What is the Burnside Problem ?

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The young William Burnside became an orphan – something that is neither easy nor fun. Influence of his work is far-reaching on any group-theorist's research or teaching. The 'Burnside problem' was a beacon illuminating a future path as only he can.

In the November 1996 issue of *Resonance*, the second author had mentioned the Burnside problem [1] in passing and said that it was beyond the scope of that article. "The time has come now," as the chivalrous professor said, "to talk of such things". If G is a *torsion group* – that is, a group in which all elements are of finite order, must the group be finite? Not so, as the simple example of the group of all roots of unity on the unit circle shows. This is easily seen not to be a finitely generated group. One may wonder whether the assumption of finite generation forces a torsion group to be finite. In general, the answer is in the negative. What if the finitely generated group is n*-torsion* for some n (that is, the orders of all the elements are divisors of a fixed number n). In 1902 Burnside raised the question of its finiteness – it turns out that the group could still be infinite. That requires rather deep methods.

Let us start with some simple cases where the answer is actually in the affirmative. First of all, if a group is 2-torsion, it is trivial to see that it is abelian. Thus, if a 2-torsion group is also finitely generated, it must be finite. As we shall show, finitely generated n-torsion groups are finite for $n = 3$ as well as for $n = 4$. These results are due, respectively, to Burnside himself and Sanov. An affirmative answer was given for the case $n = 6$ also by M Hall. The case $n = 5$ is still open as far as the authors know.

Keywords
Torsion, Burnside problem, Burnside lemma, unipotent matrix, nilpotent group, Gupta–Sidki example.

One can formulate the question for the 'universal' m-generated, n-torsion group. In other words, one can formulate the Burnside problem as the question as to whether the Burnside group $B(m, n)$ – defined as the quotient of the free group F_m on m generators by the normal subgroup generated by the relations $x^n = 1$ – is finite. For a big enough odd n, Adian and Novikov proved that there exist finitely generated, infinite n-torsion groups; their bound of 4381 has since been brought down. The present-day method of attack involves 'van Kampen diagram techniques' due to Olshanskii and others. Rather recently, Ivanov has shown the existence of finitely generated, infinite, n-torsion groups for large even n. He also uses the diagram techniques referred to above.

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There is the *restricted Burnside problem* which asks whether there is a bound on the orders of all m-generated n-torsion *finite* groups. This was finally proved in the affirmative by Efim Zelmanov who won the Fields Medal for this work in 1990. The restricted Burnside problem can be reformulated in terms of profinite groups (the latter are built out of finite groups quite akin to building the Galois group of the algebraic closure of Q from the various finite Galois groups). The advantage of that is that one could use topological techniques as a profinite group has a natural topology.

We start with a discussion of the linear case of the Burnside problem where the answer is in the affirmative and go on to prove the positive cases of $n = 3$ and $n = 4$. At the end, we recall a beautiful construction of Narain Gupta and Said Sidki [2] which gives a counterexample to the general Burnside problem. The construction shows the existence of an infinite group which is finitely generated and all of whose elements have finite p-power order for some prime p. The orders of the elements are unbounded, and thus, this is not a counterexample to the Burnside problem where the torsion is bounded.

Burnside's contributions to group theory and especially to the study of their representations are fundamental to the subject. Ironically, the most popular result through which he is often known to the lay-mathematician (an

Burnside's contributions to group theory and especially to the study of their representations are fundamental to the subject.

internet search demonstrates this), is an elementary counting lemma erroneously known as Burnside's lemma which is not due to him !

1. Burnside Problem for Matrix Groups

We first start with the case of matrix groups where the general Burnside problem has a positive solution. This is a consequence of a lemma known (rightly) as the Burnside lemma. As usual, for any field K, $M(n, K)$ denotes the set of all $n \times n$ matrices over K and $GL(n, K)$ denotes the group of invertible ones among them. We first recall that a matrix $g \in GL(n, K)$ is said to be *unipotent* if all its eigenvalues are 1. Equivalently, over the algebraic closure of K, g is conjugate to an upper triangular matrix with all diagonal entries 1.

