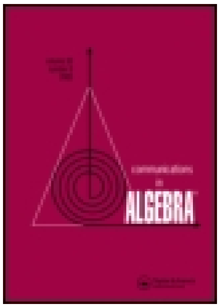

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CENTRAL EXTENSIONS OF p-ADIC GROUPS; A THEOREM OF TATE.

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§ 1 INTRODUCTION :

This note concerns itself with the study of abstract central extensions of p-adic groups by a finite group. We show, for some groups, that abstract central extensions by a finite group are automatically topological. More precisely, let A be a finite Abelian group and let G be a p-adic Lie group acting trivially on A. We denote by $H^2_{abs}(G, A)$ and $H^2_{cont}(G, A)$ respectively, the groups of abstract and topological central extensions of G by A. The first result is:

Theorem A

If G is a solvable p-adic Lie group, then

$$H^2_{abs}(G, A) \cong H^2_{cont}(G, A)$$

The method of proof is completely elementary and might even work for other p-adic groups. However, we have not worked it out in complete generality. On the other hand, if k is a nonarchimedean local field of characteristic 0, we use Theorem A to prove that :

Theorem B

$$H^2_{abs}(SL_2(k), A) \cong H^2_{cont}(SL_2(k), A)$$

Indeed, the same result holds with SL_2 replaced by a connected, simply-connected, quasisplit algebraic group over k as can be seen from the works of Moore, Matsumoto and Deodhar (see remark after 3.3). This enables us to conclude the following fact first noted by Tate ([Ta]):

Corollary C

If we write $K_2(k)_{abs} \cong K_2(k)_{cont} \bigoplus D$, then D is a divisible abelian group.

§ 2 Proof of Theorem A

In this section , we start with an elementary method of studying abstract central extensions of a p-adic group by a finite group and apply it to prove Theorem A on solvable groups .

Let $m = | A |$, the cardinality of A.

If we have a central extension

$$1 \to A \to E \xrightarrow{\pi} G \to 1, \qquad \cdots (2.1)$$

We want to give a topology on E such that E becomes a topological group and 2.1 becomes a topological central extension. Since G is a p-adic Lie group, it has a filtration by open compact subgroups ([Se], LG 4.24) : $G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ such that $\bigcap G_i = \{e\}$ and, the map

$$\phi_m : G \to G$$
$$x \mapsto x^m$$

is an isomorphism of analytic manifolds from G_i onto G_{i+n} ([loc. cit. 4.25, 4.26]) for large enough i and some fixed n.

Let $\hat{G}_i := \pi^{-1}(G_i)$ and $E_i = \phi_m(\hat{G}_i)$.
The sets E_i satisfy the following:

(i) $\forall E_i,\ \exists E_j$ such that $E_i^{-1} \subseteq E_j$.

(ii) $\forall\ x \in E$ and $\forall\ E_i,\ \exists E_j$ such that $E_i \subseteq xE_jx^{-1}$.

(iii) $\cap E_i = \{1\}$, for, $x \in \bigcap E_i \Rightarrow x = \hat{x}_i^m \Rightarrow \pi(x) = \pi(\hat{x}_i)^m \in \bigcap G_i = \{1\}$.
But, for $i >> 0$, ϕ_m is an isomorphism of G_i on G_{i+n}.
Therefore $\pi(\hat{x}_i) = 1$ for $i >> 0$
i.e. $\hat{x}_i \in A$ for $i >> 0$ i.e. $\hat{x}_i^m = 1$ for large enough i. i.e. $x = 1$.
If we also show that

(iv) $\forall E_j, \exists E_i s.t E_i.E_i \subseteq E_j$,

then $\{E_i\}$ form a system of neighbourhoods of $\{1\}$ so that E becomes a topological group and 2.1, a topological central extension.

Note that this is the only topology on E which is compatible with the possibility of 2.1 being topological (i.e. $\hat{G}_i$ should be open and φ_m should be open on $\hat{G}_i$) so that the map $H^2_{cont}(G, A) \to H^2_{abs}(G, A)$ is injective.

