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UNITRIANGULAR FACTORIZATIONS OF CHEVALLEY GROUPS

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Lately, the following problem attracted a lot of attention in various contexts: find the shortest factorization $G = UU^-UU^- \dots U^{\pm}$ of a Chevalley group $G = G(\Phi, R)$ in terms of the unipotent radical $U = U(\Phi, R)$ of the standard Borel subgroup $B = B(\Phi, R)$ and the unipotent radical $U^- = U^-(\Phi, R)$ of the opposite Borel subgroup $B^- = B^-(\Phi, R)$. So far, the record over a finite field was established in a 2010 paper by Babai, Nikolov, and Pyber, where they prove that a group of Lie type admits the unitriangular factorization $G = UU^-UU^-U$ of length 5. Their proof invokes deep analytic and combinatorial tools. In the present paper, we notice that from the work of Bass and Tavgen one immediately gets a much more general result, asserting that over any ring of stable rank 1 one has the unitriangular factorization $G = UU^-UU^-$ of length 4. Moreover, we give a detailed survey of triangular factorizations, prove some related results, discuss prospects of generalization to other classes of rings, and state several unsolved problems. Another main result of the present paper asserts that, in the assumption of the Generalized Riemann's Hypothesis, Chevalley groups over the ring $\mathbb{Z}\left[\frac{1}{p}\right]$ admit the unitriangular factorization $G = UU^-UU^-UU^-$ of length 6. Otherwise, the best length estimate for Hasse domains with infinite multiplicative groups that follows from the work of Cooke and Weinberger, gives 9 factors. Bibliography: 67 titles.

In the present paper, we show that comparing results of Hyman Bass [14] with those of Oleg Tavgen [10], one immediately gets the following result, which is both more general and more precise than all recent results pertaining to unitriangular factorizations.

Theorem 1. *Let Φ be a reduced irreducible root system and R be a commutative ring such that* $\mathrm{sr}(R) = 1$. *Then a simply connected Chevalley group $G(\Phi, R)$ admits the unitriangular factorization*

$$G(\Phi, R) = U(\Phi, R)U^-(\Phi, R)U(\Phi, R)U^-(\Phi, R)$$

of length 4.

On the other hand, since $U^-(\Phi, R) \cap B(\Phi, R) = 1$, one has

$$T(\Phi, R) \cap U(\Phi, R)U^-(\Phi, R)U(\Phi, R) = 1.$$

In other words, 1 is the *only* element of the torus $T(\Phi, R)$ that admits a unitriagular factorization of length < 4. Thus, if the ring R has at least one nontrivial unit, the factorization obtained in Theorem 1 is best possible.

What is truly amazing here, is that the usual *linear* stable rank condition works for groups of all types! Under somewhat stronger stability conditions, a similar result holds also for twisted Chevalley groups. However, the analysis of twisted groups requires much more detailed calculations for groups of Lie ranks 1 and 2, where one cannot simply invoke known results. The proofs for this case are relegated to the next paper by the authors.

Clearly, for rings of dimension ≥ 1 in general there is no way of obtaining unitriangular factorizations of length 4. However, for some particularly nice rings of dimension 1 one can obtain unitriangular factorizations of length 5 or 6. The second main result of the present paper gives the simplest example of such a kind.

Theorem 2. *Let Φ be a reduced irreducible root system and $p \in \mathbb{Z}$ be a rational prime. Under the assumption of the Generalized Riemann's Hypothesis the simply connected Chevalley group $G\left(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\right)$ admits the unitriangular factorization*

$$G\Big(\Phi, \mathbb{Z}\Big[\frac{1}{p}\Big]\Big) = \Big(U\Big(\Phi, \mathbb{Z}\Big[\frac{1}{2}\Big]\Big)U^-\Big(\Phi, \mathbb{Z}\Big[\frac{1}{2}\Big]\Big)\Big)^3$$

of length 6.

Presently, we are working toward generalization of this result to other Hasse domains and plan to return to this topic in a separate paper.

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Since there are a considerable number of papers addressing some aspects of unitriangular factorizations, and many of these papers give no reference to previous works, where similar – or even stronger! – results are established, we start our paper with a brief survey of known results, as we understand them. This is done in Secs. 1–4. After that in Secs. 5 and 6 we introduce basic notation. In Sec. 7, we prove a version of Tavgen's theorem on rank reduction, which immediately implies Theorem 1. In Sec. 8, we discuss the connection of unitriangular factorizations with Bruhat and Gauss decompositions. In Sec. 9, we consider Chevalley groups over arithmetic rings and prove Theorem 2. Finally, in Sec. 10, we mention some further related topics and state several unsolved problems.

1. Existence of unitriangular factorizations

Recently, the following problem cropped up anew, in several independent contexts: find the shortest factorization $G = UU^-UU^- \dots U^\pm$ of a Chevalley group $G = (\Phi, R)$, in terms of the unipotent radical $U = U(\Phi, R)$ of the standard Borel subgroup $B = B(\Phi, R)$ and the unipotent radical $U^- = U^-(\Phi, R)$ of the opposite Borel subgroup $B^- = B^-(\Phi, R)$.

Let us mention three such extensive clusters of various subjects where it occurred. In the following sections we see that although information between various subjects within one cluster is transmitted with serious delays, still communication is *incredibly* faster and more effective than information transfer between clusters themselves.

- Algebraic K-theory, structure theory of algebraic groups, theory of arithmetic groups.
- Theory of finite and profinite groups, asymptotic group theory, finite geometries.
- Computational linear algebra, wavelet theory, computer graphics, control theory.

Let us list some existing results concerning factorizations of the form $G = UU^-UU^- \dots U^\pm$. First, one has to establish the *existence* of such factorizations and, second, to estimate their *length*.

Clearly, the existence of a unitriangular factorization is equivalent to the following two conditions.

- The Chevalley group $G(\Phi, R)$ coincides with its elementary subgroup $E(\Phi, R)$ spanned by elementary generators.
- The width of the elementary Chevalley group $E(\Phi, R)$ with respect to elementary generators is bounded.

The answer to both of these questions is, in general, negative beyond retrieve, so that one can only expect the existence of unitriangular factorizations for some very special classes of rings. There is *vast* literature devoted to both of these problems, and we will not even attempt to address them in full. Instead, we confine ourselves to brief excerpts of the introduction to [59], referring to [61, 46] for a broader picture and further references.

- In the case where the ring R is semilocal – for instance, where it is of finite dimension over a field – for simply connected Chevalley groups, one has the equality $E(\Phi, R) = G(\Phi, R)$. The bounded width in terms of elementary generators immediately follows from the Gauss decomposition.
- It is classically known that expressions of a matrix in the group $\mathrm{SL}(2, R)$ over a Euclidean ring R as a product of elementaries correspond to continued fractions. The existence of arbitrary long division chains in $\mathbb{Z}$ shows that the group $\mathrm{SL}(2, \mathbb{Z})$ cannot have bounded width in elementary generators. Further such results are discussed in [22, 23].
- On the other hand, David Carter and Gordon Keller [15, 16] and Oleg Tavgen [9, 10, 61] demonstrated that for a Dedekind ring of arithmetic type R, the simply connected Chevalley groups $G(\Phi, R) = E(\Phi, R)$ of rank ≥ 2 have bounded width in elementary generators.
- Wilberd van der Kallen [30] made a striking discovery that, in general, even Chevalley groups of rank ≥ 2 over a Euclidean ring may have infinite width in elementary generators. More precisely, he proved that $\mathrm{SL}(3, \mathbb{C}[t])$ – and, since $\mathrm{sr}(\mathbb{C}[t]) = 2$, all groups $\mathrm{SL}(n, \mathbb{C}[t])$ for $n \geq 4$ – do not have bounded width in elementary generators.