Lemma 1.1 (Burnside) *Let K be any field and let $G \subset GL(n, K)$ be a subgroup such that the set $\{tr(g) : g \in G\}$ is finite, of cardinality r, say. Assume also that no nontrivial element of G is unipotent. Then, G must be finite, of cardinality $\leq r^{n^2}$.*

Corollary 1.2 *Let K be any field and let N be a natural number which is not a multiple of the characteristic of K. If $G \subset GL(n, K)$ is an N-torsion group, then G must be finite, of cardinality $\leq N^{n^3}$.*

Proof. Without loss of generality, we may assume that K is algebraically closed, for, replacing K by its algebraic closure neither affects the hypotheses nor the conclusions. We shall observe that the hypotheses of the lemma are satisfied. The vector subspace $V \subset M(n, K)$ generated by G has dimension at the most n^2; let $g_1, g_2, \cdots, g_d$ be elements of G which give a basis of V. For each $g \in G$, since $g^N = I$, the eigenvalues of g are N-th roots of unity; thus the trace of g has at the most N^n possibilities. Suppose, if possible, that $I \neq g \in G$ is unipotent. After conjugating by a matrix in $GL(n, K)$, we may assume that g is upper triangular with all diagonal entries 1. Let $g_{ij} \neq 0$ with $j - i \geq 1$ and $j - i$ the least possible. Now looking at the (i, j)-th entry of $I = g^N$, one has

$$0 = N g_{ij}.$$

This is a contradiction since characteristic of K does not divide N. Thus, the lemma implies the assertion of this corollary. ■

Proof of Burnside's Lemma

As before, we let $\{g_1, \cdots, g_d\}$ be elements in G which form a basis for the vector subspace of $M(n, K)$ spanned by the elements of G. To be able to 'count' the elements of G, we associate to each $g \in G$, the ordered d-tuple

$$(tr(g_1 g), tr(g_2 g), \cdots, tr(g_d g)).$$

If the same tuple were associated to elements $x, y \in G$, then we would have $tr(g_i(x - y)) = 0$ for all $i \leq d$. Now, for any $k \geq 0$, $(I - x^{-1}y)^k x^{-1} = \sum_{i=1}^{d} \lambda_i g_i$ for some $\lambda \in C$. Therefore, multiplying the i-th equation $tr(g_i(x - y)) = 0$ by λ_i and adding all of them, we get $tr((I - x^{-1}y)^{k+1}) = 0$. Since this holds for all $k \geq 0$, we must have $I - x^{-1}y$ to be a nilpotent matrix h; that is, all eigenvalues of h are 0. Hence $x^{-1}y$ is $I - h$, which is clearly unipotent. This means $x = y$. Hence the association

$$g \mapsto (tr(g_1 g), tr(g_2 g), \cdots, tr(g_d g))$$

is one-to-one. As the traces of elements of G take at the most r values, the set of d-tuples above has cardinality at the most $r^d \leq r^{n^2}$. This completes the proof.

One can refine the corollary by dropping the condition of bounded torsion when the group is finitely generated. One has :

Proposition 1.3 *Let $G \subset GL(n, K)$ be a finitely generated torsion group such that the orders of all elements are not multiples of Char K. Then, G is finite.*

A finitely generated group G all of whose nontrivial elements are of order 3 must be finite.

Proof. We only have to show that the torsion is automatically bounded. Choosing a finite set of generators of G, one can consider the smallest subfield E of K containing all their matrix entries. Obviously, $G \subset GL(n, E)$. Now, the field E_{alg} of elements in E which are algebraic over the prime subfield P of K, form a finite extension field of P. Note that since elements of G have finite order, their eigenvalues are in E_{alg}. As E can have only finitely many roots of unity, there is a bound on the order of elements of G. Then, the proposition follows from the corollary. ■

