Of grandaunts and Fibonacci

The beautiful identity

$$\prod_{s=1}^{(n-1)/2} 2Cos(\frac{\pi s}{n}) = 1$$

for odd n, appears in article 88.62 and is termed grandma's identity by Steven Humble ([1]). However, it has been well-known in some quarters as the author says. Indeed, for several years, I have known this identity as well as two identities - a twin sister and an Italian cousin perhaps ! The Italian cousin alluded to is :

$$\prod_{r=1}^{(n-1)/2} (3 + 2Cos(2\pi r/n)) = F_n$$

for odd n, where F_n is the n-th among the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$ The twin sister to the grandma identity is :

$$\prod_{s=1}^{2k} 2Cos(\frac{2\pi s}{4k+1}) = \prod_{s=1}^{2k-1} 2Cos(\frac{2\pi s}{4k-1}) = (-1)^k.$$

We call this the twin sister because it is seen to be equivalent to the grandma identity on using the substitution formula $Cos(\pi - \theta) = -Cos(\theta)$.
The starting point of my discussion is the following identity which I have observed and used in a number of ways (see [2], [3]); it can be proved by induction on n :

$$\sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} x^r (1+x)^{n-2r} = 1 + x + x^2 + \cdots + x^n.$$

Using this, we have the identities

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} x^r (1+x)^{n-1-2r} = 1+x+\cdots+x^{n-1} = \prod_{s=1}^{[(n-1)/2]} (x^2 - 2xCos(\frac{2\pi s}{n}) + 1).$$

Now, let us evaluate both sides of

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} (\frac{x}{(1+x)^2})^r (1+x)^{n-1} = \prod_{s=1}^{[(n-1)/2]} (x^2 - 2xCos(\frac{2\pi s}{n}) + 1)$$

at a value t of x for which $\frac{t}{(1+t)^2} = -1$; that is, $t^2 + 3t + 1 = 0$. The point is that the left hand side becomes $(1+t)^{n-1} \sum_{r=0}^{(n-1)/2} \binom{n-1-i}{i}$. It is well-known and easy to prove by induction on n that $\sum_{r=0}^{(n-1)/2} \binom{n-1-i}{i}$ is nothing but the n-th Fibonacci number. This is true for every n; one just uses the recursion $F_{n+1} = F_n + F_{n-1}$. On the other hand, the right hand side is

$$\prod_{r=1}^{(n-1)/2} (-3t - 2tCos(2\pi r/n)) = (-t)^{(n-1)/2} \prod_{r=1}^{(n-1)/2} (3 + 2Cos(2\pi r/n)).$$

As $-t = (1+t)^2$, this proves the remarkable identity

$$\prod_{r=1}^{(n-1)/2} (3 + 2Cos(2\pi r/n)) = F_n$$

for any odd n.

Just as Humble obtained grandma's identity by substituting $x = -1$ for odd n, one could substitute $x = i (= \sqrt{-1})$ for odd n in both expressions, to get the identities

$$(2i)^{(n-1)/2} \sum_{r=0}^{(n-1)/2} (\frac{-1}{2})^r \binom{n-1-r}{r} = (-2i)^{(n-1)/2} \prod_{s=1}^{(n-1)/2} Cos(\frac{2\pi s}{n});$$

these are both equal to 1 or i according as to whether $n \equiv \pm 1$ mod 4. Thus, we have a twin sister to grandma's identity :

$$\prod_{s=1}^{2k} 2Cos(\frac{2\pi s}{4k+1}) = \prod_{s=1}^{2k-1} 2Cos(\frac{2\pi s}{4k-1}) = (-1)^k$$

and the grandaunt identity

$$\sum_{r=0}^{(n-1)/2} (\frac{-1}{2})^r \binom{n-1-r}{r} = (-1)^{(n-1)/2} \prod_{s=1}^{(n-1)/2} Cos(\frac{2\pi s}{n}).$$

Note that Humble's grandma identity is equivalent to its twin as seen by using the observation $Cos(\pi - \theta) = -Cos(\theta)$.

[1] Steven Humble, *Grandma's identity*, Mathematical Gazette, Article 88.62, P.524, November 2004.
[2] B.Sury, *A parent of Binet's formula ?*, Mathematics Magazine, Vol. 77(2004) 308-310.
[3] James McLaughlin & B.Sury, *Powers of a matrix and combinatorial identities*, Integers, Electronic Journal of Combinatorial Number theory, Vol. 5(2005) Article A 13.

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