104.27 An additive analogue of an unsolved multiplicative problem

Consider the equation $a!\,b! = c!$ in positive integers $b \geqslant a > 1$. Apart from the infinitely many 'trivial' solutions with $b + 1 = a!$ (which gives $c = b + 1$), the only nontrivial solution known is $6!\,7! = 10!$. In fact, more generally, under the assumption of Baker's explicit 'abc' conjecture (a deep conjecture), Saranya Nair and T. N. Shorey (see [1]) proved that the only solutions to the equation

$$a_1!\,a_2! \dots a_t! = n!$$

in integers $n > a_t \geqslant a_{t-1} \geqslant \dots \geqslant a_1 > 1$, $t > 1$ are the trivial solutions $n = a_t + 1, a_t = a_1! \dots a_{t-1}! - 1$ apart from the following ones

$$7!\,3!\,3!\,2! = 9!, \qquad 7!\,6! = 10!, \qquad 7!\,5!\,3! = 10!, \qquad 14!\,5!\,2! = 16!$$

That these are all the solutions, is a conjecture due to Hickerson (see [2, p. 70]). The above authors also proved that the Hickerson conjecture holds (without assuming any conjecture) for $n \leqslant e^{80}$. Their method involves sharpening the lower bounds on the greatest prime factor of a product of consecutive positive integers. In general, Hickerson's conjecture is still open.

The additive analogue of the first problem above would be to find all positive integers a, b, c satisfying

$$(1 + 2 + \dots + a) + (1 + 2 + \dots + b) = 1 + 2 + \dots + c.$$

That is, one wishes to find all sums of two triangular numbers which are triangular: $t_a + t_b = t_c$ where $t_a = \frac{1}{2}a(a + 1)$. Analogous to the infinitely many 'trivial' solutions of the multiplicative version above, here also we have infinitely many solutions by taking $b + 1 = t_a$ (for which $c = b + 1$). Since $8t_a + 1 = 4a(a + 1) + 1 = (2a + 1)^2$, the equation $t_a + t_b = t_c$ is equivalent to

$$(2a + 1)^2 + (2b + 1)^2 = (2c + 1)^2 + 1.$$

By the above-mentioned observation, for any a, there is a solution $b + 1 = t_a$; that is, $2b + 1 = a^2 + a - 1$ and $2c + 1 = 2b + 3 = a^2 + a + 1$. In other words, $\left(2t + 1, t^2 + t - 1, t^2 + t + 1\right)$ is a polynomial parametrisation of the hyperboloid

$$x^2 + y^2 - z^2 = 1.$$

There are also other polynomial parametrisations of the hyperboloid. For instance, if we think of a^2 as the sum of the first a odd numbers, we obtain the polynomial parametrisation $\left(2t,\ 2t^2 - 1,\ 2t^2\right)$.

In terms of integer points, these give the families

$$\left(x,\ \frac{x^2 - 5}{4},\ \frac{x^3 + 3}{4}\right)$$

and

$$\left(x,\ \frac{x^2 - 2}{2},\ \frac{x^2}{2}\right)$$

of integer solutions when x is odd or even respectively.

More generally, any 3×3 matrix that preserves the quadratic form $x^2 + y^2 - z^2$ carries a point on the hyperboloid to another one.

Although one can obtain parametrisations such as above, one cannot expect a description of all integer points akin to Pythagorean triples for the circle problem. In fact, the problem is to be regarded in the same spirit as the notoriously difficult Gauss circle problem. The number of integer points (x, y, z) on the hyperboloid with $|x|, |y|, |z| \leqslant t$ can be estimated as follows.

If $r_d(n)$ denotes the number of ordered tuples $(x_1, \dots, x_d)$ of non-negative integers with $n = \sum_{I=1}^{d} x_i^2$, then consider the solutions $x^2 + y^2 = n = z^2 + 1$ for a given n. The number of solutions

$$\sum_{n \leqslant t} r_2(n) r_1(n - 1)$$

is known to have asymptotic growth $c\sqrt{t} \log t$. In fact, more generally the asymptotics of the number of integer points on a hyperboloid can naturally be obtained only by viewing the problem as one in the ergodic theory of homogeneous spaces (see [3]); in their paper, Hee Oh and Nimish A. Shah obtained the same asymptotics. However, the import of our brief note is to show how an obvious additive analogue of an unsolved multiplicative arithmetic problem leads to an old problem that has been studied for its own sake.

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References

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