VARIOUS DECOMPOSITIONS FOR GL(n)

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§ 1 Cartan and Iwasawa decompositions for GL(n,R) and GL(n,C)

Theorem (Polar decomposition)

Suppose $M \in GL(n, \mathbf{R})$. Then, M can be written in a unique manner as a product $K \cdot S$ of an orthogonal matrix K and a positive definite symmetric matrix S. Moreover, the polar decomposition

$$(K, S) \mapsto K \cdot S$$

is a homeomorphism of $O(n) \times \mathcal{P}$ onto $GL(n, \mathbf{R})$, where $\mathcal{P}$ denotes the space of symmetric, positive-definite matrices.

Proof : We start with the proof of the existence. If we had $M = KS$ as required, then $^tM = SK^{-1}$ so that we would have had $^tMM = S^2$ i.e. S would be a 'square root' of ^tMM. Therefore, we start with M and consider ^tMM which is a positive-definite symmetric matrix. As such,

$$^tMM = K_1 \cdot diag(\lambda_1, \cdots \lambda_n) \cdot K_1^{-1}$$

with $K_1 \in O(n)$ and the eigenvalues λ_i are real and positive.

Recall briefly how a matrix $S \in \mathcal{P}$ may be diagonalised by an orthogonal matrix. If $\lambda_1, \cdots \lambda_r$ are the distinct eigenvalues of a matrix $S \in \mathcal{P}$, then

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we define $V_i = \{v : Sv = \lambda_i v\}$. Now $< V_i, V_j >= 0$ for $i \neq j$, since $< Sv_i, v_j >=< v_i, Sv_j >= \lambda_i < v_i, v_j >= \lambda_j < v_i, v_j >$ which gives $< v_i, v_j >= 0$ for $v_i in V_i$ and $v_j \in V_j$. Now, we can easily see that $\mathbf{R}^n = \bigoplus_1^r V_i$. We can choose orthonormal bases for each V_i, and we are through.

If we define $S = K_1 \sqrt{D} K_1^{-1}$ where $D = diag(\lambda_1, \cdots \lambda_n)$, then $S \in \mathcal{P}$. Finally, if we put $K = MS^{-1}$, then clearly $K \in O(n)$. Thus, the existence of the decomposition is assured. The uniqueness can be seen from the uniqueness of the positive-definite 'square root' of ${}^t MM$, as follows:

Suppose P is a polynomial such that $P(\lambda_i) = \sqrt{\lambda_i}$ $(i = 1, \cdots n)$ - for e.g. the Lagrange (interpolation) polynomial.

Briefly recall how to get such a polynomial. If we are given n distinct points $\alpha_1, \cdots, \alpha_n$ and we want a polynomial which takes the values $\beta_1, \cdots, \beta_n$, we proceed in this way. First, we get for each i, a polynomial P_i of degree $n - 1$ with $P_i(\alpha_j) = \beta_i \delta_{i,j}$. Then, the required polynomial would be the polynomial $P = \sum P_i$ again of degree $n - 1$. It is clear how to get each P_i. We merely take $P_i = c_i \prod_{j:j \neq i}(X - \alpha_j)$ where c_i is to be determined so that $P_i(\alpha_i) = \beta_i$. In fact, we get $\beta_i = P_i(\alpha_i) = c_i \prod_{j:j \neq i}(\alpha_i - \alpha_j)$. Thus, we get

$$P(X) = \sum_{i=1}^{n} \beta_i \prod_{j:j \neq i} \frac{X - \alpha_j}{\alpha_i - \alpha_j}$$

So, we can get a polynomial P such that $P(\lambda_i) = \sqrt{\lambda_i}$ $(i = 1, \cdots n)$. Then $P({}^t MM) = S$. If $S_1 \in \mathcal{P}$ and $S_1^2 = {}^t MM$, then S_1 commutes with ${}^t MM$ and, therefore, with S which is a polynomial in ${}^t MM$. The matrices S and S_1 are simultaneously diagonalisable and since their eigen values are positive and their squares are the same, $S = S_1$.

Aliter (for uniqueness) $S^2 = S_1^2 = {}^t MM, S = kDk^{-1}, S_1 = k'D'(k')^{-1}$.

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Then $hD^2 = (D')^2 h$, where $h = (k')^{-1} k$. Comparing the (i,j)-th terms, we get $(d_j^2 - d_i'^2) h_{ij} = 0$. Since $d_j, d_i' > 0$, we get $S = S_1$.

Finally, note that the continuity of the function $(K, S) \mapsto KS$ is evident and, it remains to prove that if $\{M_n\} = \{K_n S_n\}$ converges to $M = KS$, then $\{K_n\} \to K$ and $\{S_n\} \to S$. Since $O(n)$ is compact (see lemma below), $\{K_n\}$ has a limit point K'; say $\{K_{n_r}\}$ is a subsequence converging to K'. Then $\{S_{n_r}\} \to (K')^{-1} M = S'$, which is in $\mathcal{P}$, and $M = K'S'$. The uniqueness of the polar decompositionshows thwn that $K' = K$ and $S' = S$.

Remark The set $\mathcal{P}$ is homeomorphic to a Euclidean space (see Proposition below). It is a convex semi-cone i.e. $S_1, S_2 \in \mathcal{P} \Rightarrow \lambda S_1 + \mu S_2 \in \mathcal{P} \ \forall \lambda, \mu > 0$ and so the topological properties of $GL(n, \mathbf{R})$ are deduced from those of $O(n)$.

Lemma

$O(n)$ *is compact.*

Proof: $O(n)$ is the inverse image of the identity, under the continuous map $M \mapsto {}^t MM$; it is therefore closed in $M(n, \mathbf{R})$.

If we define $\| M \| = \mathrm{tr}\ ({}^t MM)^{1/2}$, then $\| K \| = \sqrt{n} \ \forall K \in O(n)$, and so $O(n)$ is a bounded subset of $M(n, \mathbf{R})$.

The exponential map

For $A \in M(n, K); (K = \mathbf{R}\,\mathrm{or}\,\mathbf{C})$, the exponential of A is the matrix defined by the convergent series

$$exp\ (A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Properties

(1) If $A \cdot B = B \cdot A$, then $exp\,(A + B) = exp\,(A) \cdot exp\,(B)$. In particular, $exp\,(A)^{-1} = exp\,(-A)$.

