Howlett-Lehrer's theorem from 1979-80

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ATM Workshop, TIFR
December 17, 2011

Cusp form philosophy

The irreducible unitary (infinite-dimensional) representations of a semisimple Lie group and the complex representations of a finite group of Lie type have common features governing them.
"*Philosophy of cusp forms*":
Theme popularized famously by Harish-Chandra in a paper titled "Eisenstein series over finite fields".
Classical theory of cusp forms contain a philosophy which can also be adopted to the study of representations of finite groups of Lie type.

T.A.Springer wrote a more detailed exposition titled “Cusp forms for finite groups”, Algebraic groups and related finite groups, Springer LNM, 1968).
A brief, rough description of the philosophy following Springer:

Consider the finite group $G(k)$ for a reductive, algebraic group G over a finite field k.
The group algebra $\mathbf{C}[G(k)]$ of $G(k)$ is the set of complex-valued functions on $G(k)$ under the convolution product.
For each k-parabolic subgroup $P = MU$ of G and each function f in the group algebra, define

$$f_P(g) := \sum_{u \in U(k)} f(gu) \ \ \forall \ \ g \in G(k)$$

If f_P is the zero function for every *proper* k-parabolic subgroup P, one calls f a cusp form.

The subset $C(G)$ of all cusp forms, forms a two-sided ideal in $\mathbf{C}[G(k)]$.
If $P = MU$ is a k-parabolic subgroup and V is any representation of $M(k)$ occurring in $C(M)$, then one can extend it to $P(k)$ by extending trivially on U. It turns out that this induces a representation $Ind_{P(k)}^{G(k)}(V)$ of $G(k)$ which is independent of the choice of P containing M as a Levi component and depends only on the equivalence class of V.

Here, and elsewhere, we abuse notation to write $I_M^G(V)$ to mean the representation induced from $P(k)$ after extending the representation V of $M(k)$ trivially on $U(k)$.
$\mathbf{C}[G(k)]$ decomposes into a direct sum of two-sided ideals I_P where the irreducible representations of $G(k)$ occurring in I_P are the constituents of $I_M^G(V)$ as V varies over the irreducible representations of $M(k)$.

Therefore, one may break up the study of representations of $G(k)$ into two problems :
(i) determine $C(G)$ and
(ii) decompose $I_M^G(V)$ into irreducibles for $V \in C(M)$, where M is the Levi component of a proper parabolic k-subgroup of G (with the understanding that $C(M)$ is "known" for proper k-parabolic subgroups).
As these Levi subgroups are groups of the same type as G and have smaller rank, this would give an inductive procedure to determine all representations of G.

The first problem was solved by Deligne-Lusztig and the second one by Howlett & Lehrer; we discuss the latter. Roughly speaking, their result shows that the centralizer algebra (also called a generalized Hecke algebra) of an induced representation of a cuspidal representation (= cusp form) for a proper parabolic k-subgroup, is a twisted form of the group algebra of a "ramification group" - which is '*almost*' a reflection group.
The Howlett-Lehrer theorem gives a presentation of the Hecke algebra which implies an isomorphism with a suitable group algebra (this had been conjectured earlier by Springer).

Howlett-Lehrer's presentation theorem has been generalized by Geck-Hiss-Malle to representations in positive characteristic also; their presentation is not as neat but Ackermann later gave a proof to show that a very similar presentation exists.
A classial result of Mackey shows :
The irreducible representations of $G(k)$ occurring in $I_M^G(V)$ are in bijection with the simple modules of the centralizer algebra of $I_M^G(V)$ in $\mathbf{C}[G(k)]$.

Notations and setting

Recall:
A group G has a split (B, N)-pair of characteristic $p > 0$ and Weyl group W, if G has a (B, N)-pair with Weyl group W, B is the semidirect product of a normal p-subgroup U and the subgroup $T = B \cap N$ is an abelian group of order prime to p and $T = \bigcap_{n \in N} nBn^{-1}$.
The finite groups of Lie type of characteristic p and semi-simple algebraic groups over $\overline{\mathbf{F}_p}$ have split (B, N)-pairs of characteristic p.

G - reductive algebraic defined over a finite field k of characteristic p; we look at complex representations of the finite group $G^{Frob} = G(k)$.

An irreducible character χ of $G(k)$ is *cuspidal* if $\sum_{u \in U(k)} \chi(up) = 0$ for every proper parabolic k-subgroup $P = UM$ of G and each $p \in P(k)$.

Using Frobenius reciprocity, equivalently:

$< \chi, 1_{U(k)}^{G(k)} >= 0$ for every proper parabolic k-subgroup $P = UM$ of G and also:

$\sum_{u \in U(k)} \chi(ug) = 0$ for all $g \in G(k)$ and $P = UM$ any proper, standard parabolic k-subgroup of G.

For each parabolic k-subgroup P of G, each irreducible character ϕ of $M(k)$ extends to an irreducible character $\phi_{P(k)}$ by extending it trivially on $U(k)$.
Induced it to $G(k)$ (usually called parabolic induction or Harish-Chandra induction).
The actual induction from $M(k)$ to $G(k)$ gives representations which are too big.

Easy fact:
Let χ be an irreducible character of $G(k)$. Then, there exists a standard parabolic k-subgroup $P = UM$ and a cuspidal irreducible character ϕ of $M(k)$ such that $< \chi, \phi_{P(k)}^{G(k)} > \neq 0$.
In fact, this standard parabolic P_J corresponds to the minimal subset J of the set of k-simple roots such that $(\chi_{U_J(k)}, 1_{U_J(k)}^{G(k)}) \neq 0$.

Look at the permutation module $1_{U(k)}^{G(k)} = \mathbf{C}[G(k)/U(k)] = \mathbf{C}[G(k)]e_U$ where the idempotent $e_U = \frac{1}{|U(k)|}\sum_{u \in U(k)} u$. It can be shown (using the Mackey formula and induction on the semisimple rank) that it is isomorphic to $1_{V(k)}^{G(k)}$ where M is the Levi subgroup of two different parabolic k-subgroups UM and VM.

In fact, if UM is a standard k-parabolic, VM is a k-parabolic with ${}^{w}(VM)$ standard, then the map

$$\mathbf{C}[G(k)]e_{wVw^{-1}} \to \mathbf{C}[G(k)]e_U \ ; \ x \mapsto xe_{wVw^{-1}}we_U$$

is an $\mathbf{C}[G(k)]$-isomorphism. This will yield a $\mathbf{C}[G(k)] - \mathbf{C}[M(k)]$-bimodule isomorphism.

The theorem of Mackey alluded to is :
Let H be any finite group and let χ_1, χ_2 be characters of complex representations of subgroups Q_1, Q_2. Write χ_i^H for the induced characters on H and, for each $x \in H$, write ${}^x\chi_2$ for the character of ${}^xQ_2$ defined as

$$ {}^x\chi_2({}^xq_2) = \chi_2(q_2) $$

If R is a set of double coset representatives for $Q_1 \backslash H / Q_2$, Mackey's formula asserts that

$$ (\chi_1^H, \chi_2^H)_H = \sum_{x \in R} (\chi_1, {}^x\chi_2)_{Q_1 \cap {}^xQ_2} $$

In particular, if σ_i are irreducible, then the dimension is the cardinality of the set of $Q_1 - Q_2$ double coset representatives x for which ${}^x\sigma_1$ is equivalent to σ_2.