2. Burnside Problem for 3-Torsion Groups

This is the first nontrivial case where the Burnside problem has an affirmative solution in general (that is, without assumptions of linearity). In other words, a finitely generated group G all of whose nontrivial elements are of order 3 must be finite. A posteriori, such a group is a 3-group and is hence, nilpotent. As one can guess, the argument is special to the exponent 3 and proceeds by proving first that the group must be nilpotent; this involves some play (hopefully enjoyable) with commutators. Recall that the *descending central series* of a group G is the sequence of subgroups

$$G = C_0(G) \supset C_1(G) \supset C_2(G) \cdots\cdots\cdots$$

where $C_i(G) = [G, C_{i-1}(G)]$. Here, a notation of the form $[A, B]$ stands for the group generated by all the 'commutators' $aba^{-1}b^{-1}$ with $a \in A, b \in B$. A group G is *nilpotent* if the descending central series terminates in finitely many steps, i.e., if $C_l(G) = \{1\}$ for some l. In any group, one denotes the commutator $xyx^{-1}y^{-1}$ by $[x, y]$ and the conjugate xyx^{-1} by y^x. The higher order commutators $[x_1, x_2, \cdots, x_n]$ are defined recursively by

$$[x_1, x_2, \cdots, x_n] = [[x_1, x_2, \cdots, x_{n-1}], x_n].$$

We now prove a result which is more general than an affirmative solution of the Burnside problem for finitely generated, 3-torsion groups.

A subgroup of index *m* of an *r*-generated group, can be generated by *m*(r–1)+1 elements.

Theorem 2.1 *Any* 3*-torsion group* G *must be nilpotent; in fact,* $C_3(G) = \{1\}$ *and* $C_1(G)$ *is abelian. Consequently, a finitely generated* 3*-torsion group must be finite.*

Proof. We first show that the last assertion of the theorem is a consequence of the first one. For such a group G, the group $G/C_1(G)$ must be finite, as it is a finitely generated abelian group which is torsion. Thus, $C_1(G)$ is finitely generated as well, as it is of finite index in the finitely generated group G. This last fact is a nice exercise which is elementary but not obvious; in fact, the reader is urged to show that a subgroup of index m of an r-generated group, can be generated by $m(r-1)+1$ elements. Now, we also have $C_1(G)$ to be abelian from the first assertion. Being torsion, it has to be finite; thus G itself must be finite.

Therefore, we need to prove only the first assertion.

We break the rest of the proof into 2 or 3 steps each of which is interesting in its own right. First, we start with a remarkable fact which holds in any group. This is a commutator identity which can be verified by first principles quite easily. ■

Observation. *In any group* G,

$$[x^y, x] = ((xy^{-1})^y)^3 (y^2x^{-1})^3 ((y^{-1})^x)^3.$$

As an immediate consequence, we note :

Lemma 2.2 *In any* 3*-torsion group, every element commutes with all its conjugates.*

The next result tells us of an interesting property of all groups which have the property inferred in the previous lemma:

Lemma 2.3 *Let G be any group in which each element commutes with each of its conjugates. Then, for all $x, y, z \in G$, one has*

$$[x, y, z] = [z, x, y] = [y, z, x].$$

Proof. For convenience, for any $x \in G$, let us write $C(x)$ for the set consisting of all products of conjugates of x and x^{-1}. Note that any two elements of $C(x)$ commute. Moreover, $C(x)$ is stable under conjugation by any element. Finally, observe that for any $x_1, \cdots, x_n \in G$, we have $[x_1, \cdots, x_n] \in C(x_i)$ for all $i \leq n$. In particular, for any $g \in C(x_i)$ for some i, we have

$$[x_1, \cdots, x_n]^g = [x_1, \cdots, x_n].$$

We shall now consider the elements of the form $[x, y, z]$; these evidently generate the subgroup $C_2(G)$ in any group G. The idea is to prove that in our case these elements are contained in the center; that is, we shall show that