(2) $P \cdot exp\,(A) \cdot P^{-1} = exp\,(PAP^{-1})$. (3) The eigenvalues of $\exp\,(A)$ are the exponentials of those of A. Consequently, $\det\,(\exp A) = \exp\,(\operatorname{tr} A)$.

The first two properties are readily verified. The third one can seen by triagonalising any matrix.

Proposition

Let S (respectively $\mathcal{P}$) denote the space of $n \times n$ symmetric (respectively positive-definite symmetric) matrices over $\mathbf{R}$. Then, $\exp : S \to \mathcal{P}$ is bijective.

Proof : Let $s \in \mathcal{P}$. Then, as we saw, $\exists k \in O(n)$ such that $s = k \cdot d \cdot k^{-1}$ for some $d = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$. Since λ_i are real and positive, we can consider $\log$ (d) $:= \operatorname{diag}\,(\log \lambda_1, \cdots \lambda_n)$. Clearly, $S = k \cdot \log(d) \cdot k^{-1} \in S$ satisfies $\exp\,(S) = s$. To show injectivity, we take any $S_1, S_2 \in S$ such that $\exp\,(S_1) = \exp\,(S_2) = s$. Writing $S_1 = kdk^{-1}$ with $d = \operatorname{diag}\,(\lambda_1, \cdots \lambda_n)$, we see by (3) that the eigenvalues of S_2 are also $\lambda_i (1 \leq i \leq n)$. As before we choose a polynomial P which takes the value λ_i at $\exp\,(\lambda_i)$. Then, $P(s) = S_1$. So, S_2 commutes with $P(\exp\,(S_2)) = S_1$. So, we can again simultaneously diagonalise S_1 and S_2; and as their eigenvalues are the same, we get $S_1 = S_2$. (We can verify the injectivity also just as we did uniqueness in the polar decomposition).

Theorem (Cartan decomposition)

Writing A for the group of diagonal matrices with positive, real entries and

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K for O(n), we have

$$GL(n, \mathbf{R}) = KAK$$

Proof : Let $g \in G$. By the polar decomposition, $g = kh$, where $k \in K$ and $h \in \mathcal{P}$. Since $h \in \mathcal{P}$, $h = k_1 a k_1^{-1}$ for some $k_1 \in K$, $a \in A$. Thus, $g = k k_1 a k_1^{-1} \in KAK$.

Remarks

1. Cartan decomposition is not unique; nevertheless it can be seen that the entries of a are the positive square roots of the eigenvalues of ${}^t g g$.

2. The interest of this decomposition, apart from gaining a hold on the structure of GL(n,$\mathbf{R}$), is that it permits us to analyse the functions on GL(n,$\mathbf{R}$) which are sufficiently invariant, completely in terms of their restrictions to A, where we may bring classical analysis into play. Thus, we can show that the space L^1 of functions on GL(n,$\mathbf{R}$) biinvariant by the action of K, forms a commutative algebra.

Theorem (Iwasawa decomposition)

The product map $(k, a, n) \mapsto kan$ is a homeomorphism of $K \times A \times N$ onto GL(n,$\mathbf{R}$).

Proof : The existence of this decomposition is just the Gram-Schmidt orthogonalisation process, which we recall now.

Given a basis $\{f_1, \cdots, f_n\}$ of $\mathbf{R}^n$, the Gram-Schmidt process finds a unique orthonormal basis $\{d_1, \cdots, d_n\}$ for which $f_i = \beta_{1,i} d_1 + \cdots + \beta_{i,i} d_i$ ($i = 1, \cdots, n$) and all $\beta_{i,i} > 0$ and such that d_i and the $\beta_{i,j}$ depend continuously on $f_1, \cdots, f_n$. This is done as follows. Suppose $E_i = \mathbf{R} f_1 + \cdots \mathbf{R} f_i$ which is of dimension i. Suppose $v_i \in E_i \setminus \{0\}$, $v_i \perp E_{i-1}$ and $< v_i, v_i > = 1$. By

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recurrence on i, we see that $\{v_1, \cdots, v_i\}$ is an orthonormal basis of E_i. In particular, $\{v_1, \cdots, v_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Write $\lambda_i = < v_i, f_i >$. Since $f_i \notin E_{i-1}$, we have $\lambda_i \neq 0$. Put $d_i = v_i \frac{|\lambda_i|}{\lambda_i}$. Then $< d_i, d_i > = < v_i, v_i > = 1$. Thus, $\{d_1, \cdots, d_i\}$ is again an orthonormal basis of E_i. If we write $f_i = \sum_{j=1}^i \beta_{j,i} d_j$, then $\beta_{j,i} = < f_i, d_j >$ and $\beta_{i,i} = |\lambda_i| > 0$. To show the continuity of the dependence, we assume that d_i and $\beta_{j,i}$ depend continuously on $f_1, \cdots, f_n$ for $i < i_0$.

Now $< f_{i_0}, f_{i_0} > = |\beta_{1,i_0}|^2 + \cdots + \beta_{i_0,i_0}^2$ and so β_{i_0,i_0} depends continuously on $f_1, \cdots, f_n$.

Also $d_{i_0} = (f_{i_0} - \beta_{1,i_0} d_1 \cdots - \beta_{i_0-1,i_0} d_{i_0-1}) \beta_{i_0,i_0}^{-1}$ so that d_{i_0} depends continuously on $f_1, \cdots, f_n$.

