CHAT TIME, SAM!

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Word games

Suppose you want to change the word

'SINK'

to

'SWIM'

through a chain of meaningful English words.

The only condition is that at each step, you are allowed to change only one letter.

For simplicity, let us say all meaningful words have a vowel.

Here is a possible chain.

SINK
PINK
PINT
MINT
MIST
LIST
LOST
LOOT
SOOT
SLOT
SLIT
SLIM
SWIM !

This may not be the shortest or the only path but there is something that we notice here. What is it?

There is at least one word with at least two vowels.

Question: is that forced on us no matter which chain we use?

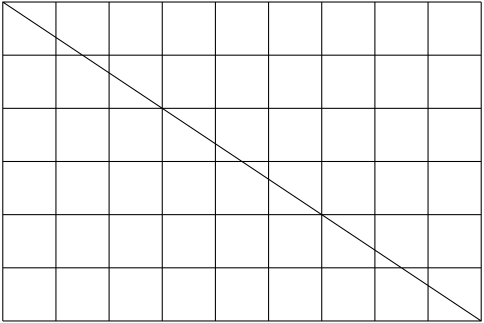
Yes! And the reason is simple.

Let us say each word has value equal to the number of vowels.
The initial value and final value are both 1 in our problem.
The value 1 can become 0,1 or 2 at the next stage.
But, at any stage, 0 is not allowed; if value 2 is reached, we are done.
So, only case to look at is if each word in the chain has value 1.

Concentrate on the vowel ‘I’ in the first word SINK.
As the value of the next word is also 1, this vowel remains a vowel at the next step.
Thus, a vowel continues in the 2nd place at each step!
But then it would persist until the end - which is not so in SWIM.

Rectangular grid

Draw a rectangular grid made up of unit squares; for example, a rectangle of length 9 units and height 6 units.
Draw a diagonal - say, from the left upper corner to the lower right corner.
How many of the small unit squares does this diagonal pass through?



What about a 18×12 rectangle?
A 42×18 rectangle?
How about a general $m \times n$ rectangle?

As an exercise in co-ordinate geometry, show that the number is $m + n - (m, n)$, where (m, n) is the greatest common divisor of m and n.

What is the shape?

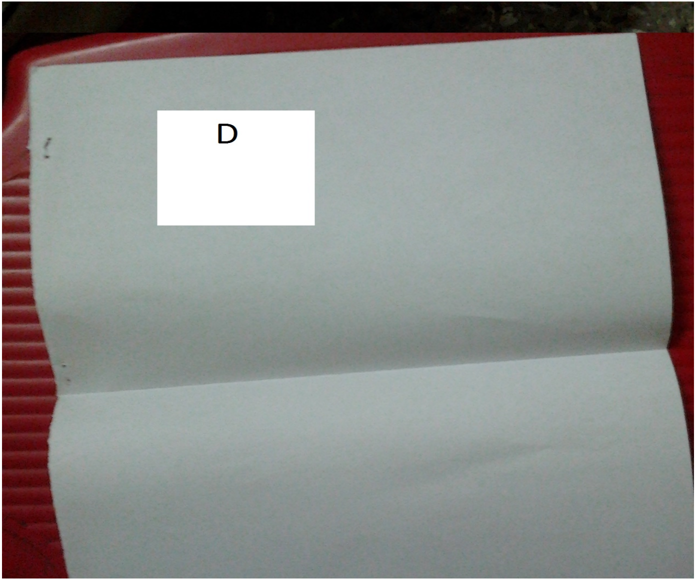
Take a thin rectangular piece of paper.
Make a knot without crumpling or tearing the paper.
Flatten the paper and fold or snip off the extra parts sticking out from the knot.
What is the shape you get?
Explain why.....

Folds

Again, take a rectangular piece of paper and make folds as follows. First, fold the paper into half by bringing the bottom edge above to match with the top edge.

On this folded sheet, perform the same operation - that is, fold into half by bringing the bottom edge on top to match the top edge.