$$[x, y, z, w] = 1 \ \ \forall \ \ x, y, z, w.$$

First, we can verify that

$$[x, yz] = [x, y][x, z]^y; [xy, z] = [y, z]^x[x, z]$$

(in any group). Therefore

$$[x, yz, z] = [[x, y][x, z]^y, z] = [[x, z]^y, z]^{[x,y]}[x, y, z].$$

Since $[x, z]$ (and therefore $[x, z]^y$) are in $C(z)$, the above expression becomes

$$[x, yz, z] = [x, y, z]. \tag{1}$$

Similarly,

$$[x, yz, y] = [[x, y][x, z]^y, y] = [[x, z]^y, y]^{[x,y]}[x, y, y] =$$
$$[[x, z]^y, y]^{[x,y]}.$$

Since $[x, y] \in C(y)$, as we observed in the beginning of the proof,

$$[x, yz, y] = [[x, z]^y, y] = [[x, z], y]^y = [[x, z], y,],$$

that is,

$$[x, yz, y] = [x, z, y]. \qquad (2)$$

Now, for each x, y, z,

$$1 = [x, yz, yz] = [x, yz, y][x, yz, z]^y = [x, z, y][x, y, z]^y$$

by (1) and (2). But, notice that $[x, y, z] \in C(y)$. Hence

$$1 = [x, z, y][x, y, z]. \qquad (3)$$

Now, we note that in our group G,

$$[x^{-1}, y] = x^{-1}.yxy^{-1} = yxy^{-1}.x^{-1} = [y, x] = [x, y]^{-1}$$

and

$$[x, y^{-1}] = xy^{-1}x^{-1}.y = y.xy^{-1}x^{-1} = [y, x] = [x, y]^{-1}.$$

Thus, $[x, z, y] = [[z, x]^{-1}, y] = [[z, x], y]^{-1} = [z, x, y]^{-1}$ by this observation.

From this and (3), we have

$$[x, y, z] = [z, x, y].$$

Changing the roles of x, y, z, we have

$$[x, y, z] = [z, x, y] = [y, z, x]. \qquad (4)$$

This proves the lemma. ■

Finally, we can complete the proof of the theorem. By the first lemma, our group satisfies the hypothesis of the pevious lemma which we shall use repeatedly.

$$\begin{aligned}
[x, y, z, w] &= [[x, y], z, w] = [z, w, [x, y]] \\
&= [[z, w], [x, y]] = [[w, z]^{-1}, [x, y]] \\
&= [[w, z], [x, y]]^{-1} = [w, z, [x, y]]^{-1} \\
&= [[x, y], w, z]^{-1} = [x, y, w, z]^{-1}.
\end{aligned}$$

Let us note the final identity

$$[x, y, z, w] = [x, y, w, z]^{-1}. \quad (5)$$

Therefore, using (5) and repeatedly using (4) as well as the basic identity $[u, v] = [v, u]^{-1}$, we have

$$\begin{aligned} & [x, y, z, w] \\ = & [x, y, w, z]^{-1} = [[x, y, w], z]^{-1} = [[w, x, y], z]^{-1} \\ = & [w, x, y, z]^{-1} = [[w, x], y, z]^{-1} = [y, z, [w, x]]^{-1} \\ = & [[y, z], [w, x]]^{-1} = [[w, x], [y, z]] = [w, x, [y, z]] \\ = & [[y, z], w, x] = [y, z, w, x]. \end{aligned}$$

Hence, using (4) and (5) for the right hand side, we have

$$[y, z, w, x] = [y, z, x, w]^{-1} = [x, y, z, w]^{-1}.$$

Hence

$$[x, y, z, w] = [x, y, z, w]^{-1} = [x, y, z, w]^2$$

which gives

$$[x, y, z, w] = 1 \ \ \forall \ \ x, y, z, w \in G.$$

Therefore, $C_2(G)$ is contained in the center, and $C_3(G) = \{1\}$ i.e., G is nilpotent. Note also that $[[x, y], [z, w]] = 1$ which means that $C_1(G) = [G, G]$ is abelian. This proves the theorem. ■

3. Burnside Problem for 4-Torsion Groups

In this section, we prove that finitely generated 4-torsion groups are finite; the proof is actually simpler than the case of 3-torsion. However, the stronger assertion in the case of 3-torsion about $C_3(G)$ etc. does not generalize.