Thus, in general one can hope to establish the existence of unitriangular factorizations only over some very special rings of dimension ≤ 1.

In the next three sections, we list some works addressing the length of unitriangular factorizations. Even this short summary shows that experts in one field are usually completely unaware of the standard notions and results in another field. It is hard to imagine, how much time and energy could have been saved, should millions of programmers and engineers learn the words *parabolic subgroup* and *Levi decomposition*, rather than persisting in retarded matrix manipulations.

The same remark applies to Application 1.1 of the work [13], where, at least for the linear case, the result was known for quite some time, in a much larger generality, and with better bound, to experts in other subjects, both in algebraic K-theory [24] and in computational linear algebra [60]. Obviously, after its driving forces are understood for the case of $\mathrm{SL}(n, R)$, its generalization to Chevalley groups does not require any serious intellectual effort, just the mastery of appropriate techniques.

2. The length of factorizations: linear groups

For the linear case, a systematic study of this problem was started in a paper by Keith Dennis and Leonid Vaserstein [24]. Originally, their interest in this problem was motivated by the following observation.

• Any element of the group $U(n, R)$, $n \geq 3$, over an arbitrary associative ring R is a product of no more than two commutators in the elementary group $E(n, R)$, Lemma 13. Earlier, van der Kallen [30] noticed that the elements of $U(n, R)$ are products of no more than three commutators.

Let us state several typical results of [24] pertaining to the length of unitriangular factorizations.

• If for a ring R the group $E(m, R)$, $m \geq 2$, is represented as a finite product $UU^-UU^- \dots U^{\pm}$ with L factors, then all groups $E(n, R)$, $n > m$, can be represented as the product of the same form, with the same number of factors, Lemma 7.

• If the stable rank of the ring R equals 1, then, as is classically known from the work of Hyman Bass [14], the group $E(2, R)$ – and thus by the previous item all groups $E(n, R)$ – admit the unitriangular factorization

$$E(n, R) = U(n, R)U^-(n, R)U(n, R)U^-(n, R)$$

of length 4.

To the authors of [24], as to all experts in algebraic K-theory, this fact is obvious to such an extent that in [24] it is not even separately stated, and just casually mentioned inside the proof of Theorem 6. Nevertheless, since this is one of the key steps in the proof of Theorem 1, and the *only one* that invokes the stability condition, we reproduce its proof, which is an adaptation of a more general similar calculation, implemented in [14].

Recall that a ring R has **stable rank** 1 if for all $x, y \in R$, which generate R as a *right* ideal, there exists a $z \in R$ such that $x + yz$ is *right* invertible. In this case we write $\mathrm{sr}(R) = 1$.

The simplest and most characteristic example of a ring of stable rank 1 is given by semilocal rings. However, there are many far less trivial examples, such as the ring of all algebraic integers. Many further examples and further references can be found in [64].

It is classically known that rings of stable rank 1 are actually *weakly finite* (the Kaplansky–Lenstra theorem), so that in their definition one could from the outset require that $x + yz \in R^*$. Since for the linear case the result is well known, and Chevalley groups of other types only exist over commutative rings, henceforth we assume that the ring R is commutative, in which case the proof below demonstrates at the same time that $\mathrm{SL}(2, R) = E(2, R)$.

In the following proof, and in the sequel, discussing linear case, we use the standard matrix notation. In particular, e denotes the identity matrix, e_{ij}, $1 \leq i, j \leq n$, denotes a standard matrix unit, i.e., the matrix which has 1 in the position (i, j) and zeroes in the other positions. Further, for $1 \leq i \neq j \leq n$ and $\xi \in R$ one denotes by $t_{ij}(\xi)$ the elementary transvection $e + \xi e_{ij}$. The matrix entry of g in the position (i, j) will be denoted by g_{ij}, whereas the corresponding entry of the inverse matrix g^{-1} will be denoted by g'_{ij}.

Lemma 1. *Let R be a commutative ring of stable rank 1. Then*

$$\mathrm{SL}(2, R) = U(2, R)U^-(2, R)U(2, R)U^-(2, R).$$

Proof. Let us trace how many elementary transformations one needs to put an arbitrary matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, R)$ in the form e. We will not introduce new notation at each step, but rather replace the matrix g by its current value, as is common in computer science. Obviously, its entries a, b, c, d should be also reset to their current values at each step.

Step 1. Multiplication by a single *lower* elementary matrix on the right allows one to make the element in the South-West corner invertible. Indeed, since the rows of the matrix are unimodular, one has $cR + dR = R$. Since $\mathrm{sr}(R) = 1$, there exists an $z \in R$ such that $c + dz \in R^*$. Thus,

$$gt_{21}(z) = \begin{pmatrix} a + bz & b \\ c + dz & d \end{pmatrix},$$

where $c + dz \in R^*$.

Step 2. Thus, we may assume that $c \in R^*$. Multiplication by a single *upper* elementary matrix on the right allows us to make the element in the South-East corner equal to 1. Indeed,

$$gt_{12}(c^{-1}(1-d)) = \begin{pmatrix} a & b + ac^{-1}(1-d) \\ c & 1 \end{pmatrix}.$$

Step 3. Thus, we may now assume that $d = 1$. Multiplication by a single *lower* elementary matrix on the right allows one to make the element in the South-West corner equal to 0. Indeed,

$$gt_{21}(-c) = \begin{pmatrix} a - bc & b \\ 0 & 1 \end{pmatrix}.$$

Since $\det(g) = 1$, the matrix on the right-hand side is equal to $t_{12}(b)$. Bringing all elementary factors to the right-hand side, we see that any matrix g with determinant 1 can be expressed as a product of the form $t_{12}(*)t_{21}(*)t_{12}(*)t_{21}(*)$, as claimed. □

Now, let us return to the Dennis–Vaserstein paper [24].

• If R is a Boolean ring, in other words, if $x^2 = x$ for all $x \in R$, then the elementary group $E(n, R)$, $n \geq 2$, admits the unitreiangular factorization

$$E(n, R) = U(n, R)U^-(n, R)U(n, R)$$

of length 3. For commutative rings the converse is obviously true: if for some $n \geq 2$ the group $E(n, R)$ admits the unitriangular factorization of length 3, then the ring R is Boolean, Remark 15.

• Let $d = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in R^*$, be a diagonal matrix such that $\varepsilon_1 \dots \varepsilon_n = 1$. Then

$$d \in U(n, R)U^-(n, R)U(n, R)U^-(n, R),$$

Lemma 18. The proof in [24] is based on general position arguments. In Sec. 8, we give another proof of a slightly stronger fact, based on induction on the rank, which can be easily generalized to all Chevalley groups.

• If the stable rank of the ring R is finite and for some $m \geq 2$, the group $E(m, R)$ has finite width with respect to the elementary generators, then for all sufficiently large n one has the unitriangular factorization

$$E(n, R) = (U(n, R)U^-(n, R))^3$$

of length 6.