For an irreducible, cuspidal (complex) representation (V, ϕ) of $M(k)$ (trivial on $U(k)$), denote the induced module as $I_M^G(\phi)$. It is $\mathbf{C}[G(k)]e_U \otimes_{\mathbf{C}[M(k)]} V$ and is usually realized as a space of functions:

$$I_M^G(\phi) = \{f : G(k) \to V | f(pg) = \phi(p)(f(g))\}$$

with the right action by $G(k)$.

The above discussion shows that we have two problems to solve:
(i) Construct the irreducible cuspidal representations of $G(k)$;
(ii) Decompose the representations $I_M^G(\phi)$ of $G(k)$ induced by cuspidal irreducible representations of $M(k)$ for Levi subgroups M of proper parabolic k-subgroups.
The work of Deligne-Lusztig studied in the workshop addresses the first problem. The irreducible cuspidal representations are realized on the étale cohomolgy groups of Deligne-Lusztig varieties.
We discuss the second problem now.

The corresponding endomorphism ring
$H(M,\phi) := End_{\mathbf{C}[G(k)]} I_M^G(\phi)$ is called the Iwahori-Hecke algebra of ϕ.
Its study tells us about the submodules and quotients of $I_M^G(\phi)$.
In particular, standard representation theory gives:
The simple $\mathbf{C}[G(k)]$-submodules of $H(M,\phi)$ are in bijection with the isomorphism classes of simple components of $I_M^G(\phi)$ (see proposition below).

Note first:
An irreducible representation of $P_J(k)$ for a standard parabolic k-subgroup P_J, is isomorphic to $\mathbf{C}[P_J(k)]e$ for some primitive idempotent e in this group algebra.
The induced representation $I_{M_J}^G(\phi)$ of $G(k)$ is thus isomorphic to $\mathbf{C}[G(k)]e$ which means that the centralizer algebra $H(M_J, \phi)$ is isomorphic to $e\mathbf{C}[G(k)]e$.

Proposition.
Call χ_J, the character of $I_{M_J}^G(\phi)$. Then, the restriction map sending characters of $\mathbf{C}[G(k)]$ to characters of $H(M_J, \phi) = e\mathbf{C}[G(k)]e$, is a bijection between irreducible characters χ of $G(k)$ satisfying $(\chi_J, \chi) \neq 0$ and irreducible characters of $e\mathbf{C}[G(k)]e$. In particular, the dimension of an irreducible representation E of $H(M_J, \phi)$ is (χ_J, χ_E) where χ_E is the restriction of χ on $\mathbf{C}[G(k)]$ to $e\mathbf{C}[G(k)]e$.

Proof.

Denote the left $\mathbf{C}[G(k)]$-module $I_{M_J}^G(\phi)$ by M_0 for short; it is a simple module.

We claim that $\mathrm{Hom}_{\mathbf{C}[G(k)]}(\mathbf{C}[G(k)]e, M_0) \cong eM_0$.

In fact, if a homomorphism belonging to the left side takes e to m, then $e = e^2$ goes to em which means $m = em \in eM_0$.

Thus, associated to any homomorphism α in $\mathrm{Hom}_{\mathbf{C}[G(k)]}(\mathbf{C}[G(k)]e, M_0)$, we have the element $\alpha(e) \in eM_0$.

Conversely, given $m \in eM_0$, we have clearly $em = m$ and so, there is a unique α in $\mathrm{Hom}_{\mathbf{C}[G(k)]}(\mathbf{C}[G(k)]e, M_0)$ with $\alpha(e) = m$.

Now, $\dim eM_0 = \dim \mathrm{Hom}_{\mathbf{C}[G(k)]}(\mathbf{C}[G(k)]e, M_0) = (\chi_J, \chi)$.

We need to show that eM_0 which is clearly an $e\mathbf{C}[G(k)]e$ - module, is also simple as such.

Suppose $0 \neq m \in eM_0$. Clearly, $em = m$ which means

$$e\mathbf{C}[G(k)]em = e\mathbf{C}[G(k)]m = eM_0.$$

Thus, we have shown that eM_0 is simple as an $e\mathbf{C}[G(k)]e$-module as well.

The character of this simple $e\mathbf{C}[G(k)]e$-module is given by the trace map on $e\mathbf{C}[G(k)]e$ and since $tM_0 \subseteq eM_0$ for any $t \in e\mathbf{C}[G(k)]e$, we have

$$\chi(t) = trace_{M_0}(t) = trace_{eM_0}(t)$$

This means that χ restricted to $e\mathbf{C}[G(k)]e$ is the character of the irreducible $e\mathbf{C}[G(k)]e$-module eM_0.

One may also show that each irreducible character of $e\mathbf{C}[G(k)]e$ is the restriction of an irreducible character of $\mathbf{C}[G(k)]$ by breaking up the corresponding primitive idempotent in $e\mathbf{C}[G(k)]e$ as a sum of orthogonal idempotents in $\mathbf{C}[G(k)]$ and obtaining a contradiction.

Examples

Let $G = GL_n$ and $|k| = q$. Then, $M = T$, the diagonal subgroup, V is the trivial $\mathbf{C}[M(k)]$-module. Then the Iwahori-Hecke algebra $H(M, \phi)$ has $n-1$ generators $T_1, \cdots, T_{n-1}$ and the relations:

$$T_i^2 = q.1 + (q-1)T_i$$

$$T_iT_j = T_jT_i \quad \mathit{for} \quad |i-j| > 1$$

and the braid relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad (1 \leq i \leq n-2)$$

The main contention of Howlett-Lehrer's work that we are discussing is that the endomorphism algebra $H(M, \phi)$ can be naturally identified with the Iwahori-Hecke algebra associated to a certain 'extended' Coxeter group $W(M, \phi)$. Consequently, by a general result due to Tits, it is a deformation of the group algebra of $W(M, \phi)$.

Recall: Given a Coxeter group W with the Coxeter matrix $(m_{ij})_{i,j\leq r}$ and an r-tuple $(v_1,\cdots,v_r)\in \mathbf{C}^r$ with $v_i=v_j$ whenever s_i,s_j are conjugate, the Iwahori-Hecke algebra $H(W,v_1,\cdots,v_r)$ with parameter $(v_1,\cdots,v_r)$ is defined by r generators $T_{s_1},\cdots,T_{s_r}$ and relations

$$T_{s_i}^2 = v_i.1 + (v_i-1)T_{s_i} \qquad (i\leq r)$$

$$T_{s_i}T_{s_j}\cdots = T_{s_j}T_{s_i}\cdots \qquad (i\neq j)$$

where the last relations have m_{ij} terms on each side.
When the vector $(v_1,\cdots,v_r)=(1,\cdots,1)$, this is just the group algebra of W.