Theorem 3.1 *If L is a finite subgroup of a 4-torsion group M such that $M =< L, x >$ for some $x^2 \in L$, then M is finite, of cardinality $\leq |L|^{|L|+1}$. Consequently,* *a finitely generated* 4*-torsion group must be a finite* 2*-group.*

A finitely generated 4-torsion group must be a finite 2-group.

Proof. We first deduce the latter assertion from the former. We proceed by induction on the minimal number d of generators needed to generate a 4-torsion group G. Trivially, if $d = 1$, then $|G| \leq 4$. Suppose $d > 1$ generators $g_1, \cdots, g_d$ generate a 4-torsion group G and that the $(d-1)$-generated subgroup $H =< g_1, \cdots, g_{d-1} >$ is finite. Consider $K :=< H, g_d^2 >$; then $G =< K, g_d >$.

From the first assertion of the theorem, one can conclude that K is finite and, therefore, G is finite.

Let us now prove the first assertion.

As $x^2 \in L$, any element of M can be written as

$$g = l_1 x l_2 x l_3 \cdots l_{n-1} x l_n$$

with $l_i \in L$ nontrivial for $1 < i < n$.

We shall show that if n is minimal for such an expression, then $n \leq |L| + 1$.

The idea is to get many expressions of length n for g and deduce that if $|L|$ is small compared to n, two such expressions coincide and give rise to a cancellation within the expression and that this would yield for g an expression of smaller length.

For any $l \in L$, we have

$$1 = (xl)^4 = xlxlxlxl,$$

which implies

$$xlx = l^{-1}x^{-1}l^{-1}x^{-1}l^{-1} = l^{-1}x^3l^{-1}x^3l^{-1} = l^{-1}xl'xl^{-1},$$

where $l' = x^2l^{-1}x^2 \in L$.

Thus, we see that in the expression

$$g = l_1 x l_2 x l_3 \cdots l_{n-1} x l_n,$$

we may replace any xl_ix by $l_i^{-1}xl'xl_i^{-1}$ which leads to the expression

$$g = l_1xl_2 \cdots xl_{i-1}l_i^{-1}xl'xl_i^{-1}l_{i+1} \cdots xl_n.$$

In other words, l_{i-1} has got replaced by $l_{i-1}l_i^{-1}$ but l_j for any $j < i-1$ did not change and the length of the new expression remains n. We use this in the following manner.

Start with any $g \in G$ and an expression

$$g = l_1xl_2xl_3 \cdots l_{n-1}xl_n$$

with n minimal. If we rewrite xl_3x as above (which entails changing l_2 to $l_2l_3^{-1}$), then we have an expression of the form

$$g = l_1xl_2l_3^{-1}xl_3'xl_4' \cdots xl_n'$$

of the same length.

Starting with the original expression

$$g = l_1xl_2xl_3 \cdots l_{n-1}xl_n$$

and rewriting xl_4x changes l_3 to $l_3l_4^{-1}$ but does not change l_2; that is,

$$g = l_1xl_2xl_3l_4^{-1}xl_4'' \cdots xl_n''.$$

If we rewrite $xl_3l_4^{-1}x$ in the above expression, we would have an expression where l_2 changes to $l_2(l_3l_4^{-1})^{-1} = l_2l_4l_3^{-1}$.

