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Central Extensions of a p-adic Division Algebra

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1. Introduction

Let k be a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers and suppose D is a finite-dimensional central division algebra over k.

Then, the group $G = SL_1(D)$ consisting of elements of reduced norm 1 in D acquires a topology from k and is a compact, totally disconnected (i.e., a profinite) group. We are interested in finding the possible (topological) central extensions

$$1 \to A \to E \to G \to 1.$$

General nonsense tells us that the set of central extensions is determined by a group, denoted $H^2(G, \mathbb{R}/\mathbb{Z})$, which, in this case, is known to be finite by some deep work of Raghunathan on the congruence subgroup problem ([R]). It is expected (although unknown as yet) that, barring a handful of exceptions, $H^2(G, \mathbb{R}/\mathbb{Z}) \cong \mu(k)_p$, the finite cyclic group of p-th power roots of unity in k. In 1988, Gopal Prasad & M. S. Raghunathan proved ([PR]) that $H^2(G, \mathbb{R}/\mathbb{Z})$ is a finite cyclic group containing an isomorphic copy of $\mu(k)_p$ and is trivial if $\mu(k)_p$ is trivial.

These results were sufficient for their original motivation to solve the so-called metaplectic problem which comes up in the congruence subgroup problem. However, the general computation of H^2 is still open.

Our aim here is to stretch the method of [PR] and study the p^2-torsion in $H^2(G, \mathbb{R}/\mathbb{Z})$ with a view to proving that if H^2 has an element of order p^2, then k contains a primitive p^2-th root of unity. The computations are rather cumbersome, and we carry them out fully only in a special case when $p = 3$ and D is the quaternion division algebra although we have partial results in more generality.

One probably needs new ideas along with the work of [PR] if one wants to compute H^2 in general. Perhaps, on the other hand, the seminal work of Lazard ([L]) on compact p-adic Lie groups has not been exploited sufficiently enough.

2. Basic structure of D

The structure of p-adic division algebras had been investigated by C. Riehm in [Ri]. Let us briefly recall some details.

D contains R, its maximal compact subring which, in turn, contains a unique (two-sided) maximal ideal P. The group $G = SL(1, D)$ of elements of reduced norm 1 in D, is a profinite group which is normal in D^*.

G admits a filtration $G_i = \{g \in G : g \equiv 1 \bmod P^i\}$ for $i \geq 1$. In fact, G_i are all normal in D^*. G_1 is a pro-p group and G/G_1 is a finite, cyclic group of order prime to p.

For $i, j \geq 1$, we have $[G_i, G_j] \subseteq G_{i+j}$. In particular, G_i/G_{i+1} is an abelian group for $i \geq 1$.

By local class field theory, there exists a uniformising parameter π in P which normalises the maximal unramified extension K of degree d over k. Also, the automorphism of K given by the conjugation by π generates the Galois group $\mathrm{Gal}(K/k) \cong \mathrm{Gal}\ (F/f)$ where F, f are the residue fields of K, k respectively.

Moreover, the non-zero elements F^* can be identified with $\mu(K)_{\mathrm{tame}}$, the cyclic group of prime-to-p roots of unity in K.

Each element $g \in R$ can be uniquely expressed as $g = g_0 + \sum_{n \geq 1} g_n \pi^n$ with $g_n \in \mu(K)_{\mathrm{tame}} \cup \{0\}$.

A little computation shows also that the abelian group G_i/G_{i+1} can be identified via the map $\rho_i : 1 + \sum_{n \geq i} g_n \pi^n \mapsto g_i$ with $F(i)$ which is either $E := \{x \in F : Tr_{F/f}(x) = 0\}$ or the whole of F according as $d|i$ or $d \nmid i$.

It is also quite easy to show that $G = G_1(G \cap \mu(K)_{\mathrm{tame}})$.

Note that each $F(i)$ is a module for $G \cap \mu(K)_{\mathrm{tame}}$ under the action $\phi \cdot x = \frac{\phi}{\sigma^i(\phi)} x$ where we have identified $\mu(K)_{\mathrm{tame}}$ with $F \setminus \{0\}$. A consequence of Hilbert's theorem 90 is that there is a nontrivial homomorphism (of modules) from $F(i)$ to $F(j)$ if, and only if, $i \equiv j \bmod d$.

3. Conditions for roots of unity

It is easy to write down necessary and sufficient conditions for k to contain a primitive p^2-th root of unity.

Recall that $k \subset K \subset D$ and π is a uniformising parameter in D normalising K. Let e denote the ramification index of k over $\mathbb{Q}_p$ and d denote the degree of D over k. Then π^d is a uniformising parameter for k and, one can expand p over k as

$$p = \theta \pi^{de} + \theta_d \pi^{de+d} + \cdots$$

for some θ's in f.