• It is extremely suggestive to compare this result with the *Sharpe decomposition.* Recall that it is established in [54] that for any associative ring R, the finitary elementary group $E(R) = \varinjlim E(n, R)$, $g \mapsto g \oplus 1$, admits the decomposition

$$E(R) = B(R)N(R)U(R)U^-(R) = B(R)U^-(R)U(R)U^-(R) = U(R)D(R)U^-(R)U(R)U^-(R),$$

where $B(R)$, $N(R)$, $U(R)$, $U^-(R)$, and $D(R)$ denote inductive limits of the corresponding groups of finite degree with respect to the same embedding. Since $D(R) \subseteq U(R)U^-(R)U(R)U^-(R)$, it follows that for an arbitrary associative ring, the finitary elementary group $E(R)$ admits the unitriangular factorization

$$E(R) = (U(R)U^-(R))^3$$

of length 6.

• Recently, motivated by applications in the factorization of integer matrices, Thomas Laffey [31, 32] explicitly calculated the bound in the Dennis–Vaserstein theorem for the ring $R = \mathbb{Z}$. In particular, the unitriangular factorization

$$\mathrm{SL}(n, \mathbb{Z}) = (U(n, \mathbb{Z})U^-(n, \mathbb{Z}))^3$$

of length 6 holds for any $n \geq 82$.

3. The length of factorizations: Chevalley groups

The proof that Chevalley groups over Dedekind rings of arithmetic type have finite width in elementary generators, given by Oleg Tavgen [10], relied on a reduction to groups of rank 2. The base of this reduction is the following fact (see [10, Proposition 1]).

• If all Chevalley groups of a certain rank l admit a unitriangular factorization $G = (UU^-)^L$ of certain length, then all groups of larger ranks also admit unitriangular factorizations of the same length. Actually, our Theorem 3 is a minor elaboration of Tavgen's idea.

• Another elaboration of the same idea can be found in a recent paper by Andrei Rapinchuk and Igor Rapinchuk [51]. There, the same idea is used to establish that Chevalley groups over a local ring R can be factored as

$$G(\Phi, R) = (U(\Phi, R)U^-(\Phi, R))^4.$$

For the last decade, there was a vivid interest in explicit calculation of the length over fields, especially over finite fields. Let $\mathbb{F}_q$, $q = p^m$, be a finite field of characteristic p. In this case the group $U(\Phi, q) = U(\Phi, \mathbb{F}_q)$ is a Sylow p-subgroup of the Chevalley group $G(\Phi, q) = G(\Phi, \mathbb{F}_q)$. Thus, in this case the calculation of the minimal length of the unitriangular factorization is essentially equivalent to the calculation of the minimal length of the factorization of a finite simple group of Lie type in terms of its Sylow p-subgroups, in the defining characteristic.

• Martin Liebeck and Laszlo Pyber in [41, Theorem D] prove that finite Chevalley groups admit the unitriangular factorization

$$G(\Phi, q) = (U(\Phi, q)U^-(\Phi, q))^6 U(\Phi, q)$$

of length 13. Actually, they also consider twisted Chevalley groups and obtain for them a similar bound, with the only exception of the senior Ree groups ${}^2\mathrm{F}_4(q)$, for which they prove the existence of the factorization $G = (UU^-)^{12}U$.

• Laszlo Babai, Nikolay Nikolov, and Laszlo Pyber [13, Application 1.1] prove that finite Chevalley groups admit the unitriangular factorization

$$G(\Phi, q) = U(\Phi, q)U^-(\Phi, q)U(\Phi, q)U^-(\Phi, q)U(\Phi, q)$$

of length 5. They also obtain a similar result with the same bound for twisted groups.

As opposed to [41], the proofs in paper [13] are of extremely mysterious nature. In the final count, they rely on the product growth estimates of the following sort. Let $X, Y \subseteq G$ be two nonempty subsets of a finite group G. Then

$$|XY| \geq \min\left(\frac{|G|}{2}, \frac{m|X| \cdot |Y|}{2|G|}\right),$$

where m denotes the smallest dimension of a nontrivial real representation of the group G. In particular, in the case where the group G is simple and the orders $|X|$ and $|Y|$ are large enough, but still much smaller than $|G|$, products rapidly grow: the order $|XY|$ is much larger than $\max(|X|, |Y|)$.

In this connection, the authors of [13] make the following claim: "For the most part we can argue by the size of certain subsets, ignoring the structure and thus greatly simplifying the proofs and at the same time obtaining considerably better, nearly optimal bounds."

Frankly, the proposal to completely ignore the structure of the objects in question, seems to us excessive. Asymptotic methods are in their place in the study of profinite groups, or groups of infinite rank. On the other hand, when applied to the finite groups proper, or to groups of Lie type of finite rank, asymptotic methods should be considered as a *surrogate* of structural algebraic methods, as a method of verifying the result we are interested in, which is *identical* to a real proof.

Certainly, in many situations such a genuine algebraic proof does not exist. What is worse, for many interesting problems such algebraic solutions are not in sight, since they would either require *enormous* case by case analysis, or cannot be obtained by the present day methods.

As an example of the first situation, we can cite the remarkable paper by Martin Liebeck and Aner Shalev [36], where it is proved that – apart from three exceptional series, the Suzuki groups and the groups $\mathrm{Sp}(4, 2^m)$ and $\mathrm{Sp}(4, 3^m)$ – almost all finite simple groups are $(2,3)$-generated. For each individual group of Lie type, and at that not only over a finite field, but also over finitely generated rings, it is basically clear how to prove its $(2,3)$-generation, or the lack thereof. In fact, for important classes of groups, including all groups of sufficiently large Lie ranks (in the order of magnitude of several dozens, for finite simple groups) such constructive algebraic proofs have been obtained. However, a complete analysis of groups of small ranks requires computational effort so extensive that it has not been completed yet, despite enormous work by many authors, over more than two decades.

As an example of the second situation, we could cite recent papers on the verbal width of finite simple groups. An amazing general result by Aner Shalev [53] asserts that for *any* nontrivial word $w \neq 1$, the verbal width of almost all finite simple groups G equals 3. In other words, every element of the group G can be represented as a product of no more than three values of the word w. See also [28, 34, 37, 40, 52], where one can find further results in the same style, an improvement of the estimate for verbal width to 2, in some cases, and references to preceding works. As far as we know, similar results are unavailable not only over rings, but even over infinite fields. What is worse, such similar results are n unavailable not just for all words, but for simplest *specific* words, such as powers. What is still worse, there is no clear understanding, how one could prove such results at all.

Nevertheless, we believe that whenever one can apply algebraic methods, coming from the structure theory and representation theory of algebraic groups, they would *invariably* give stronger and more general results. For those results that hold over arbitrary fields they should also give better bounds, and *hopefully* have simpler proofs.

4. The length of factorizations: computational linear algebra

Definitely, unitriangular factorizations for the group $\mathrm{SL}(n, K)$ are so natural and obvious that there is little doubt that they should have been known to experts in linear algebra for quite some time. This is however the earliest reference we could trace.

- Gilbert Strang [60] noticed that all groups $\mathrm{SL}(n, K)$ over a field K admit the unitriangular factorization

$$\mathrm{SL}(n, K) = U^-(n, K)U(n, K)U^-(n, K)U(n, K)$$

of length 4. This is what experts in computational linear algebra call the LULU-factorization, where mnemonically L should be interpreted as the first letter of the word 'lower', whereas U should be interpreted as the first letter of the word 'upper'.