In the same way, if we start with the original expression, rewrite xl_5x, then l_4 would change to $l_4l_5^{-1}$ and l_2, l_3 would be the same. Then, rewriting $xl_4l_5^{-1}x$, we would have an expression where l_3 changes to $l_3(l_4l_5^{-1})^{-1} = l_3l_5l_4^{-1}$ and l_2 remains as it is. Rewriting $xl_3l_5l_4^{-1}x$, l_2 changes to $l_2(l_3l_5l_4^{-1})^{-1} = l_2l_4l_5^{-1}l_3^{-1}$; that is, we have an expression of the form

$$g = l_1xl_2l_4l_5^{-1}l_3^{-1}x\tilde{l_3}x \cdots x\tilde{l_n}.$$

Continuing in this manner, we would have $n-2$ expressions for g of length n, viz.,

$$g = l_1 x k_2 x k_3 x \cdots x k_n,$$

where $k_i \in L$ and k_2 could be any one of the elements

$$l_2, l_2 l_3^{-1}, l_2 l_4 l_3^{-1}, l_2 l_4 l_5^{-1} l_3^{-1} \cdots$$

The last element in the list above is either

$$l_2 l_4 \cdots l_n l_{n-1}^{-1} l_{n-3}^{-1} \cdots l_3^{-1}$$

or

$$l_2 l_4 \cdots l_{n-1} l_n^{-1} l_{n-2}^{-1} \cdots l_3^{-1}$$

according as to whether n is even or odd. These are $n-2$ possible elements; we claim that these must be distinct. If not, then we may cancel off common terms from both sides and conclude that an expression of the form

$$l_{2d+2} \cdots l_{2r} l_{2dr\pm1}^{-1} \cdots l_{2d+3}^{-1}$$

is the identity element, for some $r > d$.

But, since it is possible to start with the original expression for g and get another expression of the same length n where l_{2d+2} is changed to the above expression representing the trivial element, it means that this part can be cancelled off and we can get an expression of smaller length. This contradiction shows that the above elements must be distinct; that is, $n-1 \le |L|$. Therefore, $|M| \le |L|^{|L|+1}$. ■

4. General Burnside Problem; An Example

In this section, we give a counterexample which shows that a finitely generated group, all of whose elements have finite p-power order (for a fixed prime p), can be infinite. This beautiful construction is due to Narain Gupta and Said Sidki [2]. It should be noted that the orders of elements in this example are unbounded. As of

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now, known counterexample constructions to the bounded torsion version of the Burnside problem involves complicated van Kampen diagram techniques due to Olshanskii and others. We do not discuss them here.

Let p be a fixed odd prime and let X be the set of all finite strings of symbols from the set of alphabets $\{0, 1, \cdots, p-1\}$. Here the empty string is of length 0. For $r \geq 0$, we write 0^r to denote the string of length r consisting of r zeros. Whenever we add or subtract two symbols from the alphabet set, it should be read modulo p. Define two permutations t and z on X as follows. They fix the empty string and on nonempty strings, their actions are :

(i) t changes the first symbol i to $(i+1)$ and leaves the rest of the string unchanged.

(ii) For a string of the form $0^r ijw$ with $i \neq 0$ and $r \geq 0$,

$$(0^r ijw)^z = 0^r i(j+i)w.$$

Thus, z only changes the symbol j which follows the first nonzero symbol i (if any) to $j+i$.

Let G be the group of permutations of X generated by t and z. Note that both t and z leave the lengths of strings invariant. So all orbits of G are finite.

Theorem 4.1 G is an infinite group and all its elements have finite, p-power order.

Proof. Note that each of z and t is of order p. Set $S = \{s_h = t^{-h} z t^h : 0 \leq h < p\} \subset G$ and let H be the subgroup of G generated by S. Then each element of S has order p and H is a normal subgroup of G containing z. A key observation we shall shortly make is that H acts on the subset of X consisting of strings starting with 0, exactly as G does on the whole of X, and this is the fact that would imply that G is infinite.

For $0 \leq k < p$, the subsets $X_k = \{kw : w \in X\}$ of strings in X starting with k, together with the subset of X consisting of the empty string, form a partition of X. A simple calculation shows that for $kw \in X_k$:

$$(kw)^{s_h} = \begin{cases} k(w)^z & \text{if } k = h \\ k(w)^{t^{k-h}} & \text{if } k \neq h \end{cases}. \tag{6}$$

The notation $k(w)^z$ above means the string starting with k followed by the string defined by the action of z on w.

