Journal of Number Theory **72**, 137–139 (1998)
Article No. NT982255

On a Conjecture of Chowla *et al.*

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Communicated by M. Waldschmidt

Received September 29, 1997

We prove some congruences for the numbers $N(m,n)=\sum_k \binom{n}{k}^2 \binom{n+k}{k}^m$. In particular, we show that the numbers $a_p=\sum_k \binom{p}{k}^2 \binom{p+k}{k}^2$ are congruent to 5 modulo p^3 for any prime $p \geqslant 5$, thereby proving a conjecture of Chowla *et al.* (*J. Number Theory* **12** (1980), 188–190).

INTRODUCTION

In his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Apéry made use of the sequences $a_n=\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ and $b_n=\sum_k \binom{n}{k}^2 \binom{n+k}{k}$. Chowla *et al.* [CCC] proved that $a_p \equiv 5 \bmod p^2$ and conjectured that $a_p \equiv 5 \bmod p^3$ for any prime $p \geqslant 5$. Beukers has also proved [B1, B2] some nice congruence involving these numbers. The purpose of this note is to prove some new congruences and, in particular, give a simple proof of the above conjecture.

We prove:

Theorem. *Let* $N(m,n)=\sum_k \binom{n}{k}^2 \binom{n+k}{k}^m$ *for* $m \geqslant 0$. *Then, for any prime* $p>3$

(i) $N(m,p) \equiv 1+2^m \bmod p^3$,

(ii) $N(2,p^r-1) \equiv 1 \bmod p^3$,

(iii) $N(1,p^r-1) \equiv 1 \bmod p^2$,

(iv) $N(2,2p-1) \equiv 5 \bmod p^3$, *and*

(v) $N(2,3p-1) \equiv 73 \bmod p^3$.

In particular, the conjecture mentioned earlier is true.

We make some elementary observations which are used in the proof.

0022-314X/98 $25.00

LEMMA. (i) *Let $r \geqslant 1$, and p be a prime such that $p-1$ does not divide $2r$. Then, the sum*

$$\sum_{k=0}^{p} \binom{p}{k}^{2r} \equiv 2 \qquad \text{mod } p^{2r+1}.$$

In particular, for a prime $p > 3$, $\binom{2p}{p} \equiv 2 \bmod p^3$. In fact, $\binom{ap}{bp} \equiv \binom{a}{b} \bmod p^3$, and $\binom{rp-1}{(r-1)p} \equiv 1 \bmod p^3$.

(ii) *For $0 < k < p$, let u_k denote the inverse* mod p^2. *Then, $\sum u_k \equiv 0$* mod p^2 *if $p > 3$.*

Proof. The proof is completely elementary; we indicate the proof for the first part which is the part relevant to the conjecture. The other parts are similarly proved.

For $1 \leqslant k \leqslant p-1$, write $\binom{p}{k} = pu_k$. Then, $u_k \equiv (-1)^{k-1} k^{-1} \bmod p$. Therefore, $\sum u_k^{2r} = \sum_{u \in \mathbb{F}_p^*} u^{2r}$ in $\mathbb{F}_p$. If we choose a generator g of $\mathbb{F}_p^*$, the identity $g^{2r} \sum_{u \in \mathbb{F}_p^*} u^{2r} = \sum_{u \in \mathbb{F}_p^*} u^{2r}$ implies that, for a prime p so that $p-1$ does not divide $2r$, we must have $\sum_{k=1}^{p-1} u_k^{2r} \equiv 0 \bmod p$. This proves $\sum_k \binom{p}{k}^{2r} = 2 + p^{2r} \sum u_k^{2r} \equiv 2 \bmod p^{2r+1}$.

Proof of the theorem. We prove part (i). Recall the two elementary facts:

(a) For any positive integer m, we have the identity $\sum_k \binom{m}{k}^2 = \binom{2m}{m}$.

(b) For any prime p, $\binom{p+k}{k} \equiv 1 \bmod p$ for $0 \leqslant k \leqslant p-1$.

Let p be a prime > 3. Then, the number

$$\begin{aligned}
N(m, p) &= \sum_{k=0}^{p} \binom{p}{k}^2 \binom{p+k}{k}^m \\
&= \sum_{k=0}^{p-1} \binom{p}{k}^2 \binom{p+k}{k}^m + \binom{2p}{p}^m \\
&\equiv \sum_{k=0}^{p-1} \binom{p}{k}^2 + \binom{2p}{p}^m \\
&\equiv \sum_{k=0}^{p-1} \binom{p}{k}^2 + \left\{ \sum_{k=0}^{p} \binom{p}{k}^2 \right\}^m \qquad \text{mod } p^3 \\
&\equiv \sum_{k=0}^{p} \binom{p}{k}^2 + \left\{ \sum_{k=0}^{p} \binom{p}{k}^2 \right\}^m - 1 \qquad \text{mod } p^3.
\end{aligned}$$

Using the lemma with $r = 1$, it follows immediately that

$$N(m, p) \equiv 1 + 2^m \qquad \text{mod } p^3 \qquad \text{if} \quad p > 3.$$

The proof of the remaining parts uses the lemma and the observations

$$\binom{p-1}{k} \equiv (-1)^k \qquad \text{mod } p$$

$$\binom{p^r-1}{k} \equiv \pm \binom{p^{r-1}-1}{[k/p]}.$$

ACKNOWLEDGMENTS

I dedicate this paper to Professor S. Chowla for delighting generations of mathematicians through his beautiful work. I am thankful to an anonymous referee for his comments.

REFERENCES

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