Returning to the proof of the theorem, let $g \in GL(n, \mathbf{R})$ be arbitrary. If $e_1, \cdots, e_n$ is the usual basis of $\mathbf{R}^n$, let $f_i = g \cdot e_i \quad \forall i = 1, \cdot, n$. Applying the Gram-Schmidt process, get an orthonormal basis $\{d_i\}$ such that $f_i = \beta_{1,i} d_1 + \cdots \beta_{i,i} d_i$. There is a unique $k \in K$ such that $k \cdot e_i = d_i \quad \forall i$. Then, if b is the matrix $(\beta_{i,j} \in A \cdot N$, then $f_i = gk^{-1} k \cdot e_i = gk^{-1} d_i = b \cdot d_i \quad \forall i$ so that $g = bk$. Thus the map is surjective. Uniqueness follows easily from the fact that if $x \in K \bigcap AN$, then all its eigenvalues are of absolute value 1 by virtue of being in K and hence are all equal to 1 by the definition of A; but $K \bigcap N = \{Id\}$ (for, if $g \in K$, then ${}^t g = g^{-1} \in K$ while $g, {}^t g \in N \Rightarrow g = Id$). Continuity of the product map is obvious; and the fact that it is a homeomorphism follows exactly as in the polar decomposition. The theorem is proved.

Theorem (Polar decomposition for GL(n,C))
Denote by $\mathcal{H}$, the space of Hermitian, positive-definite matrices. Then, the

product map $(U, H) \mapsto U \cdot H$ is a homeomorphism of $U(n) \times \mathcal{H}$ onto $GL(n, \mathbf{C})$.

Proof : It is the same as in the case of $\mathbf{R}$ except for replacing the transpose by the conjugate-transpose and the inner product on $\mathbf{R}^n$ by that on $\mathbf{C}^n$.

Theorem (Cartan decomposition for GL(n,C))
If k denotes $U(n)$, then $GL(n, \mathbf{C}) = KAK$.

Theorem (Iwasawa decomposition for GL(n, C))
The product map $(k, a, n) \mapsto kan$ is a homeomorphism of $K \times A \times N$ onto $GL(n, \mathbf{C})$.
Again, the proofs are exactly as in the real case, except for replacing the inner product on $\mathbf{R}^n$ by that on $\mathbf{C}^n$.

Exercise: Prove an Iwasawa decomposition for $GL(n, \mathbf{H})$, where $\mathbf{H}$ denotes the quaternions of Hamilton.

§ 2 On Haar measures

In this section, we first recall some basic facts about the Haar measure on a locally compact group, and later find out the relation between the Haar measures of GL(n) and its subgroups K,A,N in the Iwasawa decomposition.

Basics on Haar measures

Let G be a locally compact topological group.

(1) Let $s \in G$, f a function on G. Then sf and fs are defined, respectively, as $(sf)(x) = f(s^{-1}x)$ and $(fs)(x) = f(xs^{-1})$.

(2) A measure μ on G is left-invariant, if $\mu(sf) = \mu(f)$ i.e.

$$\int f(s^{-1}x)d\mu(x) = \int f(x)d\mu(x) \quad \forall s \in G$$

From (3) onwards, we suppose μ is a left invariant measure on G.

(3) The right modulus of μ is the function $\Delta_r : G \to \mathbf{R}_+^*$ defined by

$$\int f(xt^{-1})d\mu(x) = \Delta_r(t) \int f(x)d\mu(x)$$

It is a continuous homomorphism; it is independent of the choice of μ.

(4)

$$\int f(x^{-1})d\mu(x) = \int \frac{f(x)}{\Delta_r(x)}d\mu(x)$$

(5) Let $\sigma \in Aut(G)$. Then, there exists a unique $\delta(\sigma) \in \mathbf{R}_+^*$ such that

$$\int f(\sigma^{-1}(x))d\mu(x) = \delta(\sigma) \int f(x)d\mu(x)$$

$\delta : Aut(G) \to \mathbf{R}_+^*$ gives a continuous homomorphism. Moreover, the same formula holds (i.e. with the same number $\delta(\sigma)$) if μ is replaced by a right invariant mesure.

(6) $\delta(\text{Int } s) = \Delta_r(s)^{-1} \ \forall \ s \in G.$

(7) $\Delta_r(\sigma(s)) = \Delta_r(s).$

(8) Let $B = T \cdot U$ be a semi-direct product where $t \in T$ acts on U by σ_t. If dt, du are bi-invariant Haar measures on T, U respectively, then $dt \ du$ is a left Haar measure on B, and a right invariant measure on B is $\delta(\sigma_t) \ dt \ du$, i.e. $\Delta_r^B(t, u) = \delta^U(\sigma_t)^{-1}.$

(9) If B, T, U etc. are as in Iwasawa decomposition, we have for $t = \text{diag}(t_1, \cdots, t_n),$

$$\delta^U(\sigma_t) = \Delta_r^B(t, u)^{-1} = \prod_{i<j} \frac{t_i}{t_j}$$

Proofs

(3) Define $\tau(f) = \mu(ft)$ i.e.

$$\tau(f) = \int f(xt^{-1})d\mu(x)$$

τ is also left invariant, since $\tau(sf) = \mu((sf)t) = \mu(s(ft)) = \mu(ft) = \tau(f).$ So, we get some number $\Delta_r(t)$ as written. Choosing f such that $\int f(x)d\mu(x) = 1$, we see that

$$\Delta_r(t) = \int f(xt^{-1})d\mu(x)$$

from which we can see that Δ_r is continuous. To show it is a homomorphism, we consider

$$\Delta_r(st) \int f(x)d\mu(x) = \int f(xt^{-1}s^{-1})d\mu(x) = \int g(xt^{-1})d\mu(x)$$

(where $g(x) = f(xs^{-1})$)

$$= \Delta_r(t) \int g(x) d\mu(x) = \Delta_r(t) \int f(xs^{-1}) d\mu(x) = \Delta_r(s)\Delta_r(t) \int f(x) d\mu(x)$$

(4) We show that both sides define rightinvariant functionals. First, define $\tau_1(f) = \int f(x^{-1}) d\mu(x)$. Then

$$\tau_1(fs) = \int f(x^{-1}s^{-1}) d\mu(x) = \int \tilde{f}(sx) d\mu(x)$$

(where $\tilde{f}(x) = f(x^{-1})$)

$$= \int \tilde{f}(x) d\mu(x) = \tau_1(f)$$

Now, we define $\tau_2(f) = \int \frac{f(x)}{\Delta_r(x)} d\mu(x)$. Then,

$$\tau_2(fs) = \int \frac{f(xs^{-1})}{\Delta_r(x)} d\mu(x) = \Delta_r(s)^{-1} \int \frac{f(xs^{-1})}{\Delta_r(xs^{-1})} d\mu(x) = \Delta_r(s)^{-1} \int g(xs^{-1}) d\mu(x)$$

(where $g(x) = \frac{f(x)}{\Delta_r(x)}$).

$$= \Delta_r(s)^{-1} \Delta_r(s) \int g(x) d\mu(x) = \tau_2(f)$$

So, there is some constant $c > 0$ such that $\tau_1 = c\tau_2$. Evaluating on a function f which is symmetric about x and x^{-1} i.e. for which $f(x) = f(x^{-1})$, and noting that Δ_r is continuous, it is easy to see that $c = 1$.