The above observation shows that X_k are H-invariant. In particular, $t \notin H$ and so H is a proper subgroup of G. Since $G = \langle H, t\rangle$, G/H has order p. Now, (6) implies that the restriction of H to X_0 contains the permutations $0w \mapsto 0(w)^z$ and $0w \mapsto 0(w)^t$, and so contains a copy of G. Since H is a proper subgroup of G, this is possible only if G is infinite.

Next we prove that each element of G has p-power order. Using the identities $z^i t^j = t^j s_j^i$, each $x \in G$ can be written in the form

$$x = t^a s_{i_1} \cdots s_{i_m}, \tag{7}$$

where $0 \leq a < p$. Here, the notation s_j^i stands for the i-th power of the permutation s_j. We choose an expression for x in the form (7) with smallest m. We use induction on m to prove that x has p-power order. This is clear if $m = 0$, since t has order p. Assume that $m > 0$ and that the result is true for all elements x of the form (7) with a product of fewer than m of the s_i.

Case (I): Suppose that $a = 0$. If the subscripts i_h in (7) are all equal, say i, then $x = s_i^m$ and so order of x divides p. So either x is the identity element or it has order p. Now, assume that i_h are not all equal. Since $x \in H$, each X_i is x-invariant. By (6), for each string $kw \in X_k$ we have $(kw)^x = k(w)^u$, where u has the form $t^b s_{j_1} \cdots s_{j_n}$ and $n = |\{i_h : i_h = k\}|$. Thus for each k, by induction, x acts on X_k as a permutation whose order is a power of p. So as a permutation of the whole of X, the order of x is a power of p.

Case (II): Suppose that $0 < a < p$. Set $y = s_{i_1} \cdots s_{i_m}$. Then

$$x^p = (t^a y)^p = (t^a y t^{-a})(t^{2a} y t^{-2a}) \cdots (y). \qquad (8)$$

Now, $t^r y t^{-r} = (t^r s_{i_1} t^{-r})(t^r s_{i_2} t^{-r}) \cdots (t^r s_{i_m} t^{-r}) = s_{l_1} \cdots s_{l_m}$. So x^p can be written as a product of pm terms of the s_i $(0 \leq i < p)$. Since p does not divide a, the exponents $a, 2a, \cdots, 0$ in the expression (8) for x^p correspond to a full set of residue classes modulo p. So each s_i appears as a factor in x^p exactly m times. Applying (6) again, for each k we have $(kw)^{x^p} = k(w)^v$, where v (depending on k) is a product of pm factors consisting of either z or powers of t. Also, z occurs as a factor exactly m times and the total power to which t occurs is $b = m(1+2+\cdots+(p-1)) = m(p-1)p/2$. By using identities of the form $s_i t^r = t^r s_{i+r}$, v can be rewritten in the form $v = t^b s_{j_1} s_{j_2} \cdots s_{j_m}$. Since p is odd, p divides b and so $t^b = 1$. Now the argument in the second step of Case (I) can be applied to conclude that v acts on X_k as a permutation whose order is a power of p. Since this is true for each k, it follows that x^p has p-power order; so x has p-power order as well. This completes the proof. ■

Remark 4.2 *The construction above does not work – as it is – for $p = 2$. A corresponding theorem for $p = 2$ can be obtained with a small change. Take X to be the set of all finite strings over $\{0, 1, 2, 3\}$, define t as above and modify the definition of z as follows. For any string of the form $0^r ijw$ with $i \neq 0$:*

$$(0^r ijw)^z = 0^r i(j+i)w \quad \textit{if} \quad i = 1 \;\; \textit{or} \;\; 3, \quad \textit{and}$$

$$(0^r 2jw)^z = 0^r 2jw.$$

Suggested Reading

[1] B Sury, Combinatorial group theory, *Resonance*, Vol.1, No.11, pp.42-50, 1996.

[2] N D Gupta and S Sidki, On the Burnside problem for periodic groups, *Math. Z.*, Vol.182, pp.385-388, 1983.

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