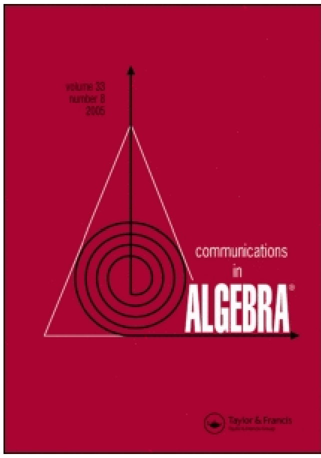

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

Bounded generation does not imply finite presentation

B. Sury [a]
[a] School of Mathematics, Tata Institute of Fundamental Research, Bombay, India

Online Publication Date: 01 January 1997
To cite this Article: Sury, B. (1997) 'Bounded generation does not imply finite presentation', Communications in Algebra, 25:5, 1673 — 1683
To link to this article: DOI: 10.1080/00927879708825944
URL: http://dx.doi.org/10.1080/00927879708825944

COMMUNICATIONS IN ALGEBRA, 25(5), 1673-1683 (1997)

BOUNDED GENERATION DOES NOT IMPLY FINITE PRESENTATION

B.Sury*

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India.
e-mail: sury@tifrvax.tifr.res.in

§ 0 Introduction

An abstract group G is said to have bounded generation, if, there are elements $g_1, g_2, \cdots, g_k$ (not necessarily distinct) in G such that

$$G = < g_1 >< g_2 > \cdots < g_k >$$

In particular, G is finitely generated. The least k is sometimes referred to as the degree of bounded generation. It is obvious that finite groups have bounded generation. It is also obvious that the free group on two or more generators does not have bounded generation. For arithmetic groups (in characteristic 0), this abstract group-theoretic property has been proved to imply the so-called congruence subgroup property ([P − R], [L]). In this note, we give an example of a solvable group which has bounded generation but which - by a criterion of Bieri and Strebel [B − S] - does not have a

*Math Reviews Subjects Classification : 20 F 05, 22 E 40, 20 F 28, 20 G 30 finite presentation, thereby answering a question posed by V.Kumar Murty. Further, this group as well as some others are examples of groups G with bounded generation such that the commutator subgroup $[G,G]$ is not even finitely generated. Further, we raise a number of relevant questions which might be of interest and also prove some results on bounded generation among which one is that the automorphism group $\mathrm{Aut}(F)$ of a free group of rank at least 3 does not have bounded generation.

§ 1 Bounded generation Vs finite presentation

We first record here some well-known lemmata.

Lemma 1.1
Let H be a subgroup of finite index in a group G. Let $\mathcal{P}$ denote any one of the properties of finite generation (f.g.), finite presentation (f.p.) or bounded generation (BG). Then, if G has property $\mathcal{P}$, then, so does H.

Proof
For f.g. and f.p., this is proved by the well-known Reidemeister-Schreier rewriting process (see Cor. 2.7.1, Cor. 2.8 of $[M-K-S]$). For BG, Tavgen (Prop.7, $[T]$) first proves this when H is normal in G, by again making use of the Reidemeister-Schreier rewriting process. From this, clearly BG follows for any subgroup of finite index. Another proof has been given by Kumar Murty $[M]$ which gives a much better bound for the degree of BG.

Lemma 1.2
Let N be a normal subgroup of a group G such that both N and G/N have bounded generation. Then, G also has bounded generation.
A similar result holds good with bounded generation replaced by finite generation or by finite presentation.

Proof
For f.g. and BG, this is evident. For f.p., this is proved in (P.33, $[Ro]$).

Lemma 1.3
Every finitely generated nilpotent group has bounded generation.
Every finitely generated nilpotent group has a finite presentation.
Therefore, for nilpotent groups, finite generation, finite presentation, and BG are all equivalent.

Proof
Let N be a finitely generated nilpotent group. It is well-known that all subgroups of N are finitely generated (Th.2.7, $[Rag]$). After this observation, it is straightforward to prove the lemma by induction on the length of the central series, using lemma 1.2.