Now, well-known properties of the p-th power map (see [M], P. 167–168) tells us that k has a primitive p^2-th root of unity if, and only if, $p(p-1)$ divides e and there exists some $Y = 1 + Y_{de/p(p-1)} \pi^{de/p(p-1)} + Y_{de/p(p-1)+d} \pi^{de/p(p-1)+d} + \cdots$ such that $Y_{de/p(p-1)} \neq 0$ and such that $Y^{p^2} \equiv 1 \bmod \pi^{de/p(p-1)+de+d}$.

This gives (using the expression of p) certain polynomial equations in the Y_i's with coefficients as some θ's. We get finitely many polynomials over f.

Our aim is, therefore, to deduce the simultaneous solvability of these equations by somehow getting information over f that can be derived from the assumption that H^2 has p^2-torsion.

It should be pointed out that the corresponding calculation with p in place of p^2 is much easier and was carried out in [PR].

4. Strategy of studying H^2 (after [PR])

In this section, we recall the basic method as well as the results of Prasad & Raghunathan from [PR] which we shall be using.

Fact 1. Using the fact that G is profinite, it is easy to deduce that $H^2(G, \mathbb{R}/\mathbb{Z}) \cong H^2(G, J)$ where J is the subgroup of $\mathbb{Q}/\mathbb{Z})$ consisting of p-power order (considered with the discrete topology).

We shall be using the Hochschild-Serre spectral sequence for the situation $G_{i-1}/G_i \leq G/G_i$ for $i > 1$.

A very useful property is that given a central extension $C \subseteq E \to A$ where A is abelian is that one has a lifted 'commutator' map; if $a, b \in A$, then for arbitrary lifts x, y of $a, b \in E$, the commutator $[x, y] = xyx^{-1}y^{-1}$ lands inside C and, is independent of the lifts. One often writes $[a, b]$ for this element of C. When C is a divisible group (like our J), the extension is 'trivial' if, and only if, E itself is abelian.

In our case, we shall use it for G_i/G_j which is abelian if $2i \geq j$.

Fact 2. $H^2(G, J) \overset{\lim}{=\!\!\to} H^2(G/G_i, J)$ and, the 'inflation' maps inf $(i) : H^2(G/G_i, J) \to H^2(G, J)$ are injective if, and only if, $d \not| i$. Moreover, $H^2(G, J)$ is the union of the images $H^2(G)_i$ over all i under the inflation maps; $H^2(G)_i$ is an increasing filtration.

If $d|i$, then there is a natural identification of Ker inf(i) with the vector space E of elements of trace zero in F over f.

Thus, one needs to compare $H^2(G)_i$ and $H^2(G)_{i-1}$ for various i.

Fact 3. If $d \not| i$, $H^2(G)_i = H^2(G)_{i-1}$.

In fact, one can show that the same equality holds if $dp \not| i$; this uses some commutator identities due to P. Hall which are valid in any group.

(P. Hall) In any group G, for elements a, b, c one has

$$[[a, b],^b c]][[b, c],^c a]][[c, a],^a b]] = 1.$$

Look at the inflation maps

$$H^2(G/G_{r-1}, J) \to H^2(G/G_r, J) \to H^2(G, J).$$

If $d|r$, since the inflation inf (r) is not injective, it is necessary to know when some $c \in H^2(G/G_r, J)$ inflates in $H^2(G, J)$ to an element which comes from $H^2(G/G_{r-1}, J)$. This happens if c restricts to the trivial extension over the subgroup G_{r-1}/G_r. More precisely, using the Hochschild-Serre sequence corresponding to $G_{r-1}/G_r \lhd G/G_r$, we see that:

Fact 4. c comes from $H^2(G/G_{r-1}, J)$ if, and only if, it is in Ker$(H^2(G/G_r, J) \to H^2(G_{r-1}/G_r, J)) \cap E_\infty^{1,1}$.

This is understood better as follows.

For any c in $H^2(G/G_r, J)$, let $J \subseteq E \to G/G_r$ denote the corresponding central extension. The $E_2^{1,1}$-term is

$$H^1(G/G_{r-1}, H^1(G_{r-1}/G_r, J)) = H^1(G_1/[G_1, G_1], \text{Hom}\ (F(r-1), J))^{G/G_1}$$
$$= \text{Hom}\ (F(1), \text{Hom}\ (F(r-1), J))^{G \cap \mu(K)_{\text{tame}}}.$$

As we noted, this is nontrivial only if $d|r$. If $d|r$, this is just the set of all (equivariant) bimultiplicative maps from $F(1) \times F(r-1)$ to f_0 where we shall write f_0 for the prime field $\mathbb{Z}/p$.

