injective, and the vector bundle $f^*(E) \to Y$ is a direct sum of line bundles.*

**Proof of existence of Adams operations.**

Clearly, the identity

$$\Lambda^k(V \oplus V') = \sum_{d=0}^{k} \Lambda^d(V) \otimes \Lambda^{k-d}(V')$$

shows that if $E = \oplus_{i=1}^r L_i$, then

$$\lambda^k(E) = \sigma_k(L_1, \cdots, L_r),$$

the $k$-th elementary symmetric polynomial in the $L_i$'s. Indeed, we see this as follows. Consider a vector bundle $E$ and the corresponding generating function of the exterior power bundles; viz.,

$$\lambda_T(E) := \sum_{d \geq 0} \lambda^d(E) T^d \in K(X)[T].$$

Therefore, we obtain for any vector bundles $E, E'$, the relation

$$\lambda_T(E \oplus E') = \lambda_T(E)\lambda_T(E').$$

Noting that

$$\lambda_T(L) = 1 + LT$$

for any line bundle $L$, we obtain

$$\lambda_T(\oplus_{i=1}^r L_i) = \prod_{i=1}^{r}(1 + L_i T).$$

Hence, we deduce that

$$\sigma_n(L_1, \cdots, L_r) = \lambda^n(\oplus_{i=1}^r L_i).$$

From the theory of elementary symmetric functions, for variables $x_1, \cdots, x_r$, each $x_1^n + \cdots + x_r^n$ is a polynomial (these are the Newton polynomials) in the elementary symmetric polynomials $\sigma_1 := \sum_{i=1}^r x_i, \sigma_2 := \sum_{i<j} x_i x_j$, etc. Denoting this polynomial to be $s_n(x_1, \cdots, x_r)$, let us define for each $E \in Vect_{\mathbf{C}}(X)$,

$$\psi^n(E) := s_n(\lambda_1(E), \cdots, \lambda_n(E))$$

where $\lambda_r(E)$ is the $r$-th exterior power.
If $f : X \to Y$, then since $f^*(\lambda^n(E)) = \lambda^n(f^*(E))$, we have

$$f^*\psi^n(E) = \psi^n(f^*(E)).$$

Using (ii) and the splitting principle repeatedly, it follows that

$$\psi^n(E \oplus E') = \psi^n(E) + \psi^n(E')$$

assuming this is true for $E, E'$ which are direct sums of line bundles. So, we assume $E = \oplus_{i=1}^r L_i, E' = \oplus_{j=1}^s L'_j$. But, in this case it is verified by the very definition. Hence, $\psi^n$ extends to a homomorphism of abelian groups from $K(X)$ to itself. Thus, we have (ii),(iii). It is also useful sometimes to write the relation between the generating functions (which are power series over $K(X)$ in general). Indeed, since we have the power series identity

$$\frac{d}{dT}\log(1 - Tx) = \frac{-x}{1 - Tx} = -x - Tx^2 - T^2x^3 - \cdots$$

implies that one may define

$$\psi_T(E) := \sum_{i \geq 0} \psi^i(E)T^i := \psi^0(E) - T\frac{d}{dT}\log \lambda_{-T}(E).$$

To see that $\psi^n$ is a ring homomorphism, it suffices to verify at the level of vector bundles that

$$\psi^n(E \times E') = \psi^n(E)\psi^n(E').$$

Once again, by the splitting principle, it suffices to prove this when $E, E'$ are direct sums of line bundles. But then

$$\psi^n(E \otimes E') = \psi^n(\oplus_{i=1}^r \oplus_{j=1}^s (L_i \otimes L'_j)) = \sum_{i,j}(L_i \otimes L'_j)^n$$

$$= \sum_{i,j} L_i^n(L'_j)^n = (\sum_i L_i^n)(\sum_j (L'_j)^n) = \psi^n(E)\psi(E').$$

Now,

$$\psi^m\psi^n(L) = \psi^m(L^n) = (L^n)^m = L^{mn} = \psi^n\psi^m(L)$$

for each line bundle. Using additivity proved above and rhe splitting principle, (i) follows. Finally, for a prime $p$, once again using splitting principle, it suffices to verify

$$\psi^p(\oplus_{i=1}^r L_i) \equiv (L_1 + \cdots + L_r)^p \ mod \ p.$$

However, the LHS is simple $L_1^p + \cdots + L_r^p$ which equals the above RHS in $K(X)$. The proof is complete.

Now, we proceed to prove Adams's theorem after making one observation on the effect of Adams operations on the reduced K-groups.