Observe that this fact got to the household of linear algebra decades after it became standard in algebraic K-theory, and at that in a much larger generality. However, as we know, the walls between different branches of mathematics are high.

As another amusing circumstance, we could mention that, as far as we know, this result was only observed in linear algebra due to applications in computer graphics! Let us explain in somewhat more detail how it happened. It is hard to us to construe the *precise* meaning in which experts in computer algebra use the term *shear*. As a first approximation one can think of shear as an arbitrary unipotent element of $\mathrm{SL}(n, K)$. Or, at least, the majority of authors in this field call arbitrary elements of the groups $U(n, K)$ and $U^-(n, K)$ shears.

However, there is no doubt how one should interpret the term *one-dimensional shears*. These are precisely transvections, not necessarily elementary. An exclusive role is played by transvections concentrated in one row – they are called *beam shears* – or in one column – they are called *slice shears*. The point is that at the level of pixels such transvections correspond to string copy with offset, which admits *extremely* efficient hardware implementations.

Alain Paeth [50] proposed the following cute method for implementing a 2D rotation:

$$\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} 1 & \tan(\varphi/2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sin(\varphi) & 1 \end{pmatrix} \begin{pmatrix} 1 & \tan(\varphi/2) \\ 0 & 1 \end{pmatrix}.$$

As we already mentioned, the shears themselves are implemented at the level of data transfer, and this method turned out to be much more efficient than coordinate conversion.

At this point it is natural to ask, whether 3D rotations admit a similar efficient hardware implementation. Initially, one first attempted to decompose a 3D rotation into a product of three 2D rotations, and then each of them into a product of three transvections. However, shortly thereafter Tommaso Toffoli and Jason Quick [63] proposed a scheme based upon the decomposition of a 3D rotation into a product of three unipotent matrices, and designed the corresponding hardware.

Let us recall that a three-dimensional rotation g is completely determined by its Euler angles (α, β, γ), for instance, as follows:

$$g = \begin{pmatrix} c(\alpha)c(\beta)c(\gamma) - s(\alpha)s(\gamma) & -c(\alpha)c(\beta)s(\gamma) - s(\alpha)c(\gamma) & c(\alpha)s(\beta) \\ s(\alpha)c(\beta)c(\gamma) + c(\alpha)s(\gamma) & -s(\alpha)c(\beta)s(\gamma) + c(\alpha)c(\gamma) & s(\alpha)s(\beta) \\ -s(\beta)c(\gamma) & s(\beta)s(\gamma) & c(\beta) \end{pmatrix},$$

where $c(\varphi)$ and $s(\varphi)$ denote $\cos(\varphi)$ and $\sin(\varphi)$, respectively. It is easy to notice – this was the starting point of [63] – that g admits the following unitriangular factorization of length 3:

$$g = \begin{pmatrix} 1 & -\tan\left(\frac{\alpha+\gamma}{2}\right) & \cos(\alpha)\tan\left(\frac{\beta}{2}\right) \\ 0 & 1 & \sin(\alpha)\tan\left(\frac{\beta}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ \sin(\alpha+\gamma) & 1 & 0 \\ -\cos(\gamma)\sin(\beta) & -\frac{\sin\left(\frac{\alpha-\gamma}{2}\right)}{\cos\left(\frac{\alpha+\gamma}{2}\right)}\sin(\beta) & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & -\tan\left(\frac{\alpha+\gamma}{2}\right) & \frac{\cos\left(\frac{\alpha-\gamma}{2}\right)}{\cos\left(\frac{\alpha+\gamma}{2}\right)}\tan\left(\frac{\beta}{2}\right) \\ 0 & 1 & -\sin(\gamma)\tan\left(\frac{\beta}{2}\right) \\ 0 & 0 & 1 \end{pmatrix}.$$

The work by Strang [60], cited at the beginning of the section, emerged as an attempt to generalize this formula to the case of an arbitrary n. After that many subsequent papers appeared, which discussed various aspects of such decompositions. To convey some flavor of this activity, we state a couple of typical results from the next paper by Toffoli [62].

• Almost all elements $\mathrm{SL}(n, \mathbb{R})$ admit unitriangular factorizations of length 3 – the ULU-factorization, as it is called by the experts in computational linear algebra. Obviously, as opposed to the previous section, here the expression 'almost all' should be interpreted in terms of Lebesgue measure.

• For any matrix $g \in \mathrm{SL}(n, K)$ there exists a permutation matrix (π), $\pi \in \mathrm{S}_n$, such that at least one of the matrices $(\pi)g$ or $(\pi)^{-1}g$ admits the unitriangular factorization of length 3.

We will make no attempts to cover systematically subsequent literature in the field, and limit ourselves to several typical papers [19, 29, 33, 55], where one can find further references.

5. Some subgroups of Chevalley groups

Our notation pertaining to Chevalley groups is utterly standard and coincides with that used in [66, 67], where one can find many further references.

Let Φ be a reduced irreducible root system of rank l, $W = W(\Phi)$ be its Weyl group, and $\mathcal{P}$ be a weight lattice intermediate between the root lattice $\mathcal{Q}(\Phi)$ and the weight lattice $\mathcal{P}(\Phi)$. Further, we fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+ and Φ^- the corresponding sets of fundamental, positive and negative roots, respectively. Our numbering of fundamental roots follows Bourbaki. Finally, let R be a commutative ring with 1; as usual, R^* denotes its multiplicative group.

It is classically known that with these data one can associate the Chevalley group $G = G_{\mathcal{P}}(\Phi, R)$, which is the group of R-points of an affine group scheme $G_{\mathcal{P}}(\Phi, -)$, known as the Chevalley–Demazure group scheme. Since mostly our results do not depend on the choice of the lattice $\mathcal{P}$, in the sequel we usually assume that $\mathcal{P} = \mathcal{P}(\Phi)$ and omit any reference to $\mathcal{P}$ in the notation. Thus, $G(\Phi, R)$ will denote the simply connected Chevalley group of type Φ over R.

In what follows, we fix a split maximal torus $T(\Phi, -)$ of the group scheme $G(\Phi, -)$ and set $T = T(\Phi, R)$. As usual, X_α, $\alpha \in \Phi$, denotes a unipotent root subgroup in G, elementary with respect to T. We fix isomorphisms $x_\alpha : R \mapsto X_\alpha$, so that $X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}$, which are interrelated by the Chevalley commutator formula (see [8, 18, 67]). Further, $E(\Phi, R)$ denotes the elementary subgroup of $G(\Phi, R)$ generated by all root subgroups X_α, $\alpha \in \Phi$.

In the sequel, the elements $x_\alpha(\xi)$ are called root unipotents. Now, let $\alpha \in \Phi$ and $\varepsilon \in R^*$. As usual, we set $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$, where $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$. The elements $h_\alpha(\varepsilon)$ are called semisimple root elements. For a simply connected group, one has

$$T = T(\Phi, R) = \langle h_\alpha(\varepsilon),\ \alpha \in \Phi,\ \varepsilon \in R^* \rangle.$$

Finally, let $N = N(\Phi, R)$ be the algebraic normalizer of the torus $T = T(\Phi, R)$, i.e., the subgroup, generated by $T = T(\Phi, R)$ and all elements $w_\alpha(1)$, $\alpha \in \Phi$. The factor-group N/T is canonically isomorphic to the Weyl group W, and for each $w \in W$ we fix its preimage n_w in N.