(8) First, we check that $dt\,du$ is left invariant.

$$\int f((sv)^{-1}tu) dt\,du = \int f(v^{-1}tu) dt\,du = \int g((t^{-1}vt)^{-1}u) du\,dt$$

(where $g(u) = f(tu)$)

$$= \int g(u) du\,dt = \int f(tu) du\,dt$$

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The right modulus is given as follows:

$$\int f(tu(sv)^{-1})du\ dt = \int f(tuv^{-1}s^{-1})du\ dt = \int f(ts^{-1}\cdot suv^{-1}s^{-1})du\ dt = \int f(tsuv^{-1}s^{-1})du\ dt =$$

(where $g(u) = f(tsus^{-1})$)

$$= \int f(tsus^{-1})du\ dt = \int h(sus^{-1})du\ dt = \delta^U(\sigma_s)^{-1}\int h(u)du\ dt$$

(where $h(u) = f(tu)$)

$$= \delta^U(\sigma_s)^{-1}\int f(tu)du\ dt$$

Therefore $\Delta_r^B(s,v) = \delta^U(\sigma_s)^{-1}$. The proof is complete, since we know a right invariant measure from (4).

(9) If we identify $U \to F^{\frac{n(n-1)}{2}}$ via the map $\theta : u = (u_{ij} \to (u_{ij}; i < j)$, the Lebesgue measure pulls back to a Haar measure on U, which we can choose to be du. Since σ_t takes u_{ij} to $\frac{t_i}{t_j}u_{ij}$, by the change of variable formula in $F^{\frac{n(n-1)}{2}}$, we get

$$\int \cdots \int f(tut^{-1})dt\ du_{ij} = \prod_{i>j}\frac{t_i}{t_j}\int \cdots \int f(u_{ij})du_{ij}$$

(recall $\int f(Av)dv = det(A)^{-1}\int f(v)dv$). Thus, $\delta(\sigma_t)^{-1} = \prod_{i>j}\frac{t_i}{t_j}$, and we get that $\prod_{i<j}\frac{t_i}{t_j}dt\ du$ is a right Haar measure on B.

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Let $F = \mathbf{R}$ or $\mathbf{C}$. Write $K = O(n)$ or $U(n)$ accordingly as $F = \mathbf{R}$ or $\mathbf{C}$. Let $B^+ = AN$ be the subgroup of GL(n,F) consisting of all those upper triangular matrices over F which have real, positive diagonal entries. Now, we will give a Haar measure on $G = GL(n,F)$ in terms of the measures on K, A, and N.

Fact: The groups $GL(n, F), K, A, N$ are unimodular. In fact, all abelian groups, compact groups, semisimple and nilpotent Lie groups are unimodular.

Let dg, dk, da, dn denote Haar measures (necessarily biinvariant) on G, K, A, N respectively. Let $k \in K$ (resp. $b \in B^+$) operate on $GL(n, F)$ and on $K \times B^+$ by left translation by k (resp. right translation by b^{-1}). Then, the product defines a homeomorphism of $K \times B^+$ onto $GL(n, F)$ which commutes with the $K \times B^+$ -actions. The inverse image of dg is a measure invariant on the left by K and on the right by B^+; therefore equals $dk\, d_r b$ where $d_r b$ is a right Haar measure on B^+. By (9), we may write a right Haar measure on B^+ as $\rho(a)da\, dn$, and a right Haar measure on GL(n,F) is $\rho(a)dk\, da\, dn$, where

$$\rho(a) = \prod_{i<j} \frac{a_i}{a_j}$$

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♣ Consequently, for $G = SL(2, \mathbf{R})$, if we write $K = SO(2)$, $A = \{\mathrm{diag}(a, a^{-1}) : a > 0\}$, and dk, dt, dn for Haar measures on K, A, N respectively, then a right Haar measure on G is given as $a^2 \, dk \, \frac{da}{a} \, dn = a \, dk \, da \, dn$ where we have used $dt = \frac{da}{a}$ for $t = \mathrm{diag}(a, a^{-1})$.

Another way to derive this is as follows.

Write $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G$; we can assume that $x \neq 0$ as the complement has measure zero. Now, $t = \frac{1+yz}{x}$. We will determine $f(X)$ so that $\omega = f(X) \, dx \wedge dy \wedge dz$ is a left invariant differential form. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have $AX = X' = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$, where

$$x' = ax + bz$$

$$y' = ay + b\left(\frac{1 + yz}{x}\right)$$

$$z' = cx + dz$$

For left invariance, we should have

$$f(X) \, dx \wedge dy \wedge dz = f(X') \, dx' \wedge dy' \wedge dz'$$

This gives $f(X')\frac{ax+bz}{x} = f(X)$ i.e. $f(X')x' = f(X)x$. So, we can take $f(X) = \frac{1}{x}$ i.e. $\omega = \frac{1}{x} dx \wedge dy \wedge dz$. Writing

$$X = ktn = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

we have $x = a \cdot \cos\theta$, $y = au \cdot \cos\theta - a^{-1} \cdot \sin\theta$, $z = a \cdot \sin\theta$, so that $\omega = a \, d\theta \wedge da \wedge du$.

Exercise : Prove from $G = SL(2, \mathbf{R}) = KAK$, that
$\frac{a^2 - a^{-2}}{2} \, dk \, \frac{da}{a} \, dn$ *is an invariant measure on G.*

Exercise : From its Iwasawa decomposition, deduce that $SL(n, \mathbf{R})$ is connected.