The special case of Howlett-Lehrer's theorem describing the Hecke algebra $H(M,\phi)$ when the corresponding parabolic subgroup is a Borel subgroup and the representation ϕ is trivial, was proved earlier by Iwahori and Tits; it shows that the above presentation holds with the parameters $v_i = [B : B \cap s_i B s_i^{-1}]$ for all $i \leq r$.

Proof of Iwahori-Tits theorem

For this proof, we will write G etc. instead of $G(k)$ for simplicity.
The set $B\backslash G$ is a $\mathbf{C}$-basis of $I_T^G(1) = \mathbf{C}[B\backslash G]$.
Using this basis, we obtain a matrix representation of G.
A basis of $H = End_{\mathbf{C}[G]} I_T^G(1)$, is indexed by the orbits of G on $B\backslash G \times B\backslash G$.
In fact, if O is an orbit, then the basis element T_O has (i,j)-th entry to be 1 or 0 accordingly as to whether (i,j) belongs to O or not.
The orbits of are in bijection with $B\backslash G/B$; BxB corresponds to the orbit of (xB, B). By the Bruhat decomposition, $B\backslash G/B$ is in bijection with W. Thus H has basis $T_w := T_{O_w}$ where O_w is the orbit of (wB, B) as $w \in W$ varies.

Now $T_x T_y = \sum_{z \in W} a_{xyz} T_z$.
So, a_{xyz} is the entry at the position (zB, B) in $T_x T_y$. It is easy to check that

$$a_{xyz} = |zBx^{-1}B \cap ByB|/|B|.$$

Let us compute T_s^2. So, $x = y = s$. Then
$zBsB \subseteq BzsB \cup BzB$.
Moreover, $a_{ssz} \neq 0$ only if $zs = s$ (i.e. $z = 1$) or $z = s$.
So $a_{ss1} = |BsB|/|B| = [B : sBs^{-1} \cap B]$.
Als, $a_{sss} = |sBsB \cap BsB|/|B| = [B : sBs^{-1} \cap B] - 1$ since
$sBsB \subseteq B \cup BsB$ and $B \cap BsB = \emptyset$.
This completes the proof.

For $G = SL_2$, the irreducible representation V of $M = T$ is 1-dimensional and, therefore, $I_M^G(\phi)$ has dimension $|SL_2(k)/B(k)| = q + 1$. The Coxeter group corresponding to $H(M, \phi)$ is a subgroup of the Weyl group $W(SL_2, T) \cong \mathbf{Z}/2\mathbf{Z}$. There are two cases:

Case (i): $W(M, V)$ is trivial.

In this case $H(M, \phi) = \mathbf{C}$ and $I_M^G(\phi)$ is a simple module.

Case (ii): $W(M, V) = W(SL_2, T)$.

In this case $H(M, \phi) = \mathbf{C}[W(G, T)]$ and $I_M^G(\phi)$ is a direct sum of two simple modules.

Intertwining operators between induced modules

To get a basis for the Iwahori-Hecke algebra, we first find its dimension.
Indeed, if χ is the character of an irreducible, cuspidal representation ϕ of $M_J(k)$ (where J is a set of k-simple roots), Mackey's formula above (with $H = G(k), Q_1 = Q_2 = P_J(k)$) tells us:

$$(\chi_{P_J(k)}^{G(k)}, \chi_{P_J(k)}^{G(k)})_{G(k)} = \sum_{w \in N_{J,J}} (\chi_{P_J(k)}, {}^{w}\chi_{P_J(k)})_{{}^{w}P_J(k) \cap P_J(k)}$$

where for any two subsets I, J of the set of k-simple roots, $N_{I,J}$ is a set of $(W_I - W_J)$-representatives (equivalently $(P_I(k) - P_J(k))$ representatives).

Using this, one can prove that the above sum is the cardinality of the *ramification group*

$$\{w \in W : w(J) = J, {}^{w}\chi = \chi\}$$

Note that ${}^{w}\chi = \chi$ is the same as the assertion that ${}^{w}\phi$ is equivalent to the representation ϕ.
So, dim $H(M_J, \phi) = |\{w \in W : w(J) = J, {}^{w}\chi = \chi\}|$.

Thus, we will have a basis element B_w corresponding to each element w so that $w(J) = J, {}^w\chi = \chi$.
The group $\{w \in W : w(J) = J, {}^w\chi = \chi\}$ is not a reflection group itself but contains a large reflection subgroup which will be useful in determining the product of elements B_w (yet to be defined!) in $End_{\mathbf{C}[G(k)]} I_{M_J}^G(\phi)$.
We will need to go 'outside' to operators which intertwine between the spaces $I_M^G(\phi)$ and $I_{M_{w(J)}}^G({}^w\phi)$.

Before defining these intertwining operators, we introduce a notation.

We sometimes write w for a lift of w in N. We fix once for all a choice as follows.

The choice of lifts $\bar{w}$ can be made so that for any reduced expression $w = s_1 s_2 \cdots s_k$, we have $\bar{w} = \bar{s_1} \cdots \bar{s_k}$.

Therefore, if w, w' have lifts $\bar{w}, \bar{w'}$, and if $l(ww') = l(w) + l(w')$, then the lift of ww' is the product $\bar{w}\bar{w'}$.

Proposition

Let w be such that $w(J) = I$ for some subsets I, J of k-simple roots. Let ϕ be an irreducible, cuspidal representation of M_J and ${}^w\phi$ be the corresponding irreducible, cuspidal representation of $M_I(k) = wM_J(k)w^{-1}$. Then, for $f \in I_{M_J}^G(\phi)$, the map $\theta_w(f) : G(k) \to V$ defined as

$$\theta_w(f) : x \mapsto \frac{1}{|U_I(k)|} \sum_{u \in U_I(k)} f(w^{-1}ux)$$

lies in $I_{M_I}^G({}^w\phi)$. Moreover,

$$\theta_w : I_{M_J}^G(\phi) \to I_{M_I}^G({}^w\phi)$$

is a homomorphism of $\mathbf{C}[G(k)]$-modules.

In the definition, we have written $f(w^{-1}ux)$ to mean $f(\bar{w}^{-1}ux)$, and ${}^w\phi$ for ${}^{\bar{w}}\phi$ etc.

Proof.

For $m \in M_I$,

$$\theta_w(f)(mx) = \frac{1}{|U_I(k)|} \sum_{u \in U_I(k)} f(w^{-1}umx) =$$

$$\frac{1}{|U_I(k)|} \sum_{u \in U_I(k)} f(w^{-1}m(m^{-1}um)x) =$$

$$\frac{1}{|U_I(k)|} \sum_{u \in U_I(k)} f(w^{-1}mux)$$

since $m^{-1}um$ runs over $U_I(k)$ when u does. Thus, $\theta_w(f)(mx) = \frac{1}{|U_I(k)|} \sum_{u \in U_I(k)} f((w^{-1}mw)w^{-1}ux) = \phi(w^{-1}mw)(\theta_w(f)(x))$.