The following result is obvious, well known, and very useful.

Lemma 2. *The elementary Chevalley group $E(\Phi, R)$ is generated by unipotent root elements $x_\alpha(\xi)$, $\alpha \in \pm\Pi$, $\xi \in R$, corresponding to the fundamental and negative fundamental roots.*

Proof. Indeed, every root is conjugate to a fundamental root by an element of the Weyl group, while the Weyl group itself is generated by the fundamental reflections w_α, $\alpha \in \Pi$. Thus, the elementary group $E(\Phi, R)$ is generated by the root unipotents $x_\alpha(\xi)$, $\alpha \in \Pi$, $\xi \in R$, and the elements $w_\alpha(1)$, $\alpha \in \Pi$. It remains only to observe that $w_\alpha(1) = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$. □

Further, let $B = B(\Phi, R)$ and $B^- = B^-(\Phi, R)$ be a pair of opposite Borel subgroups containing $T = T(\Phi, R)$, standard with respect to the given order. Recall that B and B^- are semidirect products $B = T \leftthreetimes U$ and $B^- = T \leftthreetimes U^-$ of the torus T and their unipotent radicals

$$U = U(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^+,\ \xi \in R \rangle,$$
$$U^- = U^-(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^-,\ \xi \in R \rangle.$$

Here, as usual, for a subset X of a group G one denotes by $\langle X \rangle$ the subgroup in G generated by X. The semidirect product decomposition of B amounts to saying that $B = TU = UT$, and at that $U \trianglelefteq B$ and $T \cap U = 1$. Similar facts hold with B and U replaced by B^- and U^-. Sometimes, to speak of both subgroups U and U^- simultaneously, we denote $U = U(\Phi, R)$ by $U^+ = U^+(\Phi, R)$.

Generally speaking, with any closed subset S in Φ one can associate a subgroup $E(S) = E(S, R)$. Recall that a subset S in Φ is called *closed* if for any two roots $\alpha, \beta \in S$ the fact that $\alpha + \beta \in \Phi$ implies that already $\alpha + \beta \in S$. Now, we define $E(S) = E(S, R)$ as the subgroup generated by all elementary root unipotent subgroups X_α, $\alpha \in S$:

$$E(S, R) = \langle x_\alpha(\xi), \quad \alpha \in S, \quad \xi \in R \rangle.$$

In this notation, U and U^- coincide with $E(\Phi^+, R)$ and $E(\Phi^-, R)$, respectively. The groups $E(S, R)$ are particularly important in the case where S is a *special* = *unipotent* set of roots, in other words, where $S \cap (-S) = \varnothing$. In this case, $E(S, R)$ coincides with the *product* of root subgroups X_α, $\alpha \in S$, in a fixed order.

Let again $S \subseteq \Phi$ be a closed set of roots. Then S can be decomposed into a disjoint union of its *reductive* = *symmetric* part S^r, consisting of those $\alpha \in S$, for which $-\alpha \in S$, and its *unipotent* part S^u, consisting of those $\alpha \in S$, for which $-\alpha \notin S$. The set S^r is a closed root subsystem, whereas the set S^u is special. Moreover, S^u is an *ideal* of S; in other words, if $\alpha \in S$, $\beta \in S^u$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in S^u$. The *Levi decomposition* asserts that the group $E(S, R)$ decomposes into semidirect product $E(S, R) = E(S^r, R) \leftthreetimes E(S^u, R)$ of its *Levi subgroup* $E(S^r, R)$ and its *unipotent radical* $E(S^u, R)$.

6. Elementary parabolic subgroups

The main role in the proof of Theorem 1 is played by the Levi decomposition for elementary parabolic subgroups. Denote by $m_k(\alpha)$ the coefficient of α_k in the expansion of α with respect to the fundamental roots:

$$\alpha = \sum m_k(\alpha)\alpha_k, \quad 1 \le k \le l.$$

Now, fix an $r = 1, \dots, l$ – in fact, in the reduction to a smaller rank it suffices to employ only terminal parabolic subgroups, even only the ones corresponding to the first and the last fundamental roots, $r = 1, l$. Denote by

$$S = S_r = \{\alpha \in \Phi,\ m_r(\alpha) \ge 0\}$$

the rth standard parabolic subset in Φ. As usual, the reductive part $\Delta = \Delta_r$ and the special part $\Sigma = \Sigma_r$ of the set $S = S_r$ are defined as

$$\Delta = \{\alpha \in \Phi,\ m_r(\alpha) = 0\}, \quad \Sigma = \{\alpha \in \Phi,\ m_r(\alpha) > 0\}.$$

The opposite parabolic subset and its special part are defined similarly:

$$S^- = S_r^- = \{\alpha \in \Phi,\ m_r(\alpha) \le 0\}, \quad \Sigma^- = \{\alpha \in \Phi,\ m_r(\alpha) < 0\}.$$

Obviously, the reductive part S_r^- equals Δ.

Denote by P_r the *elementary* maximal parabolic subgroup of the elementary group $E(\Phi, R)$. By definition,

$$P_r = E(S_r, R) = \langle x_\alpha(\xi),\ \alpha \in S_r,\ \xi \in R\rangle.$$

Now the Levi decomposition asserts that the group P_r can be represented as the semidirect product

$$P_r = L_r \leftthreetimes U_r = E(\Delta, R) \leftthreetimes E(\Sigma, R)$$

of the elementary Levi subgroup $L_r = E(\Delta, R)$ and the unipotent radical $U_r = E(\Sigma, R)$. Recall that

$$L_r = E(\Delta, R) = \langle x_\alpha(\xi),\ \alpha \in \Delta,\ \xi \in R\rangle,$$

whereas

$$U_r = E(\Sigma, R) = \langle x_\alpha(\xi),\ \alpha \in \Sigma,\ \xi \in R\rangle.$$

A similar decomposition holds for the opposite parabolic subgroup P_r^-, whereby the Levi subgroup is the same as for P_r, but the unipotent radical U_r is replaced by the opposite unipotent radical $U_r^- = E(-\Sigma, R)$

As a matter of fact, we use the Levi decomposition in the following form. It will be convenient to change slightly the notation and to write $U(\Sigma, R) = E(\Sigma, R)$ and $U^-(\Sigma, R) = E(-\Sigma, R)$.

Lemma 3. *The group $\langle U^\sigma(\Delta, R), U^\rho(\Sigma, R)\rangle$, where $\sigma, \rho = \pm 1$, is the semidirect product of its normal subgroup $U^\rho(\Sigma, R)$ and the complementary subgroup $U^\sigma(\Delta, R)$.*

In other words, it is asserted here that the subgroup $U^\pm(\Delta, R)$ normalizes each of the groups $U^\pm(\Sigma, R)$, so that, in particular, one has the following four relations for products

$$U^\pm(\Delta, R)U^\pm(\Sigma, R) = U^\pm(\Sigma, R)U^\pm(\Delta, R),$$

and, furthermore, the following four obvious relations for intersections hold:

$$U^\pm(\Delta, R) \cap U^\pm(\Sigma, R) = 1.$$

In particular, one has the decompositions

$$U(\Phi, R) = U(\Delta, R) \leftthreetimes U(\Sigma, R), \quad U^-(\Phi, R) = U^-(\Delta, R) \leftthreetimes U^-(\Sigma, R).$$

7. Reduction to groups of smaller rank

The following result is a minor elaboration of Proposition 1 from the paper by Oleg Tavgen [10]. Tavgen states a slightly weaker but more general result, in terms of the existence of factorizations for *all* irreducible root systems of a certain rank.[1] On the other hand, he considers also twisted groups, apart from those of type ${}^2\mathrm{A}_{2l}$.