§ 3 **Decompositions for** $GL(n, \mathbf{Q}_p))$

Theorem (Iwasawa decomposition for $GL(n, \mathbf{Q}_p)$**)**

Let $K = GL(n, \mathbf{Z}_p)$ *and* B *be the upper triangular matrices in* $GL(n, \mathbf{Q}_p)$.
Then $GL(n, \mathbf{Q}_p) = K \cdot B = KAN$, *where*

$$A = \{\mathrm{diag}(p^{a_1}, \cdots, p^{a_n}) : a_i \in \mathbf{Z}\}$$

Moreover, the a_i *are uniquely determined.*

Proof : Start with any $g \in GL(n, \mathbf{Q}_p)$. For an element $k \in K$, let us look at the rows of $k.g$. Suppose the i-th row starts with exactly a_i zeroes. Then $0 \le a_i \le n - 1 \; \forall i$. Now, let us choose k such that for the matrix $k.g$, the number $a_1 + \cdots + a_n$ is maximal.

CLAIM : The a_i are necessarily distinct i.e. they are $0, 1, \cdots, n-1$ in some order.

Let us prove the claim. Suppose not i.e. let $i < j$ be such that $a_i = a_j$. Let α_i and α_j be the first nonzero entries in the i-th and the j-th rows respectively. Now, either $u = \frac{\alpha_i}{\alpha_j}$ or u^{-1} is in $\mathbf{Z}_p$. Consider the matrix $k_1 \in K$ defined asfollows. All its diagonal entries are 1; if $u \in \mathbf{Z}_p$, then the (i, j)-th entry is $-u$ and all other nondiagonal entries are zero; if $u^{-1} \in \mathbf{Z}_p$, then define the (j, i)-th entry to be $-u^{-1}$ and other nondiagonal entries to be zero. It is clear that $k_1 \in K$. Moreover, for $k_1.g$, $\sum a_i$ has increased

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atleast by 1, since either a_i or a_j has increased by 1 and the other a_r are not decreased either. This is a contradiction of our choice of k. Thus, the claim is proved i.e. the a_i form a permutation of $(0, 1, \cdots, n-1)$.

Further, considering the 'permutation' matrix n defined by $n_{i,a_i+1} = 1$ and other entries zero, it is clear that $n \in K$ (indeed, its determinant is 1 or -1). The action of n is that, on left multiplication of a matrix by n, the rows of the given matrix are permuted accordingly. Thus, we can see that $n.k.g$ has its $(a_1, \cdots, a_n) = (0, 1, \cdots, n-1)$ i.e. $n.k.g$ is upper triangular. This proves $g \in K \cdot B$. Since $B = T \cdot N$ where T is the subgroup of diagonal matrices in $GL(n, \mathbf{Q}_p)$, it is clear that $K \cdot T = K \cdot A$.

We prove uniqueness of the A- part as follows. Suppose, $a' = kan$ for some $a = diag(p^{a_1}, \cdots, p^{a_n})$, $a' = diag(p^{a'_1}, \cdots, p^{a'_n}) \in A$. Then $k^{-1} a' = a n$. Comparing the (i, i)-th terms, we have $k_{i,i}^{-1} p^{a'_i} = p^{a_i}$. Since $k_{i,i}^{-1} \in \mathbf{Z}_p$, we have $a_i \geq a'_i$. Interchanging the roles of a and a', we get $a_i = a'_i$, and finally that $a = a'$.

♣ For $G = GL(2, \mathbf{Q}_p)$, here is a quick proof of the Iwasawa decomposition. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Choose $u, v \in \mathbf{Z}_p$ such that $cu + dv = 0$ and u, v coprime. So, we can find $w, z \in \mathbf{Z}_p$ so that $ux - wv = 1$ i.e.

$$k = \begin{pmatrix} u & w \\ v & x \end{pmatrix} \in K$$

Now, clearly

$$gk = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & w \\ v & x \end{pmatrix} \in B(\mathbf{Q}_p)$$

Theorem (Cartan decomposition for GL(n,$\mathbf{Q}_p$))

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Let A be the set of diagonal matrices $\{diag(p^{a_1}, \cdots, p^{a_n}) : a_1 \leq \cdots \leq a_n\}$

$$GL(n, \mathbf{Q}_p) = KAK$$

Moreover, the A-part is uniquely determined.

For the proof, we'll use the following

Lemma *Let $a = {}^t (a_1, \cdots, a_n) \in \mathbf{Z}_p^n$ be a vector (i.e. a is thought of as a column vector). Then, $\exists k \in K$ such that*

$$k \cdot a = {}^t (d, 0, \cdots, 0)$$

where

$$d = g.c.d(a_1, \cdots, a_n) := p^r = \frac{1}{\mathrm{Max} \mid a_i \mid_p}$$

Proof : Let $n = 2$. Since $\gamma = -\frac{a_2}{d}$ and $\delta = \frac{a_1}{d}$ are coprime i.e. have g.c.d. 1, there exist $\alpha, \beta \in \mathbf{Z}_p$ such that $\alpha\delta - \beta\gamma = 1$. Thus,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$$

We prove the general case by induction. Assume the truth for $n - 1$. Let $a_2, \cdots, a_n$ have g.c.d δ. By the induction hypothesis, we can choose $k_1 \in GL(n - 1, \mathbf{Z}_p)$ such that

$$k_1 \cdot {}^t (a_2, \cdots, a_n) = {}^t (\delta, 0, \cdots, 0)$$

Clearly

$$k = \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & k_1 \end{pmatrix} \in K$$

and $k \cdot {}^t (a_1, \cdots, a_n) = {}^t (a_1, \delta, 0 \cdots, 0)$. Since d is the g.c.d of a_1 and δ, we have some matrix

$$k_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbf{Z}_p)$$

such that $k_2 \cdot{}^t (a_1, \delta) = {}^t (d, 0)$. Putting

$$k_0 = \begin{pmatrix} k_2 & 0_{2,n-2} \\ 0_{n-2,2} & E_{n-2} \end{pmatrix} \in K$$

$k_0 \cdot k$ does the job. The proof of the lemma is now complete.