<u>Remark 1.4</u>
(a) Note that the lemma is obvious for abelian groups from the structure theorem (and this fact is used in the induction step of the above proof).
(b) Bounded generation has, in fact, been proved more generally for finitely generated, solvable, minimax groups by Kropholler (Prop.1, [K]). This class includes all supersolvable groups.

With these facts in mind, Kumar Murty asked:

<u>Q.</u> *If a group Γ has bounded generation, does it necessarily have a finite presentation?*

A few remarks are in order here.

<u>Remark 1.5</u>
(a) If Γ is an abstract group with BG such that any subgroup Γ_0 of finite index has finite abelianisation $\Gamma_0/[\Gamma_0, \Gamma_0]$, it has been proved by Rapinchuk [Rap] that the traces of all finite dimensional complex representations of Γ are algebraic numbers. Equivalently, Γ has, upto equivalence, only finitely many completely reducible representations of a given dimension.
(b) For a linear group Γ that has the property that characters take algebraic values, Platonov makes ([Rap]) the:
<u>Conjecture (Platonov)</u> Such a Γ is of 'arithmetic type' i.e. is commensurable with a finite product of S-srithmetic groups (possibly for different S) where commensurability means the existence of isomorphic subgroups of finite indices. (A form of this also appears as a problem - Problem F 13 - in [W] and is possibly due to H.Bass who studied these 'groups of finite-representation type'.
(c) It has been proved by Platonov and Rapinchuk [$P-R$] and, independently, by Lubotzky [L] that an S-arithmetic subgroup of a semisimple group (in characteristic zero) with BG, has the S-congruence subgroup property. But, the S-congruence subgroup property implies the existence of a finite presentation.
Incidentally, arithmetic groups in positive characteristic (except possibly in an anisotropic group (necessarily of type A_n)) can not[1] have BG. The reason is that BG for a group Γ implies that its pro-p-completion is a p-adic Lie group for every prime p; but the pro-p-completions of arithmetic groups in characteristic p are not analytic (they are too huge).

[1] We are indebted to T.N.Venkataramana for this remark

(d) It is an observation due to Tavgen (Prop. 9 of [*T*], or [*Rap*]) that an abstract group with BG, which is virtually, residually-p for some prime p, has a faithful linear representation. Recall that a group is residually-p if, for each element $g \neq id$, there is a normal subgroup of p-power index which does not contain g. A group has a property 'virtually' if a subgroup of finite index has that property actually.

(e) From the above remarks, it follows that:

If Platonov's conjecture above is true, and if an abstract group Γ satisfies the three properties: (i) BG; (ii) virtually residually-p; and (iii) any subgroup Γ_0 of finite index has finite abelianisation $\Gamma_0/[\Gamma_0, \Gamma_0]$, then, Γ is finitely presented.

Thus, if one were to look for an example of a linear group with bounded generation which does not have a finite presentation, essentially the only hope is to find it among solvable, nonnilpotent groups. This is what we do.

Proposition 1.6

The group G of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & p^n & c \\ 0 & 0 & 1 \end{pmatrix}; \quad a, b, c \in \mathbb{Z}[1/p], n \in \mathbb{Z}$$

has the following properties:

- *G has bounded generation of degree ≤ 12.*
- *G is not finitely presentable.*
- *$[G, G]$ is not finitely generated.*
- *G is three-step solvable.*

In fact, even its subgroup H of matrices

$$\begin{pmatrix} p^n & c \\ 0 & 1 \end{pmatrix}; \quad c \in \mathbb{Z}[1/p], n \in \mathbb{Z}$$

has the following properties:

- *H has bounded generation of degree 3.*
- *$[H, H]$ is not finitely generated; it has index exactly $p - 1$ in $\mathbb{Z}[\frac{1}{p}]$.*
- *H is two-step solvable.*