Therefore, θ_w does map the induced module of ϕ to the asserted induced module. That θ_w is a homomorphism of $\mathbf{C}[G(k)]$-modules, is immediately seen from its definition.

The special case of the proposition when $J = I$ and ${}^w\phi$ is equivalent to ϕ occurs when w is in the ramification group $W(J, \phi) = \{w : w(J) = J, {}^w\phi \sim \phi\}$. In this case, if V is the representation space of ϕ, then for each $w \in W(J, \phi)$, there is an invertible linear automorphism $e(w)$ of V such that

$$ {}^w\phi(p) = e(w)^{-1}\phi(p)e(w) \;\; \forall \;\; p \in P_J(k) $$

Moreover, the map $e(w)$ is unique up to scalars by Schur's lemma since ϕ is irreducible. We are now in a position to define relevant elements of the Hecke algebra using the θ_w and the $e(w)$ for $w \in W(J, \phi)$.

A basis for $H(M, \phi)$

Proposition.
For $w \in W(J, \phi)$ where ϕ is an irreducible, cuspidal representation of $M_J(k)$, and for $f \in I_{M_J}^G(\phi)$, define

$$B_w(f) : G(k) \to V \; ; \; x \mapsto e(w)(\theta_w(f)(x))$$

Then $B_w \in H(M_J, \phi)(= End_{\mathbf{C}[G(k)]} I_{M_J}^G(\phi))$ for each $w \in W(J, \phi)$ and form a basis as w varies over the ramification group $W(J, \phi)$.
The Proof that B_w's belong $H(M_J, \phi)$ is straightforward; let us show that $\{B_w : w \in W(J, \phi)\}$ form a linearly independent set.

Proof

Let $\sum_w b_w B_w(f)(x) = 0$ for all $x \in G(k), f \in I_{M_J}^G(\phi)$.
Recall $I_{M_J}^G(\phi) = \{f : G(k) \to V | f(mug) = \phi(m)(f(g))\}$.
For $0 \neq v \in V$, consider the map f_v in $I_M^G(\phi)$ mapping x to $\phi(x)v$ if $x \in P(k)$ and to 0 otherwise.
Now $\sum_w b_w B_w(f_v)(w_1) = 0$ for all $w_1 \in W(J, \phi)$. That is,

$$\sum_w b_w e(w) \frac{1}{|U_J(k)|} \sum_{u \in U_J(k)} f_v(w^{-1}uw_1) = 0 \;\; \forall \; w_1 \in W(J, \phi).$$

In the above sum, we may consider only those w for which $w^{-1}uw_1 \in P_J$; that is, $w_1 \in U_j w P_J$. This means $w_1 \in W_J w W_J$.
Now, the normalizer $N_W(W_J) = C_J W_J$ with $C_J \cap W_J = (1)$ where $C_J := \{w : w(J) = J\}$. As both $w, w_1 \in C_J \leq N_W(W_J)$, we have from $w_1 \in W_J w W_J$ that $wW_J = w_1 W_J$ and so, $w = w_1$.

Thus, each of the above equations has only one term corresponding to $w = w_1$ and, gives

$$\frac{1}{|U_J(k)|} b_{w_1} e(w_1) \sum_{u \in U_J(k)} f_v(w_1^{-1} u w_1) = 0 \ \ \forall \ w_1 \in W(J, \phi)$$

By the definition of f_v, we get

$$\frac{1}{|U_J(k)|} b_{w_1} e(w_1) \sum_{u \in U_J(k) \cap^{w_1} P_J(k)} \phi(w_1^{-1} u w_1) v = 0 \ \forall \ w_1 \in W(J, \phi)$$

It can be seen using $w_1(J) = J$ that $U_J \cap^{w_1} P_J = (U_J \cap^{w_1} U_J)(U_J \cap^{w_1} M_J) = U_J \cap^{w_1} U_J$ as $U_J \cap^{w_1} M_J) = (1)$. As $\phi(w_1^{-1} u w_1) = Id$ if $u \in U_J(k) \cap^{w_1} U_J(k)$, we have

$$\frac{|U_J(k) \cap^{w_1} U_J(k)|}{|U_J(k)|} b_{w_1} e(w_1) v = 0 \ \ \forall \ \ w_1 \in W(J, \phi)$$

As $e(w_1)$ is an invertible linear transformation, $e(w_1) v \neq 0$ and, we get $b_{w_1} = 0$ for all w_1.

$B_w B_{w'}$ for $l(ww') = l(w) + l(w')$

Proposition.
Let $l(ww') = l(w) + l(w'), w'(\Delta_J) \subset \Delta$ *and* $ww'(\Delta_J) \subset \Delta$.
Then, $\theta_{ww'} = \theta_w \theta_{w'}$.

It is convenient to introduce the following notation before the proof of the proposition.
For $w \in W$, let U_w denotethe product of all root subgroups $U_\alpha(k)$ for $\alpha > 0$ such that $w(\alpha) < 0$.

Proof of proposition

Note firstly that $\theta_{w'}$ maps $I^G_{M_J}(\phi)$ to $I^G_{M_{w'J}}({}^{w'}\phi)$ which, in turn, is mapped by θ_w to $I^G_{M_{ww'J}}({}^{ww'}\phi)$.
Moreover, θ_w which was defined as
$\theta_w(f) : x \mapsto \frac{1}{|U_I(k)|}\sum_{u \in U_I(k)} f(w^{-1}ux)$ (where $w(J) = I$) has also the expression

$$\theta_w(f)(x) = \frac{1}{|U_I(k) \cap U_{w^{-1}}|} \sum_{u \in U_I(k) \cap U_{w^{-1}}} f(w^{-1}ux).$$

The latter expression for θ_w is obtained from its definition by factorizing each element $u \in U_I$ as a product with $u_1 \in U_I(k) \cap U_{w_0 w^{-1}}, u_2 \in U_I(k) \cap U_{w^{-1}}$.
Indeed, $U_I(k) \cap U_{w_0 w^{-1}}$ being the product of root subgroups U_α with $\alpha > 0, w_0 w^{-1}(\alpha) < 0$, $\alpha \notin \Phi_I$, the conjugate $w^{-1} U_I(k) \cap U_{w_0 w^{-1}} w$ is the product of $U_{w^{-1}(\alpha)}$ with α as above and so, lie in U_J as $w^{-1}(\alpha) > 0, w^{-1}(\alpha) \notin \Phi_J$. Thus, the elements $w^{-1} u_1 w \in U_J(k)$ and we have

$$f(w^{-1}u_1u_2x) = f(w^{-1}u_1ww^{-1}u_2x) = f(w^{-1}u_2x)$$

Writing out $\theta_w \theta_{w'}(f)(x)$ and using the lemma below, one can prove the proposition.

A combinatorial lemma

Lemma

Let $w, w' \in W$ *such that* $l(ww') = l(w) + l(w')$. *Then*

$$^{w}(U_{w'J} \cap U_{w'^{-1}})(U_{ww'J} \cap U_{w^{-1}}) = U_{ww'J} \cap U_{(ww')^{-1}}$$

with uniqueness.