Until now G could have been any p-adic Lie group. Now,let us first assume that G is Abelian and prove (iv).
Consider the map $G \times G \xrightarrow{\psi} A$

$$(x, y) \mapsto \hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1}$$

where $\hat{x}, \hat{y}$ are any lifts of x, y in E.

Then $\psi(x^m, y) = 1 \ \forall x, y \in G$ where $m = |A|$. But the m-power map ϕ_m is an analytic isomorphism of G_i on G_{i+n} and so

$$\psi(G_i, G) = 1 \quad for \quad i >> 0.$$

Thus $[\hat{G}_i, \hat{G}_i] = \{1\}$. i.e. ϕ_m is a homomorphism on $\hat{G}_i$ for $i >> 0$ so that $E_i = \phi_m(G_i)$ is a subgroup and hence $E_i \cdot E_i = E_i$. Thus, the central extension 2.1 is automatically topological when G is Abelian. Assume more generally now that G is solvable. Since we need only show that (2.1) splits over an open compact subgroup of G, we can assume (by replacing G if necessary) that G is compact.

We shall apply induction on the derived length.We have already proved the theorem for abelian G.

Let $H = \overline{[G, G]}$.

By the induction hypothesis,

$$1 \to A \to \pi^{-1}(H) \xrightarrow{\pi} H \to 1.........(2.2)$$

splits over some open normal subgroup N of H. As N is of index n in H, we can replace N by the closure of the group $< \{x^n / x \epsilon H\} >$, which is a characteristic subgroup of H, and which is contained in N. This subgroup is normal in G and we shall call this N from now on.

Let $\varphi = N \to \pi^{-1}(H)$ be a splitting of 2.2 over N. Since N is open and of finite index in $H = \overline{[G, G]}$, $\exists$ an open normal subgroup G_o of G such that G_o/N is Abelian.

Thus, we have a central extension

$$1 \to A \to F_o = \pi^{-1}(G_o) \xrightarrow{\pi} G_o \to 1 \quad (2.3)$$

which has a splitting over a normal subgroup N of G_o such that G_o/N is Abelian.

Claim

$\varphi(N \bigcap G_i)$ is normalised by $\pi^{-1}(G_i)$ for some open normal subgroup G_i of G_o. To show this, consider (for any $\tilde{x} \in E_o$) the new splitting

$$\begin{array}{lcl} (\tilde{x}\varphi) & : & N \to E_o \\ & & n \mapsto \tilde{x}\varphi(x^{-1}nx)\tilde{x}^{-1} \text{ where } x = \pi(\tilde{x}). \\ \text{Clearly } (\tilde{x}\varphi)\varphi^{-1} & : & N \to A \\ & & n \mapsto (\tilde{x}\varphi)(n)\varphi(n^{-1}) \end{array}$$

is a homomorphism.

Moreover, $\theta : E_o \to Hom(N, A)$

$$\tilde{x} \mapsto (\tilde{x}\varphi)\varphi^{-1}$$

is a 1-cocycle on E_o which is trivial on A. Thus θ gives an element of $H^1_{abs}(G_o, Hom(N, A))$.

But since $Hom(N, A)$ is finite and G_o is compact, $\exists$ an open normal subgroup G_i of G_o on which θ is trivial i.e. $H^1_{abs}(G_o, Hom(N, A)) \cong H^1_{cont.}(G_o, Hom(N, A))$. This can be seen as follows. If r is the order of Aut (Hom(N,A)), then $g^r \cdot \beta = \beta \; \forall \; \beta \in Hom(N, A)$. But then by the cocycle condition for any $\theta \in H^1_{abs}(G_0, Hom(N, A))$, we get $\theta(g^{mr}) = \theta(g^r)^m = 1 \; \forall g \in G_0$. Since the i-nage of ϕ_{mr} is an open sbgroup of G_0, any element of $H^1_{abs}(G_0, Hom(N, A))$ is actually in $H^1_{cont}(G_0, Hom(N, A))$. Thus $\varphi(N \cap G_i)$ is normalised by $\pi^{-1}(G_i)$. In fact, we have shown for all $\tilde{x} \in \pi^{-1}(G_i)$, $n \in N \cap G_i$ that

$$\varphi(xnx^{-1}) = \tilde{x}\varphi(n)\tilde{x}^{-1}$$

Now consider the central extension

$$1 \to A \to \pi^{-1}(G_i)/_{\varphi(N \cap G_i)} \xrightarrow{\pi} G_i/_{N \cap G_i} \to 1 \text{..............} 2.4$$