Proof of Prop.1.6

Let us call $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; and $y_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Then, we have

$$x^{-1}y_{12}x = y_{12}^{p}$$
$$xy_{23}x^{-1} = y_{23}^{p}$$

In fact, we have, for every $n, r \in \mathbb{Z}$,

$$x^n y_{12}^r x^{-n} = \begin{pmatrix} 1 & rp^{-n} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \cdots\cdots\cdots (I)$$

$$x^n y_{23}^r x^{-n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & rp^{n} \\ 0 & 0 & 1 \end{pmatrix} \qquad \cdots\cdots\cdots (II)$$

To prove bounded generation, let $g = \begin{pmatrix} 1 & ap^k & bp^l \\ 0 & p^n & cp^m \\ 0 & 0 & 1 \end{pmatrix}$; $a, b, c, k, l, m, n \in \mathbb{Z}$ be any element of G.

Then, we explicitly have

$$g = x^{n-k}y_{12}^{a}x^{m-n+k}y_{23}^{c}x^{n-m}[x^B y_{12}^{A} x^{-B}, y_{23}]$$

where we have written $[P, Q]$ to mean the commutator $PQP^{-1}Q^{-1}$, and where A and B are integers defined by $bp^l - acp^{m-n+k} = Ap^{-B}$. Indeed, the right hand side of the above expression is easily calculated using (I) and (II), and checked to be the matrix g. Thus, G has bounded generation of degree ≤ 12. It is also trivial to verify that G is the semidirect product of the unipotent part N consisting of the matrices $\begin{pmatrix} 1 & ap^k & bp^l \\ 0 & 1 & cp^m \\ 0 & 0 & 1 \end{pmatrix}$; $a, b, c, k, l, m \in \mathbb{Z}$ and the toral part T with the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & p^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $n \in \mathbb{Z}$.

Also, $[G, G] = N$;

$[N, N] \cong \mathbb{Z}[1/p]$ has the elements $\begin{pmatrix} 1 & 0 & bp^l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $b, l \in \mathbb{Z}$; and

$N_{ab} = N/[N, N] \cong \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p]$.

It is known (0.2.14, 0.2.15, 0.2.16 of $[A]$) that G does not have a finite presentation. This can be checked, as follows, by using the criterion of Bieri-Strebel (Thm.C, P.260, $[B-S]$).

Theroem: *Let $G = C \propto N$ be the split extension of a nilpotent group N by an infinite cyclic group C. Then the following are equivalent:*

(i) G is finitely presented,
(ii) $N_{ab} = N/[N,N]$ is a finitely generated C-module such that:
(a) the $\mathbb{Z}$-torsion subgroup of N_{ab} is finite,
(b) the rational vector space $V = N_{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ has finite dimension, and
(c) there is a generator t of C such that the characteristic polynomial of $t \otimes \mathbb{Q} \in End(V)$ is integral.

Under the identifications of T with $\mathbb{Z}$ and of N_{ab} with $\mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p]$, the action of $\mathbb{Z}$ on $\mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p]$ is given by $n \cdot (\alpha, \beta) = (p^{-n}\alpha, p^n\beta)$, and the above-mentioned property (ii) c fails. As a matter of fact, the relations $[y_{12}^{a/p^r}, y_{23}^{b/p^s}] = [y_{12}, y_{23}]^{ab/p^{r+s}}$; $a, b, r, s \in \mathbb{Z}$ are probably the cause for the infinite number of relations in G, as pointed out by H.Bass.

Let us now prove the assertions on H. The subgroup H can be identified with the semi-direct product $\mathbb{Z} \propto \mathbb{Z}[1/p]$ where $\mathbb{Z}$ is acting by powers of p. Indeed, the semi-direct product $\mathbb{Z} \propto \mathbb{Z}[1/p]$ is isomorphic to the group with two generators x, y and a single relation $xyx^{-1} = y^p$ and an isomorphism with H is defined by sending x and y to the matrices $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}$ respectively.
Using this identification, all the assertions on H are proved immediately. For instance, any word $x^a y^b \cdots$ reduces to a word $x^a y^b x^c$, which shows H has BG of degree 3. Further, $[H,H]$ is not finitely generated since it is of index $p-1$ in $\mathbb{Z}[\frac{1}{p}]$ since $H/[H,H] \cong < x, y \mid y^{p-1} = 1 > \cong \mathbb{Z}/(p-1)\mathbb{Z} \propto \mathbb{Z}$. This proves the proposition.