This lemma is easy to prove by looking at the root subgroups occurring on both sides and showing that those occurring in the two subgroups on the left side are disjoint and their union is the set of those appearing on the right side.

Indeed, since U_K for any subset K of roots is generated by the root subgroups corresponding to the positive roots outside Φ_K, the set S of roots contributing to $U_{ww'J} \cap U_{(ww')^{-1}}$ is :

$$\alpha > 0, (ww')^{-1}(\alpha) \notin \Phi_J, (ww')^{-1}(\alpha) < 0$$

Similarly, the set S_1 of roots corresponding to $U_{ww'J} \cap U_{w^{-1}}$ is :

$$\alpha > 0, (ww')^{-1}(\alpha) \notin \Phi_J, w^{-1}(\alpha) < 0$$

and the set S_2 of roots corersponding to $U_{w'J} \cap U_{w'^{-1}}$ is:

$$\beta > 0, w'^{-1}(\beta) < 0, w'^{-1}(\beta) \notin \Phi_J$$

so that the set $w(S_2)$ of roots corresponding to ${}^w(U_{w'J} \cap U_{w'^{-1}})$ is:

$$w^{-1}(\alpha) > 0, w'^{-1}w^{-1}(\alpha) < 0, w'^{-1}w^{-1}(\alpha) \notin \Phi_J$$

Clearly, $S = S_1 \sqcup w(S_2)$.

We wish to determine what information this lemma gives about the product $B_w, B_{w'}$.
Recall that for $w \in W(J, \phi)$, $B_w(f)(x) = e(w)(\theta_w(f)(x))$ where $e(w)$ is a linear automorphism of the vector space V underlying ϕ. However, it is useful to extend this view of $e(w)$ to that of an operator : $I_M^G(\phi) \to I_M^G({}^{w^{-1}}\phi)$ by
$e(w)(f) : x \mapsto e(w)(f(x))$.
The verification that $e(w)(f) \in I_{M_J}^G({}^{w^{-1}}\phi)$ is straightforward. Indeed, more generally, for any w', $e(w)$ defines a $\mathbf{C}[G(k)]$-module homomorphism from
$I_{M_J}^G({}^{w'}\phi) \to I_{M_J}^G({}^{w'w^{-1}}\phi)$.
Therefore, for $w \in W(J, \phi)$, we can write B_w as $e(w) \circ \theta_w$, a $\mathbf{C}[G(k)]$-module homomorphism of $I_{M_J}^G(\phi)$ to itself.

A compatibility property easily verified :

Let $w, w' \in W(J, \phi)$. *Then* $e(w) \circ \theta_{w'} : I_{M_J}^G(\phi) \to I_{M_J}^G({}^{w'w^{-1}}\phi)$ *equals* $\theta_{w'} \circ e(w)$.

Recalling that for $w \in W(J, \phi)$, $e(w)$ is uniquely defined up to scalars such that ${}^{w}\phi(m) = e(w)^{-1}\phi(m)e(w)$ for all $m \in M_J(k)$. So, for $w, w' \in W(J, \phi)$,

$$^{ww'}\phi(m) = e(ww')^{-1}\phi(m)e(ww')$$

On the other hand,

$$^{ww'}\phi(m) =^{w'} \phi(w^{-1}mw) = e(w')^{-1}\phi(w^{-1}mw)e(w')$$

$$= e(w')^{-1}({}^{w}\phi)(m)e(w') = e(w')^{-1}e((w)^{-1}\phi(m)e(w)e(w')$$

Thus, the right sides of the two expressions for ${}^{ww'}\phi(m)$ along with the uniqueness implies that there is a non-zero complex number $\lambda(w, w')$ such that

$$e(w)e(w') = \lambda(w, w')e(ww')$$

We can now prove:

Proposition.
Let $w, w' \in W(J, \phi)$ *with* $l(ww') = l(w) + l(w')$. *Then,* $B_w B_{w'} = \lambda(w, w') B_{ww'}$.
Proof.
Consider the sequence of maps

$$I^G_{M_J}(\phi) \overset{\theta_{w'}}{\to} I^G_{M_J}({}^{w'}\phi) \overset{e(w')}{\to} I^G_{M_J}(\phi) \overset{\theta_w}{\to} I^G_{M_J}({}^{w}\phi) \overset{e(w)}{\to} I^G_{M_J}(\phi)$$

This is

$$B_w B_{w'} = e(w) \circ \theta_w \circ e(w') \circ \theta_{w'}$$

$$= e(w) \circ e(w') \circ \theta_w \circ \theta_{w'}$$

$$= \lambda(ww') e(ww') \theta_{ww'} = \lambda(w, w') B_{ww'}$$

$\lambda(w, w')$ for general $w, w' \in W(J, \phi)$

In order to extend the definition of $\lambda(w, w')$ to all pairs w, w' in $W(J, \phi)$ (not just those for which $l(ww') = l(w) + l(w')$, we proceed as follows.
The idea is that the representation ϕ of $M_J(k)$ can be extended with the help of the $e(w)$'s to a projective representation of a group containing $M_J(k)$.

Indeed, consider the subgroup $K(J,\phi)$ of $N_G(M_J)$ which is generated by $M_J(k)$ and the lifts $\bar{w}$ as w varies over $W(J,\phi)$. This subgroup satisfies $K(J,\phi)/M_J(k) \cong W(J,\phi)$.
Define $\bar{\phi}$ on $K(J,\phi)$ by $m\bar{w} \mapsto \phi(m)e(\bar{w})$.
It can be checked using Schur's lemma that $\bar{\phi}(k_1k_2)$ is a scalar multiple of $\bar{\phi}(k_1)\bar{\phi}(k_2)$ for $k_1, k_2 \in K(J,\phi)$.

There exists a function $\lambda : W(J,\phi) \times W(J,\phi) \to \mathbf{C}^*$ *such that*

$$e(\bar{w}_1)e(\bar{w}_2) = \lambda(w_1, w_2)e(\overline{w_1 w_2})$$

It is straightforward to check:

Lemma.

$\lambda : W(J,\phi) \times W(J,\phi) \to \mathbf{C}^*$ *is a 2-cocycle.*

The above cocyle is uniquely determined only up to coboundaries. It is convenient to make a particular choice of the cocycle as follows.

Proposition.