If you do this a number of times, say 10, times, and unfold the paper, there will be crests and troughs (ups and downs) on the papers.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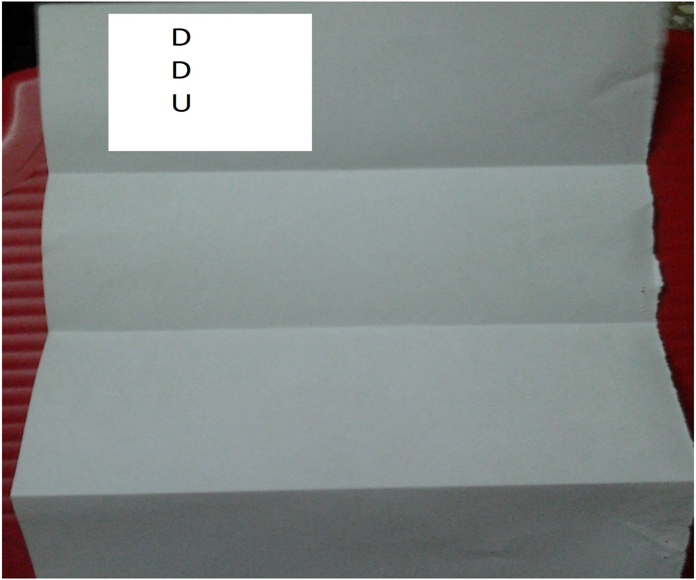
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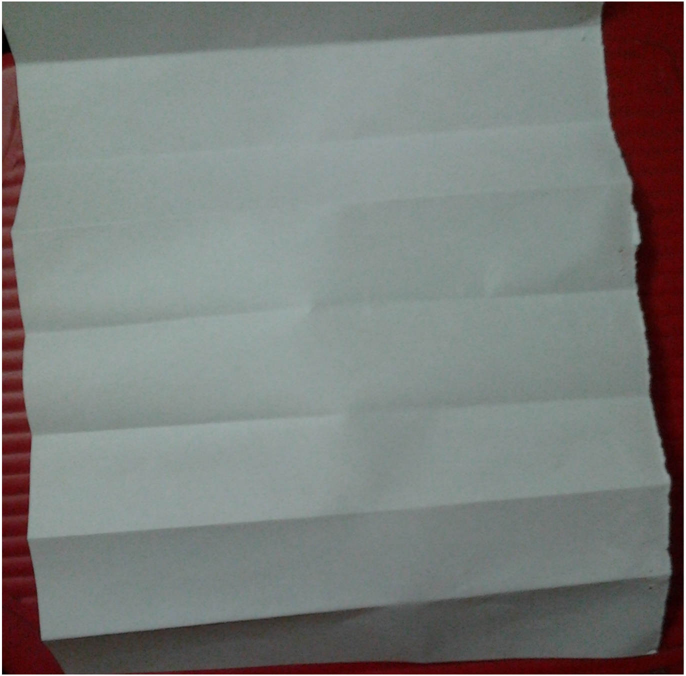

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Question : What is the number of lines?
Observing from top to bottom, what is the pattern of ‘ups’ and ‘downs’?
What is the number of lines for a general number n in place of 10?

A much more difficult question for general n, is the pattern of ups and downs...

Can you see that the pattern for $n+1$ folds starts with the pattern of n folds?

What can we say about the sequence of the odd-numbered creases?

Can you see that the $2r$-th crease is the same as the r-th crease?

Is it eventually periodic?

The answer is NO!

This sequence is mysterious - I will let you think about it.

If you can't beat them, join them

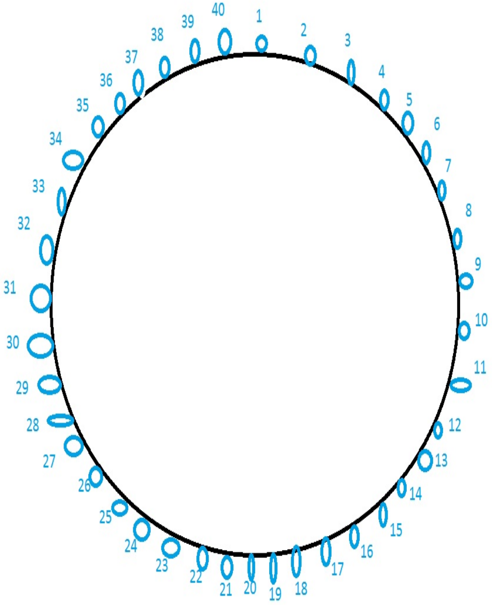
Story of Josephus:

Flavius Josephus and 39 of his comrades were surrounded when holding a revolt against the Romans during the 1st century A.D.
Rather than become slaves, they decided to kill themselves.
They arranged themselves along a circle.
Starting somewhere, they went clockwise around the circle and every 7th person was eliminated. This continued with the 7th among the surviving ones being killed at each step.
Apparently, Josephus was a clever mathematician and arranged himself in such a position that he would be the last survivor.
The story goes that he did not kill himself but came and joined the Romans!
Find out Josephus's position.

The general problem is of n people, designated by $1, 2, \cdots, n$ in clockwise order, say.

Each d-th person is eliminated going around in the clockwise direction.

What is the position of the last survivor?



FIRST FEW CASUALTIES:

7, 14, 21, 28, 35, 2, 10, 18, 26, 34, 3, 12, 22, ...

This is a problem about permutations!
If a_r is the r-th person to be eliminated, we have a permutation $a_1, a_2, \cdots, a_n$ of the numbers $1, 2, \cdots, n$.
The last survivor is a_n;
There is no known nice closed formula for a_n in terms of n and d in general although the permutation itself can be described explicitly!
Fortunately, we can find a formula when every second person is killed; that is, when $d = 2$.

This formula can be neatly expressed in binary digits, viz.
If $n = d_r d_{r-1} \cdots d_1 d_0$ is the binary expansion of n, then assuming that we start with eliminating 2, then 4 etc., we have:
$d_{r-1} d_{r-2} \cdots d_1 d_0 d_r$ as the binary digit representing the position of the last person to be killed!
For example, if the number of people is $26 = (11010)_2$, then the person to be killed last is $(10101)_2 = 21$.
So, if A to Z are sitting cyclically and every second person is killed, then U remain(s) until the end!

In Josephus's problem, the casualties in order are:

7,14,21,28,35,2,10,18,26,34,3,12,22,31,40,11,23,33,5,17,30,
4,19,36,9,27,6,25,8,32,16,1,38,37,39,15,29,13,20,24.

In general, among n people with people getting eliminated as $d, 2d$ etc., the r-th person to be eliminated is

$$(1, 2, \cdots, n)^{d-1}(2, 3, \cdots, n)^{d-1} \cdots (n-1, n)^{d-1}(r)$$

where we read from right to left.

Given enough rope

Suppose we are given an unlimited supply of identical ropes which have the following property.
Each rope burns uniformly and takes exactly an hour to burn out.
Without a watch at our disposal, what times are measurable if we are given enough rope!
For instance, if we light both ends of a rope simultaneously, the rope burns out in exactly half an hour!

The question is what durations of time are measurable given n ropes?
In fact, in an idyllic world, we may imagine an infinite number of ropes and can wonder what times are measurable!
Let us start with some simple subproblems:
How do we measure 45 minutes?
This is simple:
take two ropes; light both ends of the first rope and one end of the second rope simultaneously; when the first rope burns out (in half an hour), light the other end of the second rope.

How to measure less than half an hour? Say 15 minutes?
In the previous measurement of 45 minutes, start counting after the first rope burns out (in half an hour)!
The above idea tells us how to measure any time of $\frac{1}{2^n}$ hours; light n ropes simultaneously with the first rope being lit on both ends;
When the first rope finishes burning, light the other end of rope 2; when it burns itself out, light the other end of rope 3 etc.
Finally, $n-1$ ropes have burnt themselves out in after $\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$ hours; as the n-th rope will burn out in another $\frac{1}{2^{n-1}}$ hours but, by lighting its other end at that moment, makes it burn out in half that time (that is, in $\frac{1}{2^n}$ hours).

