A parent of Binet's formula?

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The famous Binet formula for the Fibonacci sequence $F_1 = 1 = F_2$, $F_{n+2} = F_n + F_{n+1}$ is the identity

$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}}$$

where ϕ is the golden ratio $(1+\sqrt{5})/2$.

As we all know, many identities—even quite complicated ones—once written down, can be verified by anybody who can perform elementary algebraic manipulations. However, discovering it may not be easy at all. Binet's formula too can be verified easily. As for arriving at it, one method is to look for exponential solutions to the difference equation that defines the Fibonacci numbers. Here is another way to arrive at Binet's formula by producing a polynomial identity that perhaps could be regarded as a parent of Binet's formula.

Note first that the golden ratio ϕ satisfies the identities

$$\phi - \frac{1}{\phi} = 1 \ , \ \phi + \frac{1}{\phi} = \sqrt{5}.$$

Let us look at the polynomial

$$F_n(X,Y) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} X^i Y^i (Y+X)^{n-2i}.$$

It is an easy exercise in induction on n to show that

$$F_n(X,Y) = X^n + X^{n-1}Y + \ldots + XY^{n-1} + Y^n.$$

Indeed, multiplying the identity for $n = k$ by $X + Y$ and subtracting from it the product of the identity for $n = k - 1$ by XY, one obtains the identity for $n = k + 1$.

Therefore, on the one hand, we have

$$F_n(\phi, \frac{-1}{\phi}) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} \phi^i (\frac{-1}{\phi})^i = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

On the other hand, from the identity $X^{n+1} - Y^{n+1} = (X-Y)\sum_{i=0}^{n} X^i Y^{n-i}$, we obtain

$$F_n(\phi, \frac{-1}{\phi}) = \sum_{i=0}^{n} \phi^i (\frac{-1}{\phi})^{n-i} = \frac{\phi^{n+1} - (-1/\phi)^{n+1}}{\phi + 1/\phi} = \frac{\phi^{n+1} - (-1/\phi)^{n+1}}{\sqrt{5}}.$$

Since $\sum_{i\geq 0}\binom{n-i}{i}$ satisfies the same recursion as the Fibonacci sequence and starts with F_2, F_3, it follows by induction that

$$\sum_{i\geq 0}\binom{n-i}{i} = F_{n+1} \text{ and one obtains } F_{n+1} = \frac{\phi^{n+1}-(-1/\phi)^{n+1}}{\phi+1/\phi},$$

which is Binet's formula.

This note was submitted in the beginning of 2001 and when it was accepted in October 2003, attention was drawn to two very enjoyable articles [**1, 2**] that appeared in the June 2003 issue. The authors studied a general Fibonacci-type of two-term linear recurrence:

$$g_{n+1} = ag_n + bg_{n-1},$$

where a, b are any (even complex!) constants. If we start with $g_0 = 1 = g_1$, then the analog of the formula

$$f_{n+1} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i} \text{ is } g_{n+1} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}b^i a^{n-2i},$$

as can be proved by induction. The corresponding Binet identity can be derived from the same polynomial identity above as follows. Consider the numbers defined by

$$\lambda+\mu = a\ ,\ \lambda\mu = -b.$$

$$F_n(\lambda,\mu) = \sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n-i}{i}(\lambda\mu)^i(\lambda+\mu)^{n-2i}$$

$$= \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}b^i a^{n-2i} = g_{n+1}.$$

Therefore,

$$g_{n+1} = \sum_{i=0}^{n}\lambda^i\mu^{n-i} = \frac{\lambda^{n+1}-\mu^{n+1}}{\lambda-\mu}.$$

This is Binet's formula for these general Fibonacci sequences.

It is fun to exploit the polynomial identity to derive some interesting identities involving binomial coefficients but the author would welcome a more natural motivation explaining the polynomial identity. Incidentally, one referee points out that Binet's formula appeared in De Morgan's notebooks before Binet was born.

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REFERENCES

1. D. Kalman & R. Mena, *The Fibonacci numbers—Exposed*, this MAGAZINE **76**:3 (2003), 167–181.
2. A. T. Benjamin & J. J. Quinn, *The Fibonacci numbers—Exposed more discretely*, this MAGAZINE **76**:3 (2003), 182–192.