The 2-cocycle $\lambda : W(J,\phi) \times W(J,\phi) \to \mathbf{C}^*$ *can be chosen to satisfy:*

(i) $\lambda(w,1) = \lambda(1,w) = 1$ *for all* w*;*

(ii) $\lambda(w,w^{-1}) = 1$ *for all* w*;*

(iii) $\lambda(w_1,w_2) = \frac{1}{\lambda(w_2^{-1},w_1^{-1})} = \lambda(w_2^{-1}w_1^{-1},w_1)$ *for all* w_1,w_2*;*

(iv) $\lambda(w_1,w_2)$ *is a* $|W(J,\phi)|$*-th root of unity for each* w_1,w_2*.*

We give the idea of the proof.
The 2-cocycle λ is cohomologous to λ' defined as

$$\lambda'(w_1, w_2) := \xi_{w_1}\xi_{w_2}\xi^{-1}_{w_1 w_2}\lambda(w_1 w_2)$$

for any non-zero scalars ξ_w's. The replacement of each $e(\bar{w})$ by $\xi_w e(\bar{w})$ changes λ to λ' as above. Thus, we need to choose the scalars ξ_w so that λ' would satisfy the proposition. The choice will be facilitated by introducing, for each $w \in W(J, \phi)$, a matrix $R(w) \in M_l(\mathbf{C})$ with $l = |W(J, \phi)|$, defined as $R(w)(wy, y) = \lambda(wy, y)$ and $R(w)(x, y) = 0$ otherwise. It turns out that

$$R(w_1)R(w_2) = \lambda(w_1, w_2)R(w_1, w_2)$$

Taking new matrices $R'(w) := \xi_w R(w)$ where ξ_w so chosen that $\det R'(w) = 1, R'(w^{-1}) = R'(w)^{-1}, R'(1) = I$, the λ' corresponding to R' satisfy the proposition.

A reflection subgroup of $W(J,\phi)$

The previous discussions already show that the structure of the Hecke algebra is connected closely with the group $W(J,\phi)$.
We shall observe:
There is a large normal subgroup of $W(J,\phi)$ which is the Weyl group of a quotient root system of the root system for W.
This basically follows from results proved by Howlett in an earlier paper describing a reflection group inside the normalizer of a parabolic subgroup of a Weyl group.

If W has its natural Coxeter representation on the vector space E, then for each subset J of simple k-roots, look at the subspace E_J spanned by the roots in Δ_J.
We produce a root system on the quotient space $E/E_J \cong E_J^{\perp}$.

It can happen that a k-root α outside Δ_J may still have the property that $\Delta_J \cup \{\alpha\}$ lies in *some* simple root system.
For instance, consider any k-root $\alpha \notin \Delta_J$ such that $\Delta_J \cup \{\alpha\} \subset w(\Delta)$ for some $w \in W$.
Then, $\Delta_J \cup \{\alpha\}$ lies in some simple system (may not be Δ) and we can define a nice element of the Weyl group as:

Define $w(\overline{\alpha}) := (w_0)_{J \cup \alpha}(w_0)_J$ where w_0 denotes the longest element in the corresponding Coxeter group.

Key observation:

If α satisfies the above property, then $w(\overline{\alpha})$ takes Δ_J into itself if and only if it is of order 2.

The proof uses a certain involution on the set I parametrizing the k-simple roots, called the *opposition involution*.
This is an involution $i \mapsto \overline{i}$ determined by the property that $w_0(\alpha_i) = -\alpha_{\overline{i}}$ (call α_i and $\alpha_{\overline{i}}$ opposed to each other).
Indeed, the condition $w(\overline{\alpha})(\Delta_J) \subseteq \Delta_J$, is equivalent to the condition that Δ_J is self-opposed inside $\Delta_J \cup \{\alpha\}$.
Thanks to the above observation, one naturally defines

$$\Omega := \{\alpha \in \Phi - \Delta_J : \Delta_J \cup \{\alpha\} \subseteq w(\Delta) \textit{ for some } w, w(\overline{\alpha})^2 = 1\}$$

In other words:
Ω *is the set of roots* α *outside* Δ_J *for which* $\Delta \cup \{\alpha\}$ *forms a simple system inside which* Δ_J *is self-opposed.*
Define $R_J =< w(\overline{\alpha}) : \alpha \in \Omega\}$ and $\overline{\Omega} = \{\overline{\alpha} : \alpha \in \Omega\}$, we have the theorem:

Theorem.
(i) *The map* $\alpha \mapsto \overline{\alpha}$ *from* Ω *to* $\overline{\Omega}$ *is 1-1;*
(ii) *the elements of* $\overline{\Omega}$ *normalized to unit vectors form a root system of a group with a split BN-pair;*
(iii) $w(\overline{\alpha})$ *acts on* $\bar{E} := E/E_J$ *as reflection in the hyperplane orthogonal to* $\overline{\alpha}$*;*
(iv) R_J *acts faithfully on* $\bar{E}$ *and is the Weyl group of* $\overline{\Omega}$*.*

B_w^2 for $w = w(\overline{\alpha})$

We obtained a reflection subgroup R_J of $W(J, \phi)$.
Now, we derive an expression for B_w^2 where w is one of the generators $w(\overline{\alpha})$ of R_J.
The basic idea is to replace the whole of G by the parabolic k-subgroup P_I where the subset I of simple k-roots is defined from $\Delta_J \cup \{\alpha\} = \Delta_I$.
This enables one to restrict an element B_w with $w = w(\overline{\alpha})$, to the subspace $Ind(J, I)$ of $I_{M_J}^G(V)$ consisting of those functions from $G(k)$ to V whose supports are contained in $P_I(k)$.
Then, $Ind(J, I)$ considered as an $\mathbf{C}[M_I(k)]$-module is the analogue of $I_{M_J}^G(V)$ in M_I.

The corresponding $\mathbf{C}[M_I(k)]$-endomorphism algebra of this module $Ind(J, I)$, has dimension equal to the order of

$$W_I(J, \phi) := \{w \in W_I : w(J) = J, ^w \phi = \phi\}$$

The last-mentioned group is either trivial or has order 2 (when $w(\overline{\alpha}) \in W(J, \phi)$, this is the nontrivial element).
In the latter case, let us compute B_w^2 with $w = w(\overline{\alpha})$. Towards this, we go back to the definition of B_w which involves θ_w.

Recall for any $w \in W(J, \phi)$, the operator θ_w which maps $H(M_J, \phi)$ to $H(w(J), {}^w\phi)$.
Indeed, for $n = \bar{w}$, we have

$$\theta_n(f)(g) = \frac{1}{U_{w(J)}(k)|} \sum_{u \in U_{w(J)}(k)} f(n^{-1}ug)$$

Applying this to the element $n(\overline{\alpha}) = \overline{w(\overline{\alpha})}$ and its inverse, where $\alpha \in \Delta - \Delta_J$, we have

$$H(M_J, \phi) \stackrel{\theta_n}{\to} H(M_{w(\overline{\alpha})(J)}, {}^{n(\overline{\alpha})} V) \stackrel{\theta_{n^{-1}}}{\to} H(M_J, \phi)$$

where $n = \overline{w(\overline{\alpha})}$.
So, the composite $\theta_{n^{-1}}\theta_n \in H(M_J, \phi)$ for $n = \overline{w(\overline{\alpha})}$.
As the action of this composite on $I_{M_J}^G(V)$ is given by the action on the subspace $Ind(J, I)$, we have:

Lemma.
For $\alpha \in \Delta - \Delta_J$*, there exist complex numbers* ξ, η *such that* $\theta_{n^{-1}}\theta_n$ *is* ξId *or* $\xi Id + \eta B_w$ *where* $w = w(\overline{\alpha})$ *and* $n = \bar{w}$*.*
Further, if $w(\overline{\alpha}) \in W(J, \phi)$ *(hence it has order* 2*), then* $B^2_{w(\overline{\alpha})} = \xi B_1 + \eta B_{w(\overline{\alpha})}$*.*

Proof.
The first statement is already discussed before the statement. For the latter assertion, one just uses the definition of B_w in terms of θ_w and the 2-cocycle λ chosen earlier to conclude that $B^2_{w(\overline{\alpha})} = \theta_{n^{-1}}\theta_n$. One point to note is that since $w(\overline{\alpha}) \in W(J, \phi)$ in the latter assertion, this means that its square is 1 and thus, we use the property $\lambda(w, w^{-1}) = 1$ for $w = w(\overline{\alpha})$.