Since $G_i/N \cap G_i$ is Abelian, therefore, $H^2_{abs.}(G_i/_{N \cap G_i}, A) \cong H^2_{cont.}(G_i/_{N \cap G_i}, A)$. Thus 2.4 splits over some $G_{deep}/N \cap G_i$ for some deep enough (i.e. small enough) open subgroup G_{deep} of G_i. Let ψ be such a splitting. We construct a splitting of 2.1 over G_{deep} as follows:

Let $\rho : \pi^{-1}(G_i) \to \pi^{-1}(G_i)/\varphi(N \bigcap G_i)$ be the natural map. Let $g \in G_{deep}$ and $\tilde{g}$ any lift of g to $\pi^{-1}(G_i)$. Then, $\rho(\tilde{g}) = \psi(\overline{g}) \cdot \lambda(\tilde{g})$ where $\lambda(\tilde{g}) \in A$. Set $\alpha(g) = (\tilde{g}) \cdot \lambda(\tilde{g})^{-1}$. As $\lambda(\tilde{g}a) = \lambda(\tilde{g})a$ for $a \in A$, $\alpha(g)$ is well-defined (independent of the choice of $\tilde{g}$) and one checks easily that α provides a splitting of 2.1 over G_{deep}. Thus, 2.1 splits over an open normal subgroup of G when G is solvable and Theorem A is proved.

REMARKS :

1. We must have A to be a finite group in the Theorem. For, even for a compact, Abelian group like $\mathbf{Z}_p$, we have that

$$H^1_{abs.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z}) \neq H^1_{cont.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z})$$

$$H^2_{abs.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z}) \neq H^2_{cont.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z})$$

(Here $\mathrm{Q}/\mathbf{Z}$ is considered with the discrete topology as in the theorem.)
Consider any prime $q \neq p$ and consider the homomorphism $\mathbf{Z} \to Q/\mathbf{Z}$; $1 \mapsto \frac{1}{q} \bmod \mathbf{Z}$. This extends to a homomorphism $\mathbf{Z}_p \to Q/\mathbf{Z}$ by injectivity of the group $\mathrm{Q}/\mathbf{Z}$. Since $Q/\mathbf{Z}$ is discrete, this homomorphism is not continuous, for otherwise, p^n would go to zero from some n onwards.
Therefore $\mathrm{Hom}_{abs.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z}) \neq \mathrm{Hom}_{cont.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z})$.
Moreover, for an Abelian group G and an injective group I, $H^2(G, I)$ can be identified with the group of bilinear maps from G to I modulo the symmetric bilinear ones.
Thus, we also have

$$H^2_{abs.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z}) \neq H^2_{cont.}(\mathbf{Z}_p, \mathrm{Q}/\mathbf{Z})$$

2. We also put down the general question we already mentioned in the introduction viz.

Q : For a p-adic Lie group G and a finite group A on which G acts, is $H^2_{abs}(G, A) \cong H^2_{cont}(G, A)$?

§ 3 Proof of Theorem B.

We start with any abstract central extension

$$1 \to A \to E \xrightarrow{\pi} SL(2,k) \to 1 \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Let $|A| = m$.