The above thought-process tells us how to count durations of hours of the form $\frac{1}{2^{n+r}} + \frac{1}{2^{n+r-1}} + \cdots + \frac{1}{2^n}$.
I leave you with determining all possible fractions of hours which can be determined in this manner.
For instance, can we measure $5/16$-th of an hour?

Those 9's again

The decimal expansion of $\frac{1}{7} = 0.\overline{142857}$ which has a period of length 6.
In general, for any prime $p > 5$, one can show that the decimal expansion of $1/p$ has period a divisor of $p - 1$.

As an exercise, prove that this period d is the smallest number such that $99 \cdots 9$ (repeated d times) is a multiple of p (!)

Let us note more patterns:
Write $\frac{1}{7} = 0.\overline{(142)(857)}$ where the decimal of length 6 is broken into two parts of length three: $142, 857$; their sum is 999.

What is more, in the above decimal expansion of $1/7$, one could have considered three parts $U = 14, V = 28, W = 57$ and we see $U + V + W = 10^2 - 1$.

Look at $1/17 = 0.\overline{0588235294117647}$.
The splitting of period 16 as $8 + 8$ gives the sum $10^8 - 1$;
the choice $4 + 4 + 4 + 4$ gives the sum $2(10^4 - 1)$ and
the splitting into length 2 decimals gives the sum $4(10^2 - 1)$.
A nice exercise is to prove that if the period is a multiple of r, one always has a sum of the form $k(10^r - 1)$.
Determination of k is a more difficult exercise needing more mathematics.

Interlacing shuffle

Place the cards numbered $1, 2, \cdots, 2n$ in two piles : 1 to n on the left and $n+1$ to $2n$ on the right.
Pick them up alternately from right to left as

$$n+1, 1, n+2, 2, \cdots, 2n, n$$

Again place two piles as the first n on the left and the next n on the right.
Continue interlacing as before.
After some repetition, we will have the original order!

Question : what is the minimum number of repetitions needed to reproduce the original order?

Prove that this is the smallest d such that $2^d - 1$ is a multiple of $2n + 1$.

Think of d equal piles (in place of 2) and solve he problem!

Nim

Place buttons or coins in 3 rows (there can be different number of coins in each row).
For example, there could be 9,5 and 12 coins.
Two players play this game.
First, a player takes some coins from any row (she is allowed to take coins only from one row and she has to take at least one coin but she can take any number of coins from that row including all the coins on that row also).
The next player does the same (takes coins from any one row).
This way, they alternate.
Finally, the player who removes the last coin is the winner.

Here also, the strategy is based on writing any number in binary form.
For example, the numbers $9, 5, 12$ are:

$$1001$$

$$0101$$

$$1100$$

I have deliberately written 0101 instead of 101 for 5 so that the strategy can be explained easily.
The sums of the columns from left to right are:
$2, 2, 0, 2$. So, all of them happen to be even.

Let us call this a *safe combination*.
I claim:
The person whose turn it is to play when faced with a safe combination, can be made to lose.
Moreover, if at some stage the numbers written in binary form have a non-safe combination (that is, at least one column sum is odd), then the player to play now, can play in such a way that she can make this into a safe combination.
For example, if we have $9, 5, 12$ coins and your opponent has to play. you can defeat her.

For instance, if she removes 2 coins from the 9 coins, the rows have $7, 5, 12$ which are:

$$0111$$

$$0101$$

$$1100$$

To transform the existing column sums $1, 3, 1, 2$ to a safe combination, you need to make them $0, 2, 2, 2$; that is,
the last row consisting of the biggest number should change from 1100 to 0010.
So, you should leave out just 2 coins in the last row of 12 (that is, take away 10 coins).
From this new safe combination

$$111$$

$$101$$

$$010$$

Whatever your opponent does, she has to make an unsafe combination because she can remove coins only from one row which means she makes at least 1 on that row to be 0.

To summarize, each time, you just add the columns in binary and find there is a unique binary expansion "for the largest row" which makes the combination safe.
Change your row to the number of coins corresponding to this unique binary expansion.
The same strategy is valid for any number of rows rather than just 3 but playing with 3 is complicated enough!