The determination of the constants ξ, η can be done by choosing the action on conveniently chosen functions in $I_{M_J}^G(V)$.
Indeed, ξ can be determined by computing $B_w(f)(g)$ (with v a non-zero in V and the function $f \in Ind(J, I)$ given by $f_v(1) = v$ and $g = 1$).
Taking $f = f_v$ and $g = 1$ (and observing that f_v is zero outside P_J), one obtains ξ (see lemma below).
Similarly, taking $f = f_v$ and $g = w(\overline{\alpha})$, we can determine η.

Lemma.
With $w := w(\overline{\alpha}) \in W(J,\phi)$*, we have* $B_w^2 = \xi B_1 + \eta B_w$ *with*

$$\xi = \frac{1}{|U_w(k)|},$$

$$\eta Id_V = \frac{1}{|U_w(k)|} e(n)^{-1} \sum \phi(n^{-1}u'nun)$$

where $n = \bar{w}$ *and the last sum is over all* $u, u' \in U_w(k)$ *such that* $n^{-1}u'nun \in P_J(k)$*.*
Further, $\frac{\eta}{\xi}$ *is an algebraic integer.*
Write $Ind(w)$ for the number $\frac{1}{|U_w(k)|}$.

A conceptually simpler expression for η is given as:

Proposition.

Let $\alpha \in \Delta - \Delta_J$ and suppose $w = w(\overline{\alpha})$ is in $W(J, \phi)$. Let $J \cup \{\alpha\} = I$, say. Then, the induced character $\chi(\phi)_{P_J(k) \cap M_I(k)}^{M_I(k)}$ splits into the sum of two irreducible characters χ, χ'. Their degree are related by $\chi'(1)/\chi(1) = p^c$ for some integer $c \geq 0$. Moreover, we have the expression

$$\eta = \pm \frac{p^c - 1}{\sqrt{p^c Ind(w)}}$$

Notice that $\eta = 0$ if and only if the degrees of χ, χ' are the same.

Ramification group as a semidirect product

In the earlier discussion where we got hold of a reflection group corresponding to a certain set Ω of roots, we didn't bring in the representation V (that is, the ramification group $W(J,\phi)$). It is natural to consider the effect of elements of this group on the roots in Ω; in particular, whether elements in Ω are permuted. Further, as we saw in the previous section, for such an α, the degrees of the two irreducible characters of the bigger Levi subgroup arising from $\Delta \cup \{\alpha\}$ have ratio (say, p_α) to be a power of p (possibly 1). If we know that elements w of $W(J,\phi)$ carry Ω into Ω, we can ask the relation between $p_{w(\alpha)}$ and p_α. The basic observation is:

Lemma.
If $\alpha \in \Omega$ is such that $w(\overline{\alpha}) \in W(J, \phi)$, then for any $w \in W(J, \phi)$, we have $w(\alpha) \in \Omega$ and $w(\overline{w(\alpha)}) \in W(J, \phi)$. Further, $p_{w(\alpha)} = p_\alpha$.

In view of this lemma, we have $w(\Gamma) \subseteq \Gamma$ where

$$\Gamma := \{\alpha \in \Omega, w(\overline{\alpha}) \in W(J, \phi), p_\alpha = 1\}$$

Thus, Γ consists of all roots α outside Δ_J such that $\Delta_J \cup \{\alpha\}$ is contained in some simple system of roots, $w(\overline{\alpha})$ is in $W(J, \phi)$ and has order 2 (that is, it takes Δ_J to itself, and $p_\alpha = 1$. If Γ^+ is the set of positive roots in Γ, then we define

$$C(J, \phi) := \{w \in W(J, \phi) : w(\Gamma^+) = \Gamma^+\}$$

and $R(J, \phi)$ to be the subgroup of W generated by all $w(\overline{\alpha})$ for $\alpha \in \Gamma$. using the fact that Ω is a root system (in E/E_J) with reflection group generated by $w(\overline{\alpha})$ as α varies over Ω, we have:

Proposition.
(i) $R(J,\phi)$ *is normal in* $W(J,\phi)$ *and* $W(J,\phi) = R(J,\phi)C(J,\phi)$ *with* $R(J,\phi) \cap C(J,\phi) = 1$.
(ii) $R(J,\phi)$ *is a reflection subgroup of* $W(J,\phi)$ *with root system* $\overline{\Gamma} := \{\overline{\alpha} : \alpha \in \Gamma\}$ *(inside the vector space* E/E_J*).*

The multiplication $B_w B_{w'}$ for general w, w'

There may exist simple systems which are W-equivalent to Δ_J without being equal to it.
Define $\hat{J}$ to be the set of all subsets J' of the indexing set of the simple k-roots such that $\Delta_{J'} = w(\Delta_J)$ for some $w \in W$.
Members of $\hat{J}$ are said to be *associated with* J.
For any J' associated to J, one has elements $w(\overline{\alpha}, J')$ in W for any α outside J' for which $\Delta_{J'} \cup \{\alpha\}$ is contained in a simple system of roots.

The following proposition which can be proved using the combinatorics of the Weyl group will evidently prove useful:
Proposition.
If both J', J'' are associated to J and $w(\Delta_{J'}) = \Delta_{J''}$, then

$$w = w(\overline{\alpha_r}) \cdots w(\overline{\alpha_1})$$

with $\alpha_i \in \Delta$, and $l(w) = \sum_{i=1}^{r} l(w(\overline{\alpha_1}))$ where $J_1, \cdots, J_r$ are associated to J with $w(\overline{\alpha_1})(\Delta_{J_i}) = \Delta_{J_{i+1}}$, $J' = J_1, J'' = J_{r+1}$.

If we write out the product of θ_w and $\theta_{w'}$ using the above proposition, one will obtain the product of B_w and $B_{w'}$.
Recall that we have a positive system Γ^+ of roots which defines a simple system $\overline{\Lambda}$. We denote by Λ the set of roots $\alpha \in \Gamma$ for which $\overline{\alpha} \in \overline{\Lambda}$. It can be checked that:
Λ is precisely the set of all $\alpha \in \Gamma^+$ such that α is the only positive root in Γ which is sent to a negative root by $w(\overline{\alpha})$.