In fact, it is easy to write down all its elements. These are the maps $F(1) \times F(r-1) \to f_0$; $(X, Y) \mapsto Tr_{F/f_0}(\lambda X \sigma(Y))$ for some $\lambda \in F$. With this identification, it follows that c comes from the previous level if, and only if, the corresponding λ has trace zero over f i.e., we get:

Fact 5. c in $H^2(G/G_r, J)$ is the inflation of an element of $H^2(G/G_{r-1}, J)$ if, and only if, there exists $\lambda \in F$ of trace zero over f such that c can be 'regarded' as (this is the image in the $E_2^{1,1}$-term) the map from $F(1) \times F(r-1)$ to $f_0 = \mathbb{Z}/p$ given by

$$\wedge^c(X, Y) = Tr_{F/f_0}(\lambda X \sigma(Y)) \; \forall \; X \in F(1), Y \in F(r-1).$$

It should be noted that the map above is induced by the 'commutator' map from $E_1 \times E_{r-1}$ to J where E_i is the inverse image of G_i/G_r in E.

More generally, from P. Hall's identity, one can easily show that E_i commutes with $E_{r-i+1} \; \forall \; i \leq r$. Thus the central extension splits over G_i/G_r whenever $2i > r$. Thus:

Fact 6. If $r > 2$ and $\epsilon < \frac{r}{2}$ then the restriction map $H^2(G/G_r, J) \to H^2(G_{r-\epsilon}/G_r, J)$ is the zero map.

Once again, if $\epsilon < \frac{r}{2}$, we can consider the Hochschild-Serre sequence corresponding to $G_{r-\epsilon}/G_r \lhd G/G_r$. By the above, ($E_\infty^{0,2}$ - term is zero and so) we have a homomorphism

$$H^2(G/G_r, J) \to E_\infty^{1,1} \subseteq E_2^{1,1} = H^1(G/G_{r-\epsilon}, \text{Hom } (G_{r-\epsilon}/G_r, J)).$$

One can show that the above-mentioned image in the $E_2^{1,1}$-term actually comes from $(H^1(G/G_{\epsilon+1}, \text{Hom } (G_{r-\epsilon}/G_r, J))$. More precisely, :

Fact 7. With $\epsilon < r/2$, the image in $E_2^{1,1}$ is contained in the image of

$$\inf l : H^1(G/G_{\epsilon+1}, \text{Hom } (G_{r-\epsilon}/G_r, J)) \to E_2^{1,1}.$$

A key point (discovered in [PR]) is that one can describe these 1-cocycles very explicitly. We describe this now.

Fact 8. Let ϵ, s, t be positive integers s.t. $\epsilon \leq \min.(de, \frac{1}{2}dt)$ and $s \geq \epsilon + 1$. Let $f = \left[\frac{\epsilon-1}{d}\right]$. For $(\lambda_0, \ldots, \lambda_f) \in F^{f+1}$,

$$Z_{(\lambda_0,\ldots,\lambda_f)}(a)(b) = Tr_{F/f_0}\left\{\sum_{0\leq u\leq f} \sum_{\ell+m\leq\epsilon-du} \lambda_u(\ell) a_\ell \sigma^\ell(a'_m)\sigma^{\ell+m}(b_{dt-du-\ell-m})\right\}$$

(here $a = \sum a_\ell \pi^\ell \in D_1/D_s$, $b = \sum b_m \pi^m \in D_{dt-\epsilon}/D_{dt}$) is a F-invariant 1-cocycle on D_1/D_s with values in Hom $(D_{dt-\epsilon}/D_{dt})$; these restricted to $G_1/G_s \times G_{dt-\epsilon}/G_{dt}$ and then extended to $G/G_s \times G_{dt-\epsilon}/G_{dt}$ by defining them to be zero on $(G \cap F) \times G_{dt-\epsilon}/G_{dt}$, give all the cohomology classes in $H^1(G/G_s, \text{Hom } (G_{dt-\epsilon}/G_{dt}, J))$.

Also, here $\lambda(l)$ stands for $\lambda + \sigma(\lambda) + \cdots + \sigma^{l-1}(\lambda)$.

We shall be using this only for $\epsilon = de$, $s = de + 1$ *and* $t = \frac{ep}{p-1} + e$.

Finally, let us note:

Fact 9. $H^2(G)_r = 0$ for $r < \frac{dep}{p-1}$ and $H^2(G)_{dep/(p-1)}$ constitutes the elements of order at most p in $H^2(G, \mathbb{R}/\mathbb{Z})$.

Moreover, if $H^2(G, \mathbb{R}/\mathbb{Z})$ has an element of order p^2, then $r = \frac{dep}{p-1} + de$ where $H^2(G)_r$ is the earliest where an element of order p^2 shows up.

This is because the p-th power gives $G_{\frac{dep}{p-1}} \cong G_{\frac{dep}{p-1}+de}$ and an element c of $H^2(G)_{\frac{dep}{p-1}+i}$ is of order $p^j \Leftrightarrow c^p$ in $H^2(G)_{\frac{dep}{p-1}+i-de}$ is of order p^{j-1}. Also, we have $p|e$ because cohomology 'pops up' at stages which are multiples of pd. Thus, pd/de i.e. $p|e$.