Remark 1.7
(a)The group G was constructed by H.Abels ([A]) as an example of a non-finitely presented group. Unlike G, the subgroup H is finitely presentable as we saw in the above proof.
(b) As mentioned earlier, the property of bounded generation for arithmetic groups implies the congruence subgroup property. For the above groups G and H, the congruence subgroup property can be directly verified (from knowing it for a normal subgroup and for the quotient).

In this regard, Tavgen asks the following question.
Q.1 (Tavgen) *If Γ is a group with bounded generation, and if its solvable radical is finite, then, is $\Gamma_0/[\Gamma_0, \Gamma_0]$ finite, for each subgroup Γ_0 of finite index in Γ ?*

Remark

Γ as in Q.1 above does not necessarily have Kazhdan's property T as the example $SL(2, \mathbb{Z}[1/p])$ shows. [2]

§ 2 Bounded generation and Aut(F)

In this section, we consider the automorphism group of a free group of finite rank. For the free abelian group $\mathbb{Z}^n$, the automorphism group $GL(n, \mathbb{Z})$ has been proved to have bounded generation exactly when $n \neq 2$ ([$C - K$]). Recall that for arithmetic groups (in characteristic 0), the congruence subgroup property is equivalent to the BG for the profinite completion of the corresponding arithmetic group in the sense of BG for profinite groups (Th.1 and Th.2, [$P - R$]). Let F be a free group of rank $n \geq 3$.
Our next result is:

Proposition 2.1
Aut(F) does not have bounded generation.

Remark 2.1.1
If F is free of rank 2, then also the group Aut(F) fails to have bounded generation. The reason is that the group $GL(2, \mathbb{Z})$, which is a quotient of Aut(F) (viz. the group of outer automorphisms) does not have bounded generation.

Before proving the proposition, we make a few comments. For the groups Aut(F), the following analogue of the congruence subgroup property[3] is open.

Q.2 (Ihara) *For any characteristic subgroup C of finite index in F, let $A(C)$ denote Ker (Aut(F) $\rightarrow$Aut(F/C)). Does every subgroup of finite index in Aut(F) contain some $A(C)$?*

Note that Aut(F) is known to **not** have a finite dimensional (faithful) representation if $n \geq 3$ ([$P - F$]) and, yet shares the following nice properties with the linear groups:
(a) Aut(F) is finitely presented.
(b) Aut(F) is residually finite.

---

[2]We thank Alex Lubotzky for pointing this out
[3]We are grateful to Gopal Prasad who brought this to our attention

Proof of Proposition 2.1

If we show that Aut(F) is virtually residually-p for some prime p, the proposition follows from Tavgen's observation (see remark 1.5(d)), since Aut(F) is not linear.

The proof of the fact that Aut(F) is virtually residually-p for **every** prime p is probably well-known. Since we were unable to find a reference, we give a proof here. In fact, we show, more generally:

Lemma 2.2 *If G is finitely generated, virtually residually-p for some p, then Aut(G) is also virtually residually-p.* [4]