Theorem.
$B_w B_{w'} = (\frac{ind(w)ind(w')}{ind(ww')})^{1/2}\lambda(w,w')B_{ww'}$ *in each of the following three cases:*
(i) *either* $w \in W(J,\phi), w' \in C(J,\phi)$ *or* $w' \in W(J,\phi), w \in C(J,\phi)$;
(ii) $w \in W(J,\phi), w' = w(\overline{\alpha})$ *with* $\alpha \in \Lambda$ *and* $w(\alpha) > 0$;
(iii) $w' \in W(J,\phi), w = w(\overline{\alpha})$ *with* $\alpha \in \Lambda$ *and* $w^{-1}(\alpha) < 0$.
Finally, for $\alpha \in \Lambda$, *we have*
$B^2_{w(\overline{\alpha})} = \frac{1}{ind(w(\overline{\alpha}))} \pm \frac{p_\alpha - 1}{(p_\alpha ind(w(\overline{\alpha})))^{1/2}} B_{w(\overline{\alpha})}$.

A neater basis

It turns out to be convenient to change the basis B_w by scalar multiples so that the multiplication rules look much neater. For instance, for $\alpha \in \Lambda$, recall that the last statement of the above theorem gives

$$B^2_{w(\overline{\alpha})} = \frac{1}{ind(w(\overline{\alpha}))} + \frac{\epsilon_\alpha(p_\alpha - 1)}{(p_\alpha ind(w(\overline{\alpha})))^{1/2}} B_{w(\overline{\alpha})}$$

for $\epsilon_\alpha = \pm 1$.
Define

$$T_\alpha = \epsilon_\alpha (p_\alpha ind(w(\overline{\alpha})))^{1/2} B_{w(\overline{\alpha})}$$

Then, we will get

$$T_\alpha^2 = p_\alpha 1 + (p_\alpha - 1) T_\alpha$$

We wish to define T_w for $w \in R(J, \phi)$ by writing w as a product of elements of the form $w(\overline{\alpha})$.
Here the key is:

Lemma.

Let $w \in R(J, \phi)$ have the two reduced expressions $w = w(\overline{\alpha}_1) \cdots w(\overline{\alpha}_r) = w(\overline{\beta}_1) \cdots w(\overline{\beta}_r)$. Then,

$$T_{\alpha_1} \cdots T_{\alpha_r} = T_{\beta_1} \cdots T_{\beta_r}$$

This shows that we can define T_w for $w \in R(J, \phi)$.
Define T_w for $w \in C(J, \phi)$ as $T_w = ind(w)^{1/2} B_w$.
Finally, for any $w \in W(J, \phi)$, one can write uniquely as $w = w_1 w_2$ with $w_1 \in C(J, \phi), w_2 \in R(J, \phi)$ and define $T_w = T_{w_1} T_{w_2}$.
It turns out that T_w is a scalar multiple of B_w for any w as proved by induction on the number of roots in Γ^+ made negative by w. We have:

For $w \in W(J,\phi)$*,* $T_w = \epsilon_w (p_w ind(w))^{1/2} B_w$ *where* ϵ_w *is some root of unity and* $p_w = \prod_{\alpha \in \Gamma^+, w(\alpha)<0}$*.*

To state the multiplication formulae of the T_w's, we will consider the 2-cocycle $\mu(w,w')$ (cohomologous to the $\lambda(w,w')$ chosen earlier) defined as

$$\mu(w,w') = \epsilon_w \epsilon_{w'} \epsilon_{ww'}^{-1} \lambda(w,w')$$

This 2-cocycle satisfies the property:
If $x, x' \in C(J,\phi), w, w' \in R(J,\phi)$*, then* $\mu(xw, x'w') = \lambda(w,w')$*. In particular, this is* 1 *if either* x *or* x' *is* 1*.*
We state the final theorem now.

Theorem.
(i) $T_w T_{w'} = \mu(w,w')T_{ww'}$ *if* (w,w') *or* $(w',w) \in W(J,\phi) \times R(J,\phi)$;
(ii) *Let* $\alpha \in \Lambda, w \in W(J,\phi), w' = w(\overline{\alpha})$. *Then,* $T_w T_{w'} = T_{ww'}$ *or* $p_\alpha T_{ww'} + (p_\alpha - 1)T_w$ *accordingly as to whether* $w(\alpha) > 0$ *or* < 0.
(iii) *Let* $\alpha \in \Lambda, w \in W(J,\phi), w' = w(\overline{\alpha})$. *Then,* $T_{w'} T_w = T_{w'w}$ *or* $p_\alpha T_{w'w} + (p_\alpha - 1)T_w$ *accordingly as to whether* $w^{-1}(\alpha) > 0$ *or* < 0.

Hecke algebra as twisted group algebra

Tits constructed generic algebras of this type and proved the important theorem that any two specializations which are semisimple must be isomorphic.
More precisely, in our context, consider a variable t_α corresponding to each $\alpha \in \Lambda$ such that $t_\alpha = t_{w(\alpha)}$ for each $w \in W(J, \phi)$.
That is, we have an indeterminate for each $W(J, \phi)$-orbit in Λ.
Then, there is a unique associative algebra $A(t_\alpha)$ over the polynomial ring $\mathbf{C}[t_\alpha; \alpha \in \Lambda]$ which has a basis a_w for each $w \in W(J, \phi)$ such that the multiplication is given by the formulae as in the above theorem with t_α in place of p_α for each α.

An algebra homomorphism $\sigma : A(t_\alpha) \to \mathbf{C}$ is called a *specialization*.
The specialization $t_\alpha \mapsto p_\alpha$ gives our hecke akgebra $H(M_J, \phi)$.
The specialization $t_\alpha \mapsto 1$ gives the deformed group algebra of $W(J, \phi)$.
As these algebras are semisimple, Tits's theorem implies that the Hecke algebra $H(M_J, \phi)$ is isomorphic to the deformed group algebra $\mathbf{C}[W(J, \phi)]_\mu$.

There is a relation between the irreducible characters of the generic algebra and those of its specializations.
This provides a way to compute the degree of irreducible characters of $G(k)$ if the degrees of irreducible cuspidal characters of the Levi subgroups M_J are known.
It has been proved later by Lusztig & Geck that the cocycle μ is actual trivial cohomologically.

Lawrence Morris proved a generalization of Howlett-Lehrer's theorem in the context of irreducible, admissible representations of a p-adic group.
The analogous Hecke algebra consists of smooth, compactly supported End(V)-valued functions on the group, where V is a cuspidal representation of the finite group of Lie type P/U with P, a parahoric subgroup.
The proof, although based on Howlett-Lehrer's proof, is substantially more complicated. Thus, one may refer to Howlett-Lehrer theory rather than just a theorem.

(FINALLY:)

To find all reps of $G(\mathbf{F}_q)'s$,
here's what everyone argues.
"Get each cuspidal one (for each Levi),
induce and decompose (no casualties heavy)
and that would be the end of the news!"'

Footnote: If some ends are *Lus* - *ztig* them as you have to draw *De-ligne* somewhere!

THANK YOU FOR LISTENING !