5. Outline of proof

Here is how we obtain conditions over f using the assumption that H^2 has p^2-torsion. We look at the corresponding element in

$$H^1(G/G_{1+de}, H^1(G_{dep/(p-1)}/G_{dep/(p-1)+de}, J)).$$

We consider the abelian subgroup $A = K \cap G$ of G and, as elements of the above cohomology group, we may write down the equations $[X, Y^{p^2}] = 0 \ \forall \ X, Y \in \mathcal{A}$, where $\mathcal{A}$ is the image of A in $G/G_{\frac{dep}{p-1}+de}$. These commutators are computed with the help of the above explicit expressions for $\epsilon = de, dt = \frac{dep}{p-1} + de, s = de + 1$ as written down in fact 8.

We note that the triviality of these commutators is due to their bilinearity. Thus we get equations over f from which we try to deduce the required equations (for p^2th root to exist in k) in f. Note that in the computation of commutators in the central extension corresponding to c, the λ_0 which figures is such that $Tr_{F/f}(\lambda_0) \neq 0$ since $\frac{dep}{p-1} + de$ is the smallest level where p^2-torsion occurs.

6. The prime $p = 3$

We carry out the computations only in the following special case.

We assume $p = 3, d = 2, e = p(p-1) = 6$.

As π^2 is a uniformising parameter for k, we can write $\frac{3}{\pi^{12}} = \theta + \theta_2\pi^2 + \theta_4\pi^4 + \dots$ where $\theta, \theta_2, \dots \in f$.

For a p^2-th root of 1 to exist in k, it is necessary and sufficient that $\exists Y \in U_{\frac{de}{p(p-1)}}$ such that $Y^{p^2} \in U_{\frac{dep}{p-1}+de+1}$ and such that $Y \notin U_{\frac{de}{p-1}}$ i.e. Y is not a p-th root.

In our case, we want $Y \in U_2 \setminus U_6$ such that $Y^9 \equiv 1 \bmod \pi^{31}$. Then,

$$\theta Y_2^3 + Y_2^9 = 0$$
$$\theta_2 = 0$$
$$\theta_4 = 0$$

$$\theta(Y_4 + Y_2^2)^3 + \theta_6 Y_2^3 = 0$$
$$\theta^2 Y_2 + \theta_8 Y_2^3 = 0$$
$$\theta^2(Y_4 + Y_2^2) + \theta_{10} Y_2^3 = 0$$
$$\theta^2 Y_6 + \theta Y_6^3 + 2(\theta^2 Y_2 Y_4 + \theta Y_2^3 Y_4^3) + \theta_6(Y_4^3 + Y_2^6) + \theta_{12} Y_2^3 = 0.$$

We can find such a Y if, and only if, the following 'compatibility' conditions hold

$$\theta_2 = 0$$
$$\theta_4 = 0$$
$$\theta + X^2 = 0 \text{ has a solution } \mu \text{ in } f$$
$$\theta_8^3 = \theta^5$$
$$\theta_{10}^3 + \theta^4\theta_6 = 0$$
$$X^9 + \theta^3 X^3 = \theta_6^3 + \theta_6^6 \mu^{-9} - \theta_6 \mu^6 + \theta_{12}^3 \mu^{-3} \text{ has a solution over } f.$$

If these hold, then the solutions for Y are:

$$Y_2 = \text{A solution of } Y_2^6 + \theta = 0$$
$$Y_4 = \text{A solution of } Y_4^3 = \theta_6 Y_2^{-3} - Y_2^6$$
$$Y_6 = \text{A solution of } Y_6^9 + \theta^3 Y_6^3 = \theta_6^3 + \frac{\theta_6^6}{Y_2^{27}} + \theta^3\theta_6 + \frac{\theta_{12}}{Y_2^9}.$$

Let us expand the relevant powers of Y now using the expression of p in k.