Denoting by U^+ and U^- the subgroups of upper and lower unipotents in $SL(2,k)$, we have group-theoretic sections over open compact subgroups of U^+ and U^-.Indeed, if

$$x(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in U^+$$

we can define $s : U^+ \to E$ by $s(x(a)) = \hat{x}^m$, where $\pi(\hat{x}) = x$ is the element of U^+ satisfying $x^m = x(a)$.We use, for convenience, the notation $s(x(a)) = X(a)$ and $s(y(b)) = Y(b)$ where

$$y(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

We extend this to a set-theoretic section s over the whole of $SL(2,k)$ as follows : We have the standard notations

$$w(t) = x(t)y(-t^{-1})x(t)$$

and $h(t) = w(t).w(-1)$.

We define $W(t) := s(w(t))$ to be $X(t)Y(-t^{-1})X(t)$ and $H(t) := s(h(t))$ to be $W(t).W(-1)$.

Lemma 3.1

(i) $W(t) := X(t)Y(-t^{-1})X(t) = Y(-t^{-1})X(t)Y(-t^{-1})$

(ii) $W(t)^{-1} = W(-t)$

(iii) $W(t)X(a)W(t)^{-1} = Y(-at^{-2})$

(iv) $W(t)Y(b)W(t)^{-1} = X(-bt^2)$

(v) $W(t)W(u)W(t)^{-1} = W(u^{-1}t^2)$.

Proof

(ii) follows from the definition of W.

For (iii), we write $X(a) = \hat{x}^m$ where $\pi(\hat{x})^m = x(a)$. Then,

$$\pi(W(t)\,\hat{x}\,W(t)^{-1})^m = w(t)x(a)w(t)^{-1} = y(-at^{-2})$$

gives (iii) immediately .

(iv) follows from (iii), and (v) from (iii) and (iv).

(i) follows from (iv) since $Y(-t^{-1})X(t)Y(-t^{-1}) = Y(-t^{-1})W(t)X(t)^{-1} = W(t)W(t)^{-1}Y(-t^{-1})W(t)X(t)^{-1} = W(t)$ on applying (iii) with $a = t$.

As usual, elements of $SL(2,k)$ are divided into two kinds viz. $g_1(u,t) = x(u)h(t)$ and $g_2(u,t,v) = x(u)w(t)x(v)$.We define $s(g_1(u,t)) = X(u)H(t)$ and $s(g_2(u,t,v)) = X(u)W(t)X(v)$.

Then, a cocycle representing (*) is given by

$$B : SL(2,k) \times SL(2,k) \to A$$

$$B(g,h) = s(g)s(h)s(gh)^{-1}$$

It is easy to see using Lemma 3.1 that B is determined by its values on $T \times T$ where $T = \{h(t) : t \in k^*\}$.In fact,

(a)$B(g_1(u,t), g_1(u_1,t_1)) = B(t,t_1)$

(b)$B(g_1(u,t), g_2(u_1,t_1,v_1)) = B(t,t_1)$

(c)$B(g_2(u,t,v), g_1(u_1,t_1)) = B(t,t_1^{-1})$

(d)$B(g_2(u,t,v), g_2(-v,t_1,v_1)) = B(-t,-t_1^{-1})$

(e)$B(g_2(u,t,v), g_2(u_1,t_1,v_1)) = B(tz^{-1},z^{-1})^{-1}.B(tz^{-1},t_1)$ if $z = v+u_1 \neq 0$.

The following result essentially says that B is a Steinberg cocycle (See [Mo],Ch.8 for definition) with values in A.

Lemma 3.2

$(i) B(st, r)B(s, t) = B(s, tr)B(t, r);$

$B(1, s) = B(s, 1) = 1$

$(ii) B(s, t) = B(t^{-1}, s)$

$(iii) B(s, t) = B(s, -st)$

$(iv) B(s, t) = B(s, (1 - s)t)$

Proof : Easily follows from Lemma 3.1.

If we show , in addition , that B is continuous, then it would follow from Moore's work ([Mo],Ch.10) that (*) is a topological central extension.

Proposition 3.3

For finite A, any B satisfying the properties of Lemma 3.2 (i.e. any Steinberg cocycle) is automatically continuous.