Proof of Cartan decomposition

Let $g \in GL(n, \mathbf{Q}_p)$. Write $g = p^{a_1} g_1$ where g_1 is primitive i.e.it is in $M(n, \mathbf{Z}_p)$ and has coprime entries. We will show that the double coset $K g_1 K$ has a representative of the form $\begin{pmatrix} 1 & 0 \\ 0 & g_2 \end{pmatrix}$ with $g_2 \in M(n-1, \mathbf{Z}_p)$. Thereafter, we'll use this assertion to complete the proof of the theorem. As g_1 is primitive, it has a column with coprime entries. Multiplying g_1 on the right by an appropriate permutation matrix, we get a matrix h_1 in the same double coset but whose first column has coprime entries. So, we may assume that $g_1 = h_1$. Now, we apply the lemma. We can multiply g_1 from the left by an element of K to obtain a matrix of the form

$$\begin{pmatrix} 1 & \mathbf{b} \\ 0 & g_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{b} \\ 0 & E_{n-1} \end{pmatrix}$$

where $\mathbf{b}$ is $1 \times n-1$ with p-adic integral entries. This proves our assertion. Returning to the proof of the theorem, what we have shown is that the double coset $KgK = p^{a_1} K g_1 K$ contains a representative of the form

$$g' = \begin{pmatrix} p^{a_1} & 0 \\ 0 & p^{a_1} g_2 \end{pmatrix}$$

where p^{a_1} is the g.c.d. of the entries of g and $g_2 \in M(n-1, \mathbf{Z}_p)$. It is clear that g' remains in KgK when g_2 varies in $GL(n-1, \mathbf{Z}_p)\, g_2\, GL(n-1, \mathbf{Z}_p)$. So, we can apply the same arguments to the matrix $p^{a_1} g_2$ and continue to get consequently a representative in A.

We prove the uniquenessof the A part.

Let $a = diag(p^{a_1}, \cdots, p^{a_n})$, $a' = diag(p^{a'_1}, \cdots, p^{a'_n}) \in A$ be such that $a' = k\ a\ k'$. Multiplying by a scalar matrix, we may assume that $a_1 = 0$ and all a_i , $a'_i \geq 0$. Going modulo p, we have an equality of matrices over $\mathbf{F}_p$,

$$\widehat{a'} = \widehat{k}\ \widehat{a}\ \widehat{k'}$$

Now, clearly for some r , s,

$$\widehat{a'} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \widehat{a} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$$

Comparing ranks on both sides, $r = s$. Writing

$$k = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \quad k' = \begin{pmatrix} k_{1'} & k'_2 \\ k'_3 & k'_4 \end{pmatrix}$$

we get

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{k_1} & \widehat{k_2} \\ \widehat{k_3} & \widehat{k_4} \end{pmatrix} \cdot \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \widehat{k'_1} & \widehat{k'_2} \\ \widehat{k'_3} & \widehat{k'_4} \end{pmatrix}$$

This gives $\widehat{k_1} \cdot \widehat{k'_1} = I_r$ and $\widehat{k_3} \cdot \widehat{k'_1} = 0$.

Therefore $\widehat{k_3} = 0$.

Consider now the diagonal matrix

$$P = \begin{pmatrix} I_r & 0 \\ 0 & p^{-1} I_{n-r} \end{pmatrix}$$

Multiplying the original equation on the left by P, we have the equation

$$P\ a' = P\ k\ a\ k' = P k P^{-1}\ P\ a\ k'$$

Since this decreases the a_i's and the a'_i's, we would be done by arguing inductively provided we know that $P\ k\ P^{-1} \in K$. But, clearly

$$P k P^{-1} = \begin{pmatrix} k_1 & p k_3 \\ p^{-1} k_3 & k_4 \end{pmatrix}$$

Since $\widehat{k}_3 = 0$, we have $p^{-1}k_3$ has entries from $\mathbf{Z}_p$. Since $PkP^{-1} \in M(n, \mathbf{Z}_p)$ and its determinant is the same as that of k, it is clear that $PkP^{-1} \in K$. Therefore the uniqueness follows on applying induction.

Corollary K is a maximal compact subgroup.

Proof: Since K is the inverse image of the units $\mathbf{Z}_p^*$ under the map det $:M(n, \mathbf{Z}_p) \to \mathbf{Z}_p$ and since $\mathbf{Z}_p^*$ is closed in $\mathbf{Z}_p$ while $M(n, \mathbf{Z}_p)$ is compact, it follows easily that K is compact. If $K_1 \supset K$, $K_1 \neq K$ is a subgroup of $GL(n, \mathbf{Q}_p)$, then there is $k_1 \in K_1 \setminus K$. Applying the Cartan decomposition to k_1, we get an element $a = diag(p^{a_1}, \cdots, p^{a_n}) \in K_1 \bigcap A$, $a \in K$. But then some $a_i \neq 0$ and so the set $\{a^n : n \in \mathbf{Z}\}$ is unbounded which shows that K_1 is not compact.

§ 4 Bruhat decomposition for GL(n) over any field

Let F be an arbitrary field. We are interested in the group GL(n,F). Let B denote the subgroup of all invertible upper triangular matrices with entries from F. Consider the group N of monomial matrices over F i.e. $n \in N$ if, and only if, exactly one entry is nonzero in each row and each column of n. Obviously $N \supset T$, the subgroup of diagonal matrices in B. Moreover, T is normal in N and the group $W := N/T \cong S_n$, the symmetric group. Let $\pi : N \to W$ be the natural map. Since $T \subset B$, for $w \in W$, the coset wB makes sense.

Theorem (Bruhat decomposition)

$$GL(n, F) = \bigcup_{w \in W} BwB$$

where the union is disjoint.

Proof : Let $g \in GL(n, F)$. For an element $k \in B$, let us look at the rows of $k.g$. Suppose the i-th row starts with exactly a_i zeroes. Then $0 \leq a_i \leq n - 1$ $\forall i$. Now, let us choose k such that for the matrix $k.g$, the number $a_1 + \cdots + a_n$ is maximal.

CLAIM : The a_i are necessarily distinct i.e. they are $0, 1, \cdots, n-1$ in some order.