$$\begin{aligned}
Y &= 1 + Y_2\pi^2 + Y_4\pi^4 + Y_6\pi^6 + \dots\dots \in U_2 \\
Y^3 &= 1 + (\theta\pi^{12} + \theta_2\pi^{14} + \theta_4\pi^{16} + \dots)(Y_2\pi^2 + Y_4\pi^4 + Y_6\pi^6 + \dots) \\
&\quad + (\theta\pi^{12} + \theta_2\pi^{14} + \dots)(Y_2^2\pi^4 + 2Y_2Y_4\pi^6 + (Y_4^2 + 2Y_2Y_6)\pi^8 + \dots) \\
&\quad + (Y_2^3\pi^6 + Y_4^3\pi^{12} + Y_6^3\pi^{18} + \dots) \\
&= 1 + Y_2^3\pi^6 + Y_4^3\pi^{12} + \theta Y_2\pi^{14} + (\theta Y_2^2 + \theta Y_4 + \theta_2 Y_2)\pi^{16} \\
&\quad + (Y_6^3 + 2\theta Y_2 Y_4 + \theta Y_6 + \theta_2 Y_2^2 + \theta_2 Y_4 + \theta_4 Y_2)\pi^{18} + \dots \\
Y^9 &= 1 + (\theta\pi^{12} + \theta_2\pi^{14} + \dots)(Y_2^3\pi^6 + Y_4^3\pi^{12} + \theta Y_2\pi^{14} + (\theta Y_2^2 + \theta Y_4 + \theta_2 Y_2)\pi^{16} \\
&\quad + (Y_6^3 + 2\theta Y_2 Y_4 + \theta Y_6 + \theta_2 Y_2^2 + \theta_2 Y_4 + \theta_4 Y_2)\pi^{18} + \dots) \\
&\quad + (\theta\pi^{12} + \theta_2\pi^{14} + \dots)(Y_2^6\pi^{12} + 2Y_2^3 Y_4^3\pi^{18} + \dots) + (Y_2^9\pi^{18} \bmod \pi^{36}) \\
&= 1 + (\theta Y_2^3 + Y_2^9)\pi^{18} + \theta_2 Y_2^3\pi^{20} + \theta_4 Y_2^3\pi^{22} + (\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3)\pi^{24} \\
&\quad + (\theta^2 Y_2 + \theta_2 Y_4^3 + \theta_2 Y_2^6 + \theta_8 Y_2^3)\pi^{26}
\end{aligned}$$

$$+ (\theta^2 Y_2^2 + \theta^2 Y_4 + 2\theta\theta_2 Y_2 + \theta_4 Y_2^6 + \theta_4 Y_4^3 + \theta_{10} Y_2^3)\pi^{28}$$
$$+ (\theta^2 Y_6 + \theta Y_6^3 + 2(\theta^2 Y_2 Y_4 + \theta Y_2^3 Y_4^3) + \theta_6 (Y_4 + Y_2^2)^3$$
$$+ \theta_{12} Y_2^3 + 2\theta\theta_2 Y_2^2 + 2\theta\theta_2 Y_4 + 2\theta\theta_4 Y_2 + \theta_2^2 Y_2)\pi^{30} + \dots .$$

Now $H^1\big(G/G_{1+de}, \mathrm{Hom}\,\big(G_{\frac{dep}{p-1}}/G_{\frac{dep}{p-1}+de}, J\big)\big) = H^1\big(G/G_{13}, \mathrm{Hom}\,(G_{18}/G_{30}, J)\big)$. Consider $X, Y \in K^1$ such that $Y^9 \in G_{18}/G_{30}$.

One knows then that $[X, Y^9] = [X, Y]^9 = 1$ and $[X, Y^9]$ can be calculated from our knowledge of $H^1(G/G_{13}, \mathrm{Hom}\,(G_{18}/G_{30}, J))$.

In fact, if $X = 1 + \sum_1^6 X_{2i}\pi^{2i}$ and $Y = 1 + \sum Y_{2i}\pi^{2i}$, if $Y^9 = 1 + \sum_9^{14} b_{2i}\pi^{2i}$, then

$$[X, Y^9] = Tr_{F/f_0}\left\{\sum_{0\le u\le 5}\ \sum_{\ell+m\le 12-2u} \lambda_u(\ell) X_\ell \bar{X}_m b_{30-2u-\ell-m}\right\}. \tag{1}$$

Here $X^{-1} = 1 + \sum \bar{X}_{2i}\pi^{2i}$. Since $Y^9 \in G_{18}/G_{30}$, we have $b_i = 0$ for $i \neq$ 18, 20, 22, 24, 26 or 28.

Contributions to the right side of (1) are as follows:

For $u = 5$, it is

$$Tr_{F/f_0}\{\lambda_5(2) X_2 b_{18}\} = Tr_{F/f_0}\left\{Tr_{F/f}(\lambda_5) X_2(\theta Y_2^3 + Y_2^9)\right\}.$$

For $u = 4$, it is

$$Tr_{F/f_0}\left\{\lambda_4(2)(X_2 b_{20} + X_2\bar{X}_2 b_{18}) + \lambda_4(4) X_4 b_{18}\right\}$$
$$= Tr_{F/f_0}\left\{Tr_{F/f}(\lambda_4)\left[X_2\theta_2 Y_2^3 + (2X_4 + X_2\bar{X}_2)(\theta Y_2^3 + Y_2^9)\right]\right\}.$$

For $u = 3$, it is

$$Tr_{F/f_0}\left\{\lambda_3(2)(X_2 b_{22} + X_2\bar{X}_2 b_{20} + X_2\bar{X}_4 b_{18})\right.$$
$$\left. + \lambda_3(4)(X_4 b_{20} + X_4\bar{X}_2 b_{18}) + \lambda_3(6) X_6 b_{18}\right\}$$
$$= Tr_{F/f_0}\left[Tr_{F/f}(\lambda_3)\left\{X_2\theta_4 Y_2^3 + (2X_4 + X_2\bar{X}_2)\theta_2 Y_2^3\right.\right.$$
$$\left.\left. +(2X_4\bar{X}_2 + X_2\bar{X}_4)(\theta Y_2^3 + Y_2^9)\right\}\right].$$

Note that $\lambda_3(6) = 3T_{r_{F/f}}(\lambda_3) = 0$.