**Lemma.** *The Adams operations also give operations on $\widetilde{K}(X)$ which satisfy:*
(i) $\psi^n(x * y) = \psi^n(x) * \psi^n(y)$;
(ii) *if $X = S^{2n}$, then $\psi^m$ acts on $\widetilde{K}(S^{2n})$ as multiplication by $m^n$.*
**Proof.**
By naturality, we clearly get Adams operations on reduced K-theory as:

$$\begin{array}{ccccccccc} 0 & \to & \widetilde{K}(X) & \to & K(X) & \to & K(pt) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{K}(X) & \to & K(X) & \to & K(pt) & \to & 0 \end{array}$$

where the middle and right vertical arrows are $\psi^n$ and the left vertical arrow is as a consequence. This proves (i).
For (ii), as usual $H$ denotes the dual of the canonical line bundle on $S^2$ and consider $\psi^m([H]-1)$. We have

$$\psi^m([H]-1) = [H]^m - 1 = (1 + ([H]-1))^m - 1 = m([H]-1)$$

since $[H]-1)^2 = 0$. Therefore,

$$\psi^m((([H]-1)^*)^n) = ((\psi^m([H]-1))^*)^n = ((m([H]-1))^*)^n = m^n((([H]-1))^*)^n.$$

**Proof of Adams's theorem.**
We will leave the part that $n = 1, 2$ or $4$ give Hopf invariant $\pm 1$ for the exercise session and prove the nontrivial part.
Let $f : S^{4n-1} \to S^{2n}$ have Hopf invariant $\pm 1$.
Recall the mapping cone $C_f = S^{2n} \cup_f D^{4n}$ obtained (by means of $f$) by attaching an $n$-cell to $S^{2n}$. As before, let $\alpha, \beta \in \widetilde{K}(C_f)$ be, respectively, the

image under $q^*$ of $(([H]-1))^*)^{2n}$ and a preimage of $(([H]-1))^*)^n$ under the map $i^*$ in the short exact sequence

$$0 \to \widetilde{K}(S^{4n}) \xrightarrow{q^*} \widetilde{K}(C_f) \xrightarrow{i^*} \widetilde{K}(S^{2n}) \to 0.$$

Now $\psi^m(\alpha) = m^{2n}\alpha$ and $\psi^m(\beta) = m^n\beta + a_m\alpha$ for some integer $a_m$. Apply this for $m = 2$. As we have the properties

$$\psi^2(\beta) = 2^n\beta + a_2\alpha$$

and

$$\psi^2(\beta) \cong \beta^2 \ mod\ 2,$$

we get that $a_2$ is odd because $\beta^2 = \pm\alpha$ by the assumption that Hopf invariant is $\pm 1$. Now,

$$\psi^6(\beta) = \psi^3(\psi^2(\beta)) = \psi^3(2^n\beta + a_2\alpha)$$
$$= 2^n(3^n\beta + a_3\alpha) + a_2(3^{2n}\alpha) = 6^n\beta + (2^n a_3 + 3^{2n}a_2)\alpha.$$

Also,

$$\psi^6(\beta) = \psi^2(\psi^3(\beta)) = \psi^2(3^n\beta + a_3\alpha)$$
$$= 3^n(2^n\beta + a_2\alpha) + a_3(2^{2n}\alpha) = 6^n\beta + (3^n a_2 + 2^{2n})\alpha.$$

Therefore, we have

$$(3^{2n} - 3^n)a_2 = (2^{2n} - 2^n)a_3.$$

As $a_2$ is odd, it follows that $2^n|(3^n - 1)$. An elementary exercise tells us that $n = 1, 2$ or $4$.
This finishes the proof of Adams's theorem and along with that the proof of the parallelizability/division algebras theorem modulo the splitting principle.

The splitting principle itself follows from the computation of $K(\mathbf{C}P^n)$ along with the following theorem on cell complexes.

**Leray-Hirsch theorem.** *Let $p : E \to X$ be a complex vector bundle with $E, X$ compact, Hausdorff and fibre $F$ a finite cell complex with only even-dimensional cells. Suppose there exist $c_1, \cdots, c_r \in K(E)$ that restrict to a basis for $K(p^{-1}(x))$ for each fibre $p^{-1}(x)$. Then, $K^*(E)$ is a free $K^*(X)$-module with $\{c_1, \cdots, c_r\}$ a basis.*

**Example of $\mathbf{C}P^n$.**
$K(\mathbf{C}P^n) \cong \mathbf{Z}[T]/(T-1)^{n+1}$ *by means of the map $T \mapsto L$ from the RHS where $L$ is the dual of the canonical line bundle on $\mathbf{C}P^n$.*

The resolution of the monstrous moonshine conjecture by Borcherds (for which he won the Fields medal) uses Adam operations in a nontrivial way.

The basic reference is Atiyah's lecture notes on K-theory and some online notes by A.Hatcher.