Theorem 3. *Let Φ be a reduced irreducible root system of rank $l \geq 2$ and R be a commutative ring. Suppose that for subsystems $\Delta = \Delta_1, \Delta_l$ the elementary Chevalley group $E(\Delta, R)$ admits a unitriangular factorization*

$$E(\Delta, R) = (U(\Delta, R)U^-(\Delta, R))^L.$$

Then the elementary Chevalley group $E(\Phi, R)$ admits the unitriangular factorization

$$E(\Phi, R) = (U(\Phi, R)U^-(\Phi, R))^L.$$

of the same length $2L$.

Clearly, Theorem 1 immediately follows from Lemma 1 and Theorem 3, so that it only remains to prove Theorem 3.

The leading idea of Tavgen's proof is so general and beautiful that it works in many other similar contexts. It relies on the fact that for systems of rank ≥ 2 every fundamental root falls into the subsystem of smaller rank obtained by dropping either the first or the last fundamental root. Eiichi Abe and Kazuo Suzuki [11] and [12] used the same argument in their description of normal subgroups, to extract root unipotents. A similar consideration, in conjunction with a general position argument, was used by Vladimir Chernousov, Erich Ellers, and Nikolai Gordeev in their simplified proof of the Gauss decomposition with prescribed semisimple part [21].

Let us reproduce the details of the argument. By definition,

$$Y = (U(\Phi, R)U^-(\Phi, R))^L$$

is a *subset* in $E(\Phi, R)$. Usually, the easiest way to prove that a subset $Y \subseteq G$ coincides with the whole group G consists in the following.

[1] Observe that one should read the first relation in Proposition 1 from [10] as $\mathrm{rk}({}^\sigma\Phi_0) = m$.

Lemma 4. *Assume that $Y \subseteq G$, $Y \neq \varnothing$, and let $X \subseteq G$ be a symmetric generating set. If $XY \subseteq Y$, then $Y = G$.*

Proof of Theorem 3. By Lemma 2, the group G is generated by the fundamental root elements

$$X = \{x_\alpha(\xi) \mid \alpha \in \pm\Pi,\ \xi \in R\}.$$

Thus, by Lemma 4, it suffices to prove that $XY \subseteq Y$.

Let us fix a fundamental root unipotent $x_\alpha(\xi)$. Since $\mathrm{rk}(\Phi) \geq 2$, the root α belongs to at least one of the subsystems $\Delta = \Delta_r$, where $r = 1$ or $r = l$, generated by all fundamental roots, except for the first or the last one, respectively. Set $\Sigma = \Sigma_r$ and express $U^{\pm}(\Phi, R)$ in the form

$$U(\Phi, R) = U(\Delta, R)U(\Sigma, R), \quad U^-(\Phi, R) = U^-(\Delta, R)U^-(\Sigma, R).$$

Using Lemma 3, we see that

$$Y = (U(\Delta, R)U^-(\Delta, R))^L(U(\Sigma, R)U^-(\Sigma, R))^L.$$

Since $\alpha \in \Delta$, one has $x_\alpha(\xi) \in E(\Delta, R)$, so that the inclusion $x_\alpha(\xi)Y \subseteq Y$ immediately follows from the assumption. □

8. The use of the Bruhat and Gauss decompositions

Observe that, as opposed to the proofs in [13], the proof in the paper by Liebeck and Pyber [41] is natural and transparent, and is based on the Bruhat decompsoition $g = udn_wv$, where $u, v \in U$, $d \in T$, and $w \in W$. The excessive number of factors in their result is due to the fact that they decompose d and n_w separately, and, on top of that, the way they do it is far from being optimal. For instance, to decompose an element $h_\alpha(\varepsilon)$ they rely directly on the definition $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$, rather than perform the actual calculation in $\mathrm{SL}(2, R)$. Substituting the expression of $w_\alpha(\varepsilon)$ in terms of root unipotents in the definition of $h_\alpha(\varepsilon)$, we express $h_\alpha(\varepsilon)$ as a product of 5 root unipotents as follows:

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Actually, it is well known that $h_\alpha(\varepsilon)$ is already a product of 4 root unipotents:

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 - \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon(\varepsilon - 1) & 1 \end{pmatrix}.$$

As a matter of fact, the same idea allows one to obtain immediately the unitriangular factorization of the same length 5, as in [13] for Chevalley groups over semilocal rings. The proof is based on the Gauss decomposition and the following toy version of Theorem 1.

Recall that the following analog of the Gauss decomposition was established by Eiichi Abe and Kazuo Suzuki [11, 12] and by Michael Stein [58].[2]

Lemma 5. *Let R be a semilocal ring. Then for a simply connected Chevalley group, one has the decomposition*

$$G(\Phi, R) = U(\Phi, R)T(\Phi, R)U^-(\Phi, R)U(\Phi, R).$$

As we know, the Gauss decomposition for the group $\mathrm{SL}(n, R)$ holds under a weaker assumption $\mathrm{sr}(R) = 1$ (see, *for instance*, [65]. The first author of [2] noticed that, in fact, the condition $\mathrm{sr}(R) = 1$ is *necessary* for a Chevalley group $G(\Phi, R)$ to admit the Gauss decomposition, for *all* types. For the linear case, this obvious circumstance was rediscovered some 20+ years later in [47, 20]. On the other hand, for all types, except for $\Phi = \mathrm{A}_l, \mathrm{C}_l$, all known sufficient conditions are somewhat stronger, say, something like $\mathrm{asr}(R) = 1$. Thus, there is still a gap between the necessary and sufficient conditions.

The following result is a generalization of Lemma 18 of [24], where a slightly weaker fact is proved for the case $\Phi = \mathrm{A}_l$.

[2] In terms of computational linear algebra this is the ULU-decomposition, whereas the Bruhat decomposition is the UPU-decomposition, where P should be interpreted as the first letter of the word "permutation." Though, Stein himself called this decomposition a *Bruhat type decomposition*. Erich Ellers and Nikolai Gordeev usually intend by the Gauss decomposition the LPU-decomposition, which is normally called the *Birkhoff decomposition*. At the same time, in the works on computational linear algebra the name 'Gauss decomposition' often refers either to the PLU-decomposition or to the LUP-decomposition.

Theorem 4. *Let Φ be a reduced irreducible root system and R be an arbitrary commutative ring. Then one has the following inclusion*
$$N(\Phi,R)\subseteq U(\Phi,R)U^{-}(\Phi,R)U(\Phi,R)U^{-}(\Phi,R).$$

Corollary. *Let Φ be a reduced irreducible root system and R be a commutative semilocal ring. Then the simply connected Chevalley group $G(\Phi,R)$ admits the unitriangular factorization*
$$G(\Phi,R)=U(\Phi,R)U^{-}(\Phi,R)U(\Phi,R)U^{-}(\Phi,R)U(\Phi,R)$$
of length 5.