For $u = 2$, it is

$$Tr_{F/f_0}\left\{Tr_{F/f}(\lambda_2)\begin{bmatrix} X_2(\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3) + (2X_4 + X_2\bar{X}_2)\theta_4 Y_2^3 \\ +(2X_4\bar{X}_2 + X_2\bar{X}_4)\theta_2 Y_2^3 + (X_8 + 2X_4\bar{X}_4 + X_2\bar{X}_6)(\theta Y_2^3 + Y_2^9)\end{bmatrix}\right\}.$$

For $u = 1$, it is

$$Tr_{F/f_0}\left\{Tr_{F/f}(\lambda_1)\begin{bmatrix} X_2(\theta^2 Y_2 + \theta_2 Y_4^3 + \theta_2 Y_2^6 + \theta_8 Y_2^3) \\ +(2X_4 + X_2\bar{X}_2)(\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3) \\ +(2X_4\bar{X}_2 + X_2\bar{X}_4)\theta_4 Y_2^3 + (X_8 + 2X_4\bar{X}_4 + X_2\bar{X}_6)\theta_2 Y_2^3 \\ +(2X_{10} + X_8\bar{X}_2 + 2X_4\bar{X}_6 + X_2\bar{X}_8)(\theta Y_2^3 + Y_2^9) \end{bmatrix}\right\}.$$

For $u = 0$, it is

$$Tr_{F/f_0}\left\{Tr_{F/f}(\lambda_0)\begin{bmatrix} X_2(\theta^2 Y_2^2 + \theta^2 Y_4 + 2\theta\theta_2 Y_2 + \theta_4 Y_2^6 + \theta_4 Y_4^3 + \theta_{10} Y_2^3) \\ +(2X_4 + X_2\bar{X}_2)(\theta^2 Y_2 + \theta_2 Y_4^3 + \theta_2 Y_2^6 + \theta_8 Y_2^3) \\ +(2X_4\bar{X}_2 + X_2\bar{X}_4)(\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3) \\ +(X_8 + 2X_4\bar{X}_4 + X_2\bar{X}_6)\theta_4 Y_2^3 \\ +(2X_{10} + X_8\bar{X}_2 + 2X_4\bar{X}_6 + X_2\bar{X}_8)\theta_2 Y_2^3 \\ +(2X_{10}\bar{X}_2 + X_8\bar{X}_4 + 2X_4\bar{X}_8 + X_2\bar{X}_{10})(\theta Y_2^3 + Y_2^9) \end{bmatrix}\right\}.$$

We also see (since X is of norm 1 and $p = 3$) that

$$2X_4 + X_2\bar{X}_2 = -(X_4 + X_2^2)$$
$$2X_4\bar{X}_2 + X_2\bar{X}_4 = X_2^3$$
$$2X_{10}\bar{X}_2 + X_8\bar{X}_4 + 2X_4\bar{X}_8 + X_2\bar{X}_{10} = -(X_4^3 + X_2^6).$$

We have $0 = [X, Y^9] =$ Sum of these 6 terms corresponding to the values $u = 0, 1, 2, 3, 4, 5$.

Now, we start proving the compatibility conditions hold good.

Define $\tilde{X}$ by changing X_{10} to $\tilde{X}_{10} = X_{10} + \mu$ for some μ of trace 0. Then, we can have $\tilde{X} \in K^1$ with $\tilde{X}_i = X_i$ for $i < 10$. Now,

$$\begin{aligned} 0 &= [\tilde{X}, Y^9] - [X, Y^9] \\ &= Tr_{F/f_0}\left\{Tr_{F/f}(\lambda_1)2\mu(\theta Y_2^3 + Y_2^9) + Tr_{F/f}(\lambda_0)2\mu\theta_2 Y_2^3\right\} \\ &= 4Tr_{f/f_0}\left\{Tr_{F/f}(\lambda_1)\mu(\theta Y_2^3 + Y_2^9) + Tr_{F/f}(\lambda_0)\mu\theta_2 Y_2^3\right\}. \end{aligned}$$

We have used the fact that when $d = 2$, $E \cdot E = f$.

For $Y_2 \neq 0$, since μY_2^3 could be any arbitrary element of f, not depending on Y_2^3, we must have

$$Tr_{F/f}(\lambda_1)(\theta + Y_2^6) + Tr_{F/f}(\lambda_0)\theta_2 = 0\ \forall\ Y_2 \in E.$$

If we take $\tilde{Y}_2$ whose square is not Y_2^2, then we get, on subtraction,

$$Tr_{F/f}(\lambda_1)(\tilde{Y}_2^6 - Y_2^6) = 0 \text{ i.e. } Tr_{F/f}(\lambda_1) = 0. \qquad (2)$$

Therefore (as the existence of p^2-torsion implies that $Tr_{F/f}(\lambda_0) \neq 0$), we obtain

$$\theta_2 = 0.$$

Then, the first compatibility condition for the existence of a primitive p^2-th root of unity in k is proved.