Let us prove the claim. Suppose not i.e. let $i < j$ be such that $a_i = a_j$. Let α_i and α_j be the first nonzero entries in the i-th and the j-th rows respectively. Consider the matrix $k_1 \in B$ defined asfollows. All its diagonal entries are 1; the (i, j)-th entry is $-\frac{\alpha_i}{\alpha_j}$ and all other entries are zero. It is clear that $k_1 \in B$. Moreover, for $k_1.g$, $\sum a_i$ has increased atleast by 1, since a_i has increased by 1 and the other a_r are not decreased. This is a contradiction of our choice of k. Thus, the claim is proved i.e. the a_i form a permutation of $(0, 1, \cdots, n - 1)$.

Further, considering the 'permutation' matrix n defined by $n_{i, a_i+1} = 1$ and other entries zero, it is clear that $n \in N$. The action of n is that, on left multiplication of a matrix by n, the rows of the given matrix are permuted accordingly. Thus, we can see that $n.k.g$ has its $(a_1, \cdots, a_n) = (0, 1, \cdots, n - 1)$ i.e. $n.k.g$ is upper triangular. This proves $g \in BWB$.

To prove uniqueness, suppose $n_1 \in BnB$, say $n_1 = bnb_1$. So $b = n_1 b_1^{-1} n^{-1}$. If we define for a matrix $X = (x_{ij})$, $\mathrm{Supp}(X) = \{(i, j) : x_{ij} \neq 0\}$, then $\mathrm{Supp}(b) = \mathrm{Supp}(n_1 b_1^{-1} n^{-1}) \supseteq \mathrm{Supp}(n_1 n^{-1})$. But $n_1 n^{-1}$ is monomial and b does not have any support under the diagonal; this gives $n_1 n^{-1} \in T$ i.e. $\pi(n_1) = \pi(n)$.

♣ For $SL(2, F)$, Bruhat decomposition looks explicitly as follows. Let

$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. If $z = 0$, then $X \in B$. If $z \neq 0$, then

$$X = \begin{pmatrix} -z^{-1} & -x \\ 0 & -z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & tz^{-1} \\ 0 & 1 \end{pmatrix} \in BWU$$

SOME MORE DETAILS

Basics on Haar measure

G locally compact group.

(1) Let $s \in G$, f a function on G. Then sf and fs are defined, respectively, as $(sf)(x) = f(s^{-1}x)$ and $(fs)(x) = f(xs^{-1})$.

(2) A measure μ on G is left-invariant, if $\mu(sf) = \mu(f)$ i.e.

$$\int f(s^{-1}x)d\mu(x) = \int f(x)d\mu(x) \quad \forall s \in G$$

From (3) onwards, we suppose μ is a left invariant measure on G.

(3) The right modulus of μ is the function $\Delta_r : G \to \mathbf{R}_+^*$ defined by

$$\int f(xt^{-1})d\mu(x) = \Delta_r \cdot \int f(x)d\mu(x)$$

It is a continuous homomorphism; it is independent of the choice of μ.

(4)

$$\int f(x^{-1})d\mu(x) = \int \frac{f(x)}{\Delta_r(x)}d\mu(x)$$

(5) Let $\sigma \in Aut(G)$. Then, there exists a unique $\delta(\sigma) \in \mathbf{R}_+^*$ such that

$$\int f(\sigma^{-1}(x))d\mu(x) = \delta(\sigma)\int f(x)d\mu(x)$$

$\delta : Aut(G) \to \mathbf{R}_+^*$ gives a continuous homomorphism. Moreover, the same formula holds (i.e. with the same number $\delta(\sigma)$) if μ is replaced by a right

invariant mesure.

(6) $\delta(\text{Int } s) = \Delta_r(s)^{-1} \ \forall \ s \in G$.

(7) $\Delta_r(\sigma(s)) = Delta_r(s)$.

(8) Let $B = T \cdot U$ be a semi-direct product where $t \in T$ acts on U by σ_t. If dt, du are bi-invariant Haar measures on T, U respectively, then $dt \ du$ is a left Haar measure on B, and a right invariant measure on B is $\delta(\sigma_t) \ dt \ du$, i.e. $\Delta_r^B(t, u) = \delta^U(\sigma_t)^{-1}$.

(9) In our case of B, T, U etc., we have

$$\delta^U(\sigma_t) = \Delta_r^B(t, u)^{-1} = \prod_{i<j} \frac{t_i}{t_j}$$

where $t = \text{diag}(t_1, \cdots, t_n)$.

Proofs

(3) Define $\tau(f) = \mu(ft)$ i.e.

$$\tau(f) = \int f(xt^{-1}) d\mu(x)$$

τ is also left invariant, since $\tau(sf) = \mu((sf)t) = \mu(s(ft)) = \mu(ft) = \tau(f)$. So, we get some number $\Delta_r(t)$ as written. Choosing f such that $\int f(x) d\mu(x) = 1$, we see that

$$\Delta_r(t) = \int f(xt^{-1}) d\mu(x)$$

from which we can see that Δ_r is continuous. To show it is a homomorphism, we consider

$$\Delta_r(st) \int f(x) d\mu(x) = \int f(xt^{-1}s^{-1}) d\mu(x) = \int g(xt^{-1}) d\mu(x)$$

(where $g(x) = f(xs^{-1})$)

$$= \Delta_r(t) \int g(x) d\mu(x) = \Delta_r(t) \int f(xs^{-1}) d\mu(x) = \Delta_r(s) \Delta_r(t) \int f(x) d\mu(x)$$

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(4) We show that both sides define rightinvariant functionals. First, define $\tau_1(f) = \int f(x^{-1})d\mu(x)$. Then

$$\tau_1(fs) = \int f(x^{-1}s^{-1})d\mu(x) = \int \tilde{f}(sx)d\mu(x)$$

(where $\tilde{f}(x) = f(x^{-1})$)

$$= \int \tilde{f}(x)d\mu(x) = \tau_1(f)$$

Now, we define $\tau_2(f) = \int \frac{f(x)}{\Delta_r(x)}d\mu(x)$. Then,

$$\tau_2(fs) = \int \frac{f(xs^{-1})}{\Delta_r(x)}d\mu(x) = \Delta_r(s)^{-1}\int \frac{f(xs^{-1})}{\Delta_r(xs^{-1})}d\mu(x) = \Delta_r(s)^{-1}\int g(xs^{-1})d\mu(x)$$

(where $g(x) = \frac{f(x)}{\Delta_r(x)}$).

$$= \Delta_r(s)^{-1}\Delta_r(s)\int g(x)d\mu(x) = \tau_2(f)$$

So, there is some constant $c > 0$ such that $\tau_1 = c\tau_2$. Evaluating on a function f which is symmetric about x and x^{-1} i.e. for which $f(x) = f(x^{-1})$, and noting that Δ_r is continuous, it is easy to see that $c = 1$.