Proof

If G is virtually residually-p, there is a characteristic subgroup G_0 of finite index in G which is residually-p and if Aut(G_0) is virtually residually-p, so is Aut(G) as seen by pulling back via the restriction homomorphism from Aut(G) to Aut(G_0). Without loss of generality, we, therefore, assume that G is residually-p. Let H be any characteristic subgroup of p-power index in G. Consider the p-group $P = G/H$. Now, $P/\Phi(P) \cong \bigoplus_1^r \mathbb{Z}/p$ where $\Phi(P)$ denotes the Frattini subgroup of P i.e. the intersection of all maximal proper subgroups of P. Moreover, the number r of copies of $\mathbb{Z}/p$ is bounded **independently of** H; indeed, $r \leq n$, where n is the number of generators of G. Now, by a famous theorem of Hall, Ker (Aut(P) $\to$ Aut($P/\Phi(P)$)) is a p-group. Note that Aut($P/\Phi(P)$) $\cong GL(r, \mathbb{Z}/p)$, and that there are only finitely many homomorphisms from Aut(G) to GL($n, \mathbb{Z}/p$), since there are only finitely many subgroups of index bounded by the order of GL($n, \mathbb{Z}/p$). Call $\mathcal{A}$ to be the intersection of the kernels of all the homomorphisms from Aut(G) to GL($n, \mathbb{Z}/p$). We claim that $\mathcal{A}$ is residually-p. Let $\sigma \in \mathcal{A}$, $\sigma \neq id$. So, there is g such that $\sigma(g)g^{-1} \neq id$. Let H be a characteristic subgroup of p-power index in G such that $\sigma(g)g^{-1} \notin H$; then $\sigma \notin$ Ker(Aut(G) $\to$ Aut(G/H)). Call $P = G/H$. Consider, now, the composite

$$\mathcal{A} \hookrightarrow Aut(G) \to Aut(P) \to Aut(P/\Phi(P)) \hookrightarrow GL(n, \mathbb{Z}/p)$$

By the choice of $\mathcal{A}$, the image goes into $N := Ker(Aut(P) \to Aut(P/\Phi(P)))$, a p-group. Since the image of σ is nontrivial in Aut(P), Ker($\mathcal{A} \to N$) is normal, of p-power index in $\mathcal{A}$, and does not contain σ. This completes the proof of the lemma and, hence, of the proposition as well.

[4] After this paper was written, Lubotzky informed us that this already appears as Prop.2 in his paper in J. of Algebra 63(1980)494-498.

Remark 2.2.1
The above proof shows that even for the free group of rank 2, the group Aut(F) is virtually, residually-p.

Yet another interesting question that Tavgen asks is:

Q.3 (Tavgen) *Is every residually finite group Γ with bounded generation also virtually residually-p for some prime p ?*

If both the questions 1 and 3 have an affirmative answer, and, further, if Platonov's conjecture (See Remark 1.5(b)) also has an affirmative solution, this will complete the following program:

Γ residually finite, BG, with finite solvable radical $\overset{Yes\ for\ Q.3}{\Longrightarrow}$
Γ also virtually residually-p $\Longrightarrow$
Γ also linear $\overset{Yes\ for\ Q.1}{\Longrightarrow}$
Γ has all characters algebraic $\overset{Platonov\ conj.}{\Longrightarrow}$
Γ arithmetic group with the congruence subgroup property.

From Lemma 1.2 and Prop. 2.1, it is immediate that:

Cor. 2.3
For $n \geq 3$, Ker $(Aut(F) \to GL(n, \mathbb{Z}))$ does not have bounded generation.

Remark 2.3.1
The assertion of the corollary also holds for $n = 2$. Indeed, the kernel in question is isomorphic to the free group F (Prop. 4.5 of $[L - S]$).

Similar to bounded generation is the property of polynomial index growth (PIG). A group G has PIG if, for each n, the subgroup generated by its n-th powers has index bounded by a polynomial in n of some fixed degree d independent of n. For arithmetic groups, PIG is also related to the congruence subgroup property ($[P - R], [L]$). Mann and Segal proved ($[M - S]$) that a finitely generated, residually nilpotent group with PIG is linear.
Therefore, we have also:

Cor.2.4
Aut(F) does not have PIG if $n \geq 3$.

Remark 2.5
In contrast to Q.1, Aut(F) is an example of a group without bounded generation but with finite presentation and with [Aut(F),Aut(F)] finitely generated; indeed, Aut(F)/[Aut(F),Aut(F)] is of order 2.