Let us do the same with X_8 now i.e. call $\tilde{X} = 1 + X_2\pi^2 + X_4\pi^4 + X_6\pi^6 + (X_8 + \mu)\pi^8 + (X_{10} + \mu X_2)\pi^{10} + \cdots \in K^1$.

$$0 = [\tilde{X}, Y^9] - [X, Y^9]$$
$$\Rightarrow 0 = Tr_{f/f_0}\left\{Tr_{F/f}(\lambda_2)\mu(\theta Y_2^3 + Y_2^9) + Tr_{F/f}(\lambda_0)\mu\theta_4 Y_2^3\right\}.$$

Changing μ to $\alpha\mu$ for any $\alpha \in f$, we get

$$0 = Tr_{F/f}(\lambda_2)(Y_2^9 + \theta Y_2^3) + Tr_{F/f}(\lambda_0)\theta_4 Y_2^3 \ \forall\ Y_2 \in E.$$

Again, as before, we will get

$$Tr_{F/f}(\lambda_2) = 0 \tag{3}$$

and therefore,

$$\theta_4 = 0. \tag{4}$$

This is the 2nd compatibility condition.

Now, putting $X_2 = 0$ in the original formula (which now consists of four terms since $Tr(\lambda_1) = 0 = Tr(\lambda_2)$), we get for all $X, Y \in E$

$$Tr_{f/f_0}\left\{Tr_{F/f}(\lambda_4)X(Y^9 + \theta Y^3) + Tr_{F/f}(\lambda_0)[X(\theta^2 Y + \theta_8 Y^3) + X^3(\theta Y^3 + Y^9)]\right\} = 0. \tag{5}$$

Again putting $X_4 = -X_2^2$ in the original formula, we get

$$Tr_{f/f_0}\left\{\begin{matrix} Tr_{F/f}(\lambda_5)X_2(Y_2^9 + \theta Y_2^3) + Tr_{F/f}(\lambda_3)X_2^3(Y_2^9 + \theta Y_2^3) \\ +Tr_{F/f}(\lambda_0)[X_2(\theta^2 Y_2^2 + \theta^2 Y_4 + \theta_{10} Y_2^3) + X_2^3(\theta Y_2^6 + \theta Y_4^3 + \theta_6 Y_2^3)] \end{matrix}\right\} = 0.$$

Putting $Y_2 = 0$ will give

$$Tr_{f/f_0}\{Tr_{F/f}(\lambda_0)(\theta^2 XY + \theta X^3 Y^3)\} = 0 \ \forall\ X, Y \in E. \tag{6}$$

Since $Tr_{F/f}(\lambda_0) \neq 0$ and since $Z \mapsto \theta^2 Z + \theta Z^3$ is an endomorphism of f, therefore, it follows from (6) that $\exists Z \neq 0$ in f such that $\theta^2 Z + \theta Z^3 = 0$. Thus, the 3rd compatibility condition has been proved.

In the above, instead of putting $Y_2 = 0$, let us, instead, put $Y_4 = -Y_2^2$ which will give $\forall\ X, y \in E$

$$Tr_{f/f_0}\left\{\begin{matrix} Tr_{F/f}(\lambda_5)X(\theta Y^3 + Y^9) + Tr_{F/f}(\lambda_3)X^3(\theta Y^3 + Y^9) \\ +Tr_{F/f}(\lambda_0)(\theta_{10} XY^3 + \theta_6 X^3 Y^3) \end{matrix}\right\} = 0. \tag{7}$$

We shall use this later in the proof of the fifth compatibility condition.

Next our aim is to prove $\theta_8^3 = \theta^5$ which is the 4th condition. Now since F is a quadratic extension of f and E is the trace zero elements in F, we have $E = f \cdot \alpha$ for some $\alpha \in F$ such that $\alpha^2 \in f^* \setminus (f^*)^2$. Let us expand (5) by putting $X = x\alpha$, $Y = y\alpha$ where $x, y \in f$. We get

$$0 = Tr_{f/f_0}\left\{\alpha^4 Tr(\lambda_4) Z y^2(\theta + y^6\alpha^6) + \alpha^2 Tr(\lambda_0)[Z(\theta^2 + \theta_8 y^2\alpha^2) \right. \\ \left. + \alpha^4 Z^3(\theta + y^6\alpha^6)]\right\} \tag{8}$$

where we have put $Z = xy$ and the traces inside are $Tr_{F/f}$.

