Placing Signs

Can one find a natural number n and a way to place $+$ and $-$ signs between consecutive terms of the sequence 1^k, 2^k, 3^k, $\ldots$, n^k so that the sums are zero for each of $k = 1, 2, \ldots, 100$? It turns out to be possible, and one has more generally:

Theorem. *Given any natural number n, the set $\{1, 2, 3, \ldots, 2^{n+1}\}$ can be partitioned into two subsets A_n and B_n, each of size 2^n, such that*

$$\sum_{a \in A_n} a^k = \sum_{b \in B_n} b^k \quad \textit{for } k = 1, 2, \ldots, n. \tag{1}$$

This can be proved by induction. For $n = 1$, take $A_1 = \{1, 4\}$ and $B_1 = \{2, 3\}$. If one has chosen A_n and B_n of size 2^n with $A_n \sqcup B_n = \{1, 2, 3, \ldots, 2^{n+1}\}$ and such that (1) holds, simply take

$$A_{n+1} = A_n \cup (2^{n+1} + B_n), \quad B_{n+1} = B_n \cup (2^{n+1} + A_n).$$

Here, of course, $d + S$ denotes the set $\{d + s : s \in S\}$. The proof of

$$\sum_{a \in A_n} a^k + \sum_{b \in B_n} (2^{n+1} + b)^k = \sum_{b \in B_n} b^k + \sum_{a \in A_n} (2^{n+1} + a)^k \quad \text{for } k = 1, 2, \ldots, n+1$$

is evident by the binomial expansion.

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