More precisely, we claim [Aut(F),Aut(F)] $= \pi^{-1}(SL(n,\mathbb{Z})$ where $\pi : Aut(F) \to Aut(F/[F,F]) = GL(n,\mathbb{Z})$. To see that $\pi^{-1}(SL(n,\mathbb{Z}) \subseteq$ [Aut(F),Aut(F)], let us recall an observation of Nielsen (See P.28, $[L-S]$) that Ker(π) is the normal subgroup generated by the single automorphism

$$\alpha_{112} : X_1 \mapsto X_2^{-1}X_1X_2$$

Here $X_1, \cdots, X_n$ is a basis of F and we use the above notation to mean that every other generator is fixed.
But, if we consider the automorphisms $\tau_{ij} : X_i \mapsto X_iX_j$ and $\sigma_i : X_i \mapsto X_i^{-1}$, we can easily check that

$$\alpha_{112} = [\tau_{12}, \sigma_2][\sigma_1, \tau_{12}]$$

Here we have written $[x,y]$ for $xyx^{-1}y^{-1}$.
Therefore, Ker(π) $\subseteq$ [Aut(F), Aut(F)]. Since π surjects onto GL($n,\mathbb{Z}$) and since SL($n,\mathbb{Z}$) = [GL($n,\mathbb{Z}$), GL($n,\mathbb{Z}$)] is of index 2 in GL($n,\mathbb{Z}$), it follows that [Aut(F),Aut(F)] $= \pi^{-1}(SL(n,\mathbb{Z})$ is of index 2 in Aut(F).

Acknowledgements

I would like to express my gratitude to V. Kumar Murty. The questions he asks are always very interesting. Moreover, it was his suggestion that I write this note and his encouragement that made me do it. I would also like to thank O.Tavgen for many interesting discussions in 1992 when he visited the Tata Institute. In particular, he helped me out with the proof of Prop. 2.1. I am indebted to H.Bass and to A.Lubotzky for carefully going through this note pointing out some inaccuracies.

References

$[A]$ - H.Abels, Finite presentability of S-arithmetic groups, Springer Lecture Notes 1261, 1987.
$[B-S]$ - R.Bieri-R.Strebel, Comment. Math. Helv., Vol.53 (1978), P.258-278.
$[C-K]$ - D.Carter-G.Keller, Amer.J.Math. 105(1983), P.673-687.
$[K]$ - P.H.Kropholler, Proc. Lond. Math Soc, Vol.49 (1984), P.155-169.
$[L]$ - A.Lubotzky, Invent.Math, Vol.119 (1995) P.267-295.
$[L-S]$ - R.C.Lyndon-P.E.Schupp, Combinatorial group theory, Springer, 1977.
$[M]$ - V.Kumar Murty, Bounded and finite generation of arithmetic groups, To appear in Canadian Math Society Proceedings (15 pages).

[$M-K-S$] - W.Magnus-A.Karrass-D.Solitar, Combinatorial group theory, Interscience, 1966.
[$M-S$] - A.Mann-D.Segal, Proc. Lond.Math.Soc, Vol.61 (1990) P.529-545.
[$P-F$] - C.Procesi-E.Formanek, J. of Algebra, Vol.149 (1992) P.494-499.
[$P-R$] - V.P.Platonov-A.S.Rapinchuk, Math USSR Izvestiya, Vol.40 (1993) P.455-476.
[Rag] - M.S.Raghunathan, Discrete subgroups of Lie groups, Springer-Verlag, 1972.
[Rap] - A.S.Rapinchuk, Combinatorial theory of arithmetic groups, Preprint of Byelorussian academy of Sciences, No. 20 (1990).
[Ro] - D.J.S.Robinson, Finiteness conditions and general soluble groups, Vol.I, 1972.
[T] - O.Tavgen, Math USSR Izvestiya, Vol.36(1991), P.101-128.
[W] - C.T.C.Wall (editor), Homological group theory, Durham 1977.

Received: August 1996