(5) Define $\tau(f) = \int f(\sigma^{-1}(x))d\mu(x)$. Then,

$$\tau(sf) = \int f(\sigma^{-1}(t^{-1}x))d\mu(x) = \int g(t^{-1}x)d\mu(x)$$

where $\sigma(s) = t$ and $g(x) = f(\sigma^{-1}x)$.

$$= \int g(x)d\mu(x) = \tau(f)$$

Since τ is also left invariant, we get $\delta(\sigma)$.

(8) First, we check that $dt\, du$ is left invariant.

$$\int f((sv)^{-1}tu)dt\, du = \int f(v^{-1}tu)dt\, du = \int g((t^{-1}vt)^{-1}u)du\, dt$$

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(where $g(u) = f(tu)$)

$$= \int g(u)du\ dt = \int f(tu)du\ dt$$

The right modulus is given as follows:

$$\int f(tu(sv)^{-1})du\ dt = \int f(tuv^{-1}s^{-1})du\ dt = \int f(ts^{-1}\cdot suv^{-1}s^{-1})du\ dt = \int f(tsuv^{-1}s^{-1})du\ dt =$$

(where $g(u) = f(tsus^{-1})$)

$$= \int f(tsus^{-1})du\ dt = \int h(sus^{-1})du\ dt = \delta^U(\sigma_s)^{-1}\int h(u)du\ dt$$

(where $h(u) = f(tu)$)

$$= \delta^U(\sigma_s)^{-1}\int f(tu)du\ dt$$

Therefore $\Delta_r^B(s,v) = \delta^U(\sigma_s)^{-1}$. The proof is complete, since we know a right invariant measure from (4).

(9) If we identify $U \to F^{\frac{n(n-1)}{2}}$ via the map $\theta : u = (u_{ij}) \to (u_{ij}; i < j)$, the Lebesgue measure pulls back to a Haar measure on U, which we can choose to be du. Since σ_t takes u_{ij} to $\frac{t_i}{t_j}u_{ij}$, by the change of variable formula in $F^{\frac{n(n-1)}{2}}$, we get

$$\int \cdots \int f(tut^{-1})dt\ du_{ij} = \prod_{i>j}\frac{t_i}{t_j}\int \cdots \int f(u_{ij})du_{ij}$$

(recall $\int f(Av)dv = det(A)^{-1}\int f(v)dv$). Thus, $\delta(\sigma_t)^{-1} = \prod_{i>j}\frac{t_i}{t_j}$, and we get that $\prod_{i>j}\frac{t_i}{t_j}dt\ du$ is a right Haar measure on B.

$\to$ Consequently, for $G = SL(2,\mathbf{R})$, if we write $K = SO(2)$, $A = \{\mathrm{diag}(a, a^{-1}) : a > 0\}$, and dk , dt , dn for Haar measures on K, A, N respectively, then

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a right Haar measure on G is given as $a^2 \, dk \, \frac{da}{a} \, dn = a \, dk \, \frac{da}{a} \, dn$ where we have used $dt = \frac{da}{a}$ for $t = \text{diag}(a, a^{-1})$.

Another way to derive this is as follows.

Write $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G$; we can assume that $x \neq 0$ as the complement has measure zero. Now, $t = \frac{1+yz}{x}$. We will determine $f(X)$ so that $\omega = f(X) \, dx \wedge dy \wedge dz$ is a left invariant differential form. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have $AX = X' = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$, where

$$x' = ax + bz$$

$$y' = ay + b(\frac{1+yz}{x})$$

$$z' = cx + dz$$

For left invariance, we should have

$$f(X) \, dx \wedge dy \wedge dz = f(X') \, dx' \wedge dy' \wedge dz'$$

This gives $f(X')\frac{ax+bz}{x} = f(X)$ i.e. $f(X')x' = f(X)x$. So, we can take $f(X) = \frac{1}{x}$ i.e. $\omega = \frac{1}{x} f(X) \, dx \wedge dy \wedge dz$. Writing

$$X = ktn = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

we have $x = a \cdot \cos(\theta)$, $y = au \cdot \cos(\theta) - a^{-1} \cdot \sin(\theta)$, $z = a \cdot \sin(\theta)$, so that $\omega = a \, d\theta \wedge da \wedge du$.

Exercise : Prove from $G = SL(2, \mathbf{R}) = KAK$, that
$\frac{a^2 - a^{-2}}{2} \, dk \, \frac{da}{a} \, dn$ *is an invariant measure on G.*

Exercise : From its Iwasawa decomposition, deduce that $SL(n, \mathbf{R})$ is connected.

$\rightarrow$ For SL(n,F), Bruhat decomposition looks explicitly as follows:

Let $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in SL(2,F)$. If $z = 0, then X \in B$. If $z \neq 0$,

$$X = \begin{pmatrix} -z^{-1} & -x \\ 0 & -z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & tz^{-1} \\ 0 & 1 \end{pmatrix} \in BWU$$

$\rightarrow$ For $G = GL(2, \mathbf{Q}_p)$, here is a quick proof of the Iwasawa decomposition.

Write $K = SL(2, \mathbf{Z}_p)$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We choose $u, v \in \mathbf{Z}_p$ such that $cu + dv = 0$ and u, v are coprime. So, we can find $w, x \in \mathbf{Z}_p$ so that $ux - vw = 1$ i.e.

$$k = \begin{pmatrix} u & w \\ v & x \end{pmatrix} \in K = SL(2, \mathbf{Z}_p)$$

Now, clearly

$$gk = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & w \\ v & x \end{pmatrix} \in B(\mathbf{Q}_p)$$

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