Proof. Indeed,
$$G=UTU^{-}U\leq U(UU^{-}UU^{-})U^{-}U=UU^{-}UU^{-}U.$$
□

Since this result is both less general in terms of the condition on the ground ring and weaker in terms of the resulting length than Theorem 1, we do not present the proof of Theorem 4 in general. However, for the purpose of illustration, we reproduce its proof in the linear case. Recall that in this case, the group $N(\mathrm{A}_{n-1},R)$ coincides with the group $N=\mathrm{SN}(n,R)$ of monomial matrices with determinant 1.

Proof of Theorem 4 for $\Phi=\mathrm{A}_{n-1}$. Let $g=(g_{ij})\in\mathrm{SN}(n,R)$. Let us argue by induction on n. In the case $n=1$, there is nothing to prove. Thus, let $n\geq 2$.

Case 1. First, let $g_{nn}=0$. Then there exists a unique $1\leq r\leq n-1$ such that $a=g_{rn}\neq 0$ and a unique $1\leq s\leq n-1$ such that $b=g_{ns}\neq 0$, all other entries in the sth and nth columns are equal to 0. Since g is invertible, automatically $a,b\in R^{*}$. The matrix $gt_{sn}(b^{-1})$ differs from g only in the position (n,n), where now we have 1 instead of 0. Consecutively multiplying the resulting matrix on the right by $t_{ns}(-b)$ and then by $t_{sn}(b^{-1})$, we get the matrix h that differs from g only at the intersection of the rth and nth rows with the sth and nth columns, where now instead of $\begin{pmatrix}0&a\\b&0\end{pmatrix}$ one has $\begin{pmatrix}-ab&0\\0&1\end{pmatrix}$. Observe that the determinant of the leading submatrix of order $n-1$ of the matrix h equals 1, and thus we can apply the induction hypothesis and obtain for that last matrix the desired factorization in the group $\mathrm{SL}(n-1,R)$. This factorization does not affect the last row and the last column. Thus, all factors of the above factorization of the submatrix h lie in L_{n-1}. It only remains to notice that $t_{sn}(b^{-1})\in U_{n-1}$, and $t_{ns}(-b)\in U^{-}_{n-1}$ and then to invoke Lemma 3.

Case 2. Now, let $b=g_{nn}\neq 0$. Take arbitrary $1\leq r,\ s\leq n-1$ for which $a=g_{rs}\neq 0$. Again, automatically $a,b\in R^{*}$. As in the previous case, we concentrate on the rth and nth rows and the sth and nth columns. Since there are no further nonzero entries in these rows and columns, any additions between them do not change other entries of the matrix, and only affect the submatrix at the intersection of the rth and nth rows with the sth and nth columns. Now, multiplying g by $t_{ns}(b^{-1})t_{sn}(1-b)t_{ns}(-1)t_{ns}(-b^{-1}(1-b))$ on the right, we obtain the matrix h, where this submatrix, which was initially equal to $\begin{pmatrix}a&0\\0&b\end{pmatrix}$, will be replaced by $\begin{pmatrix}ab&0\\0&1\end{pmatrix}$. At this point the proof can be completed in exactly the same way as in the previous case. □

9. Dedekind rings of arithmetic type

Let K be a global field, i.e., either a finite algebraic extension of the field $\mathbb{Q}$ or a field of algebraic functions in one variable over a finite field of constants $\mathbb{F}_q$. Further, let S be a finite set of nonequivalent valuations of K, nonempty in the functional case and containing all Archimedean valuations in the number case. For a non-Archimedean valuation $\mathfrak{p}$ of the field K we denote by $v_{\mathfrak{p}}$ the corresponding exponent.

As usual, $R=\mathcal{O}_S$ denotes the ring consisting of all $x\in K$ such that $v_{\mathfrak{p}}(x)\geq 0$ for all valuations $\mathfrak{p}$ of the field K, which do not belong to S. The ring $\mathcal{O}_S$ is called the Dedekind ring of arithmetic type determined by the set of valuations S of the field K, or, otherwise, a Hasse domain (see, for instance, [1]). We will be mostly interested in the case $|S|\geq 2$, where, by the Dirichlet unit theorem, the ring $\mathcal{O}_S$ has a unit of infinite order.

As another immediate consequence of the reduction to smaller ranks, of the positive solution for the congruence subgroup problem [1, 43, 7] and of the Cooke–Weinberger paper [23], one can state the following result. Observe that Cooke–Weinberger's calculations depend on the infinity of primes in arithmetic progressions, subject to additional multiplicative restrictions, and thus depend upon the GRH=Generalized Riemann's Hypothesis.

Theorem 5. *Let Φ be a reduced irreducible root system and $R = \mathcal{O}_S$ be a Dedekind ring of arithmetic type with an infinite multiplicative group. Under the assumption of the Generalized Riemann's Hypothesis, the simply connected Chevalley group $G(\Phi, R)$ admits the unitriangular factorization*

$$G(\Phi, R) = (U(\Phi, R)U^-(\Phi, R))^4 U(\Phi, R)$$

of length 9.

One can prove similar results also in the absence of the Generalized Riemann's Hypothesis. However, in that case the known length estimates would be much worse, and depend on the number of classes of the ring R, the number of prime divisors of its discriminant, or something of the sort. Nevertheless, we are convinced that 9 here could be actually replaced by 5 or 6.

Proof of Theorem 2. Theorem 3 implies that one has only to prove the existence of a unitriangular factorization of length 6 for the group $\mathrm{SL}\left(2, \mathbb{Z}\left[\frac{1}{2}\right]\right)$. In fact, we will prove the following slightly more precise result, where $U = U\left(2, \mathbb{Z}\left[\frac{1}{2}\right]\right)$ and $U^- = U^-\left(2, \mathbb{Z}\left[\frac{1}{2}\right]\right)$, respectively. □

Lemma 6. *Let $p \in \mathbb{Z}$ be a rational prime. Then under the assumption of the Generalized Riemann's Hypothesis, one has*

$$\mathrm{SL}\left(2, \mathbb{Z}\left[\frac{1}{2}\right]\right) = U^-UU^-UU^- \bigcup UU^-UU^-U.$$

Our proof critically depends on the recent results on Artin's conjecture, specifically, on the following result, which was first stated in the papers by Pieter Moree [44, 45]. A complete proof can be found in [35, Corollary 5.4].

Lemma 7. *Let $a \in \mathbb{Z}$, $a \neq 0, 1, -1$, be a square-free integer and $c, d \in \mathbb{Z}$, $c \perp d$, be coprime integers. Then, under the assumption of the Generalized Riemann's Hypothesis, the density of the set of primes q in the residue class $c \pmod d$, for which q is a primitive root $\mod d$, exists and is positive, with the only exception of the case where the discriminant of $\mathbb{Q}(\sqrt{a})$ divides d and every prime in the residue class $c \pmod d$ completely decomposes in $\mathbb{Q}(\sqrt{a})$.*

Let us state the following immediate consequence of this result.

Corollary. *Let $p \in \mathbb{Z}$ be a rational prime and $c \perp d$ be two coprime integers such that $p \perp d$. Then, under the assumption of the Generalized Riemann's Hypothesis, there are infinitely many primes q in the residue class $c \pmod d$, for which p is a primitive root modulo q.*

Now, we are all set to prove Lemma 6, and thus also Theorem 2.