Replacing y by $y+1$ and subtracting, we get

$$0 = Tr_{f/f_0}\begin{Bmatrix} \alpha^4 Tr(\lambda_4) Z[(1-y)\theta + \alpha^6((y+1)^8 - y^8)] \\ +\alpha^2 Tr(\lambda_0)[Z\theta_8\alpha^2(1-y) + \alpha^{10}Z^3(1-y^3))] \end{Bmatrix}.$$

Again, putting $y+1$ for y and subtracting,

$$0 = Tr_{f/f_0}\begin{Bmatrix} \alpha^4 Tr(\lambda_4) Z(-\theta - \alpha^6(1 + y^2 + y^4 + y^6)] \\ +\alpha^2 T_r(\lambda_0)(-Z\theta_8\alpha^2 - Z^3\alpha^{10}) \end{Bmatrix}.$$

Assuming that $\# f$ is large enough so that $\exists y_1, y_2$ s.t. $1 + y_1^2 + y_1^4 + y_1^6 \neq 1 + y_2^2 + y_2^4 + y_2^6$, we will have

$$0 = Tr_{f/f_0}\{\alpha^{10} Tr(\lambda_4) Z(y_1^2 + y_1^4 + y_1^6 - y_2^2 - y_2^4 - y_2^6)\}\ \forall\ Z \in f$$

and so

$$Tr_{F/f}(\lambda_4) = 0. \tag{9}$$

The equation (8) becomes
$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)[Z(\theta^2 + \theta_8 y^2\alpha^2) + \alpha^4 Z^3(\theta + y^6\alpha^6)]\}$. Putting $y+1$ for y and subtracting,

$$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)[Z\theta_8\alpha^2(1-y) + Z^3\alpha^{10}(1-y^3)]\}.$$

Putting $-y$ for y and adding

$$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)[Z\theta_8\alpha^2 + Z^3\alpha^{10}]\}.$$

This makes the previous equation

$$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)[Z\theta_8\alpha^2 y + Z^3\alpha^{10}y^3]\}.$$

Replace y by y^2 and substitute in (8) to get

$$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)(Z\theta^2 + Z^3\alpha^4\theta)\}\ \forall\ Z \in f.$$

Adding the last 2 equations

$$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)(Z\theta^2 + Z\theta_8\alpha^2 y + Z^3\alpha^4\theta + Z^3\alpha^{10}y^3)\}.$$

Write xZ in place of Z and then write $y = Z^2\alpha^{-2}$ to get

$$0 = Tr_{f/f_0}\{\alpha^2 Tr(\lambda_0)[x(\theta^2 Z + \theta_8 Z^3) + x^3(Z^3\alpha^4\theta + Z^9\alpha^4)]\} \; \forall \, x, Z.$$

For each $Z \in f$, $x \mapsto x(\theta^2 Z + \theta_8 Z^3) + x^3(Z^3\alpha^4\theta + Z^9\alpha^4)$ is an endomorphism of f. Therefore, $\forall \; Z \in f$, $\exists$ a corresponding $x \neq 0$ for which it vanishes. Take Z such that $\theta Z^3 + Z^9 = 0$ which is possible by the validity of the 3rd compatibility condition. The corresponding x satisfies

$$x(\theta^2 Z + \theta_8 Z^3) = 0 \text{ i.e. } \theta^2 Z + \theta_8 Z^3 = 0.$$

The 2 equations $\theta Z^3 + Z^9 = 0$ and $\theta^2 Z + \theta_8 Z^3 = 0$ give $\theta_8^3 = \theta^5$ which is the 4th compatibility condition.

Before proving the 5th condition, note that any other uniformising parameter of K is π^2 times some unit u of K and so, a uniformising parameter of k is π^2 times the norm of u. But, Norm(u) runs over all units of k as K is unramified. An easy computation shows that, by changing π^2, we may assume that $\theta_6 = \theta_{12} = 0$. The 6th compatibility condition then just reduces to the 3rd one. Also, the fifth condition becomes then $\theta_{10} = 0$.

To prove this holds, we start with (7) and proceed exactly as we did with (5). We get quite easily that $Tr_{F/f}(\lambda_5) = 0$ and $Tr_{F/f}(\lambda_3) = \frac{-\theta_6}{\theta} Tr_{F/f}(\lambda_0) = 0$. Therefore, (7) reduces to

$$0 = Tr_{f/f_0}\{Tr(\lambda_0)(\theta_{10} X Y^3)\} \; \forall \; X, Y \in E$$

which easily gives $\theta_{10} = 0$, the 5th condition.

Hence, we have proved:

Theorem. *Let k be an extension of $\mathbb{Q}_3$ whose ramification index is 6. Also, assume that the residue field f of k is so large that there exist $a, b \in f$ such that $(1 + a^2)(1 + a^4) \neq (1 + b^2)(1 + b^4)$. Let D be the quaternion division algebra over k. Suppose $H^2(SL(1, D), \mathbb{R}/\mathbb{Z})$ has an element of order 9. Then k contains a primitive 9-th root of 1.*

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