Proof

In fact,we will show that $B(s^2, t) = [\hat{s}, \hat{t}]$ which is bilinear. (Here $[\hat{s}, \hat{t}]$ is the commutator of any two lifts of s and t.)

To see this, note that $B(t^2, s)B(t^{-1}, t^2 s) = B(t, s)B(t^{-1}, t^2)$ by (i) $= B(t, s)$, since $B(t^{-1}, t^2) = B(t^{-1}, -t) = B(t^{-1}, 1) = 1$ by (ii) and (i). Also $B(t^{-1}, t^2 s) = B(t^{-1}, -ts) = B(t^{-1}, s) = B(s, t)$ by (iii) and (ii).

Thus $B(t^2, s) = B(t, s)B(s, t)^{-1} = [\hat{t}, \hat{s}]$.

Thus the function $\beta(s, t) := B(s^2, t)$ is bilinear and is, consequently , continuous at 1. Therefore $\beta(s^m, t) = 1$ if $m = | A |$.

But $x \mapsto x^2$ is open map of k^*. So $B(U, k^*) \equiv 1$ for a neighbourhood U of 1. Now, if $u \in U$, then $B(su, t).B(u, s) = B(u, st).B(s, t)$ ie. $B(su, t) = B(s, t) \forall s \in k^*$ whence B is continuous everywhere in k^*.The proof of the Proposition and of Theorem B is complete.

Remarks

From the results of Moore, Matsumoto, Deodhar and Deligne (See for e.g. [De]), we know that if G is simply-connected,absolutely simple,quasi-split $/k$,then $H^2_{abs.}(G(k), A) \to H^2_{abs.}(H(k), A)$ is injective, where H is a k-subgroup of G, which is k-isomorphic to SL_2, generated by the root subgroups $U_\alpha, U_{-\alpha}$ where α is a long root (in a root system of G relative to some maximal k-split torus) and also that (See [GP-MSR]; Th.5.11) $H^2_{cont.}(G(k), A) \to H^2_{cont.}(H(k), A)$ is an isomorphism. Thus, from Theorem B ,it directly follows that $H^2_{abs}(G(k), A) \cong H^2_{cont}(G(k), A)$ for quasisplit G.Similarly, from the work of M.Stein ([St]), for an open compact subgroup U in a Chevalley group of rank ≥ 4 , we can conclude that abstract and topological central extensions by a finite group coincide.

Proof of Corollary C

We recall from [Mi] that the group $K_2(k)$ is the center of $St(k) = lim_{\to} St(n, k)$, where the Steinberg group $St(n, k)$ is the abstract universal central extension of the group $SL(n, k)$. In other words $K_2(k) = H_2(SL(k))$. Since $K_2(k)_{cont.} \cong \mu(k)$ (Moore, Matsumoto), we have an exact sequence

$$0 \to C \to K_2(k) \to \mu(k) \to 0$$

Theorem B and the remarks above applied to $SL_n(k)$ for $n >> 0$ gives that any homomorphism from $K_2(k)$ to a finite group factors through $\mu(k)$. Thus every subgroup of $K_2(k)$ of finite index contains C. So C does not contain any proper subgroup of finite index as it is of finite index in $K_2(k)$ (Note that $\mu(k)$ is finite). Therefore C is an infinite Abelian group which does not contain proper subgroups of finite index.

For any prime p, consider pC. If $pC \neq C$, then consider C/pC(which has to be infinite then). This is a $\mathbf{Z}/p\mathbf{Z}-$ vector space and, as such, if non-zero, admits a non-trivial projection to $\mathbf{Z}/p\mathbf{Z}$ and hence we have a surjective homomorphism of C in $\mathbf{Z}/p\mathbf{Z}$. The kernel of this homomorphism is a proper subgroup of finite index in C, which contradicts the fact that C does not have any proper subgroup of finite index. Hence $pC = C$ for any prime p and therefore C is divisible. Thus the above exact sequence splits and,consequently, $K_2(K) \cong \mu(k) \oplus D$ where D is a divisible group.

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