Proof. Let $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}\left(2, \mathbb{Z}\left[\frac{1}{2}\right]\right)$. Since the case where at least one of the matrix entries x, y, z, w equals 0 was already considered in the above proof of Theorem 4 for the linear case, in the sequel we may assume that $xyzw \neq 0$. Now, we set

$$g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} p^\alpha a & p^\beta b \\ * & * \end{pmatrix} \in \mathrm{SL}\left(2, \mathbb{Z}\left[\frac{1}{2}\right]\right),$$

where $a, b \in \mathbb{Z}$ are not divisible by p, and $\alpha, \beta \in \mathbb{Z}$.

Case 1: $\alpha \geq \beta$. By the corollary to Lemma 7, there are infinitely many primes q of the form $p^{\alpha-\beta}a + bk$ such that p is a primitive root modulo q. Then

$$gt_{21}(k) = \begin{pmatrix} p^\beta q & p^\beta b \\ * & * \end{pmatrix}.$$

Since p is a primitive root mod q, there exists a $u \geq 1$ such that $p^u \equiv b \pmod q$. Let, say, $p^u = b + lq$. In this case,

$$gt_{21}(k)t_{12}(l) = \begin{pmatrix} p^\beta q & p^{\beta+u} \\ * & * \end{pmatrix}.$$

Then for $\theta = (1 - p^\alpha q)/p^{\beta+u}$, we get

$$gt_{21}(k)t_{12}(l)t_{21}(\theta) = \begin{pmatrix} 1 & p^{\beta+u} \\ * & * \end{pmatrix}$$

and, finally,

$$gt_{21}(k)t_{12}(l)t_{21}(\theta)t_{12}(-p^{\beta+u}) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Thus, in this case $g \in U^-UU^-UU^-$.

Case 2: $\alpha < \beta$. In this case, exactly the same argument shows that $g \in UU^-UU^-U$. □

10. Some related unsolved problems

Let us list some further problems which are intimately related to unitriangular factorizations. Our Theorem 1 allows one to improve slightly the known bounds in some of them.

• Let us mention another problem similar to the one discussed in the present paper: estimate the width of the elementary group $E(\Phi, R)$ with respect to *all* unipotent elements. From the Gauss decomposition with a prescribed semisimple part, as obtained by Erich Ellers and Nikolai Gordeev, it follows that every noncental element of a Chevalley group $G(\Phi, K)$ over a field K is a product of two unipotent elements [25].

Theorem 1 immediately implies the following result.

Corollary 1. *Let Φ be a reduced irreducible root system and R be a commutative ring such that* $\mathrm{sr}(R) = 1$. *Then every element of the simply connected Chevalley group $G(\Phi, R)$ is a product of three unipotent elements.*

For rings of dimension ≥ 1, the situation is much more delicate (see, for example, the paper by Fritz Grunewald, Jens Mennicke, and Leonid Vaserstein [27]).

• Another related problem, which has recently been considered in the paper by Martin Liebeck, Nikolay Nikolov, and Aner Shalev [38], is the width of a Chevalley group in the fundamental $\mathrm{SL}(2, R)$'s. For Chevalley groups over a (finite) field, that paper gives the estimate $5|\Phi^+|$, which follows from [13]. Our Theorem 1 immediately implies a better bound in a much more general situation.

Corollary 2. *Let Φ be a reduced irreducible root system and R be a commutative ring such that* $\mathrm{sr}(R) = 1$. *Then the simply connected Chevalley group $G(\Phi, R)$ can be written as the product of* $4|\Phi^+|$ *copies of the fundamental* $\mathrm{SL}(2, R)$.

As a matter of fact, even this estimate is far from the real one. The Bruhat decomposition immediately implies the bound $3|\Phi^+|$, and somewhat more precise calculations allow one to get the estimate $2|\Phi^+|$ for all Bezout rings. This is done in the paper by the first author and Evdokim Kovach, currently under way.

• A large number of papers are devoted to the width of Chevalley groups in commutators (see, in particular, [25, 39, 57, 59]). For Chevalley groups over a ring of stable rank 1, our Theorem 1 implies the trivial estimate 6, which is far from being optimal. We strongly suspect that in fact the commutator width of Chevalley groups over a ring of stable rank 1 does not exceed 2 or 3.

In conclusion, we state some further unsolved problems in the field.

Problem 1. *For a Chevalley group of rank ≥ 2, find a minimal L such that*

$$G(\Phi, \mathbb{Z}) = (U(\Phi, \mathbb{Z})U^-(\Phi, \mathbb{Z}))^L.$$

In fact, the precise estimate is not even known in the following special case. We believe that the estimate by Thomas Laffey [31, 32], which follows from [24], is grossly exaggerated.

Problem 2. *Find a minimal n starting from which one has the unitriangluar factorization*

$$\mathrm{SL}(n, \mathbb{Z}) = (U(n, \mathbb{Z})U^-(n, \mathbb{Z}))^3.$$

The following question is of obvious relevance, in connection with Lemma 6.

Problem 3. *Is it true that $U^-UU^-UU^- = UU^-UU^-U$?*

Most probably, in general this is not the case, but it would be very interesting to see an explicit counterexample.

Generally speaking, no unitriangular factorizations exist over rings of dimension > 1, and no such factorizations can possibly exist. Their role is taken by parabolic factorizations (see [3, 4, 59] and the references therein). These factorizations allow one to efficiently reduce problems for the group itself, to similar problems for groups of smaller ranks.

Problem 4. *Estimate the width of Chevalley groups over commutative rings of small stable ranks in terms of classical subgroups/subgroups of type* A_l.

Even for fields, such estimates should be much better than the ones obtained in [51, 52].

The following problem can easily be solved by the methods of the present paper.

Problem 5. *Calculate the width of the elementary Chevalley group* $E(\Phi, R)$ *over a semilocal ring* R *in terms of the unipotent radicals* U_P *and* U_P^- *of two opposite parabolic subgroups.*

Let us mention yet another direction in which it would be natural to generalize the results of the present paper. In [6], Victor Petrov and Anastasia Stavrova constructed the elementary subgroup $E(R)$ in an isotropic reductive group $G(R)$ and, for groups of relative rank ≥ 2, proved the normality of $E(R)$ in $G(R)$. See also the recent paper [5] by Alexander Luzgarev and Anastasia Stavrova, where it is proved that $E(R)$ is perfect, with the known exceptions for Chevalley groups of rank 2.

Despite the apparent similarity of the statements, the following problem is not at all trivial, and, to the best of our knowledge, presently there are no nontrivial estimates at all.

Problem 6. *Calculate the width of the elementary subgroup* $E(R)$ *of an isotropic reductive group* $G(R)$ *over a semilocal ring* R*, in terms of the unipotent radicals* U_P *and* U_P^- *of two opposite parabolic subgroups.*

Even less than that is known in the arithmetic context.

Problem 7. *Prove that the elementary subgroup* $E(R)$ *of an isotropic reductive group* $G(R)$ *of relative rank* ≥ 2 *has bounded width with respect to the unipotent radicals* U_P *and* U_P^- *of two opposite parabolic subgroups, in the case where* $R = \mathcal{O}_S$ *is a Dedekind ring of arithmetic type.*

The only result we are aware of, regarding this problem, is the paper by Igor Erovenko and Andrei Rapinchuk [26], which addresses the case of orthogonal groups corresponding to a form of Witt index ≥ 2.

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