The Amitsur-Levitzki Identity via Graph Theory

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If $S(X_1, \cdots X_k) = \sum sgn(\sigma) X_{\sigma 1} \cdots X_{\sigma k}$ is the polynomial in k noncommuting variables where the sum is over all permutations σ of k symbols and $sgn(\sigma)$ denotes the signature of σ, it is not difficult to show that $S(A_1, \cdots, A_{n^2+1}) = 0$ for arbitrary $n \times n$ matrices A_i. Amitsur and Levitzki proved the beautiful result that

$$S(A_1, \cdots A_{2n}) = 0$$

and that there is no nonzero polynomial in k variables with k less than $2n$ satisfied by arbitrary $n \times n$ matrices.

Here, we shall discuss a graph-theoretic method of R Swan which gives a geometric proof of this result. This graph-theoretic result is also of independent interest.

We first make the following elementary but important observation that $S(X_1, \cdots, X_k)$ depends linearly on the X_i's. Therefore, it suffices to prove that the Amitsur-Levitzki identity holds for the elementary $n \times n$ matrices $e_{i,j}$ with 1 at the (i, j)th place and 0s elsewhere.

Graph theory enters in the following manner. For us, a graph would be finite and oriented, that is, the edges would be directed. Let $A_1, \cdots A_{2n}$ be *elementary* matrices. Note that the product $e_{i,j} e_{k,l} = \delta_{j,k} \delta_{i,l} I$, where $\delta_{j,k}$ is Kronecker's delta.

Given these A_i's, we define an oriented graph Γ as follows:

Γ has n vertices $P_1, \ldots, P_n$ and has one edge e_i for each A_i where, if $A_i = e_{j,k}$, e_i has initial point P_j and terminal point P_k. Clearly, the number E of edges and the number V of vertices have the relation $E = 2V$ for this

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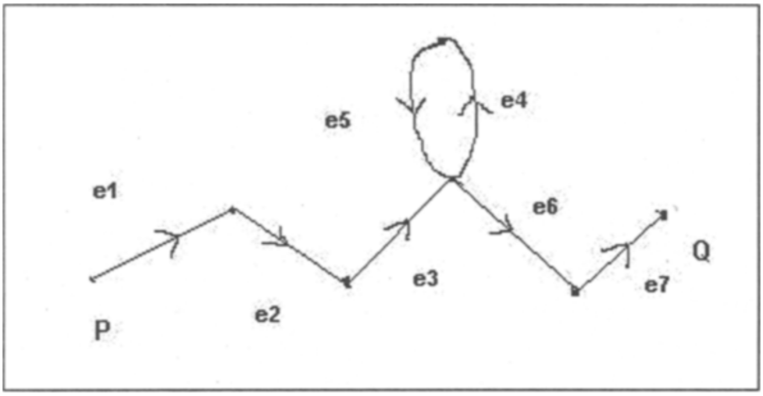


Figure 1. A unicursal path.

graph. The multiplication rule for elementary matrices now shows that a product $A_{\sigma(1)} \cdots A_{\sigma(2n)}$ has a nonzero (i,j)th entry if and only if the corresponding sequence of edges $e_{\sigma(1)} \dots e_{\sigma(2n)}$ is a unicursal path from P_i to P_j. Here, by a unicursal path (see *Figure* 1), we mean a method of walking from P_i to P_j along edges so that every edge is traversed just once, and is traversed in the proper direction. In case such a unicursal path exists, the (i,j)th entry of $A_{\sigma(1)} \cdots A_{\sigma(2n)}$ is 1.

Thus, in any graph, if we fix an ordering of the edges, each unicursal path $\omega = (e_1, \dots, e_E)$ then gives a permutation of the edges of Γ. Define $\epsilon(\omega)$ to be the sign of this permutation. Note that a unicursal path is nothing but the analogue of an Euler path in the oriented case.

In the case of our graph, we note that the (i,j)th entry of $S(A_1, \cdots A_{2n})$ is zero if, and only if, the number of unicursal paths ω from P_i to P_j for which $\epsilon(\omega) = +1$ is equal to the number of unicursal paths ω from P to Q for which $\epsilon(\omega) = -1$. In fact, this holds for more general graphs as well:

By a unicursal path, we mean a method of walking along edges so that every edge is traversed just once, and is traversed in the proper direction.

Theorem 1. *Suppose $E \geq 2V$. Let P and Q be any fixed vertices of Γ (not necessarily distinct). Then the number of unicursal paths ω from P to Q with $\epsilon(\omega) = +1$ is equal to the number of unicursal paths ω from P to Q with $\epsilon(\omega) = -1$.*

Connectedness of the graph is a necessity for the existence of a unicursal path.

As observed above, this theorem implies:

Theorem 2 (Amitsur and Levitzki). *If* $A_1, \ldots, A_{2n}$ *are any* $n \times n$ *matrices with entries in any commutative ring, then* $S(A_1, \ldots, A_{2n}) = 0$.

Before starting the proof of Theorem 1, we make a few very simple observations after recalling two definitions. We allow a vertex to be joined to itself and we also allow two vertices to be joined in many ways. If P is any vertex of Γ, the *order* of P is defined to be the total number of edges beginning or ending at P. An edge which joins P to itself will be counted twice. The *flux* of P is defined to be the number of edges beginning at P minus the number of edges ending at P.

From now on we will assume that Γ has no vertices of order 0. The reason for this assumption is that connectedness of the graph is a necessity for the existence of a unicursal path.

In an unoriented graph, we have the famous result due to Euler that if the number of vertices with odd order is more than 2, there cannot exist any Eulerian path. We have the following analogous result for the oriented graphs.

Proposition: If there is a unicursal path from P to Q, then

(a) Γ is connected.

(b) Every vertex other than P and Q has flux 0.

(c) If $P = Q$, then P has flux 0.

(d) If $P \neq Q$, then P has flux +1 and Q has flux -1.

Proof: We can prove (a) easily as, in a disconnected graph, there will not be any path that connects all edges and, so there cannot be any unicursal path. Since we must leave each vertex the same number of times that we enter it, all vertices apart from the origin and the terminus have flux 0; that is, (b) is clear. Of course, (c) is a special case of (b). (d) follows from the fact that we leave P one more time than we enter it, and the reverse holds for Q.

In an unoriented graph, we have the famous result due to Euler that if the number of vertices with odd order is more than 2, there cannot exist any Eulerian path.

Preliminary Observations:

1. Suppose that the two edges e and e' of Γ have the same initial points and the same terminal points. Then the theorem holds for Γ. To see this, we merely observe that, given any unicursal path ω, we can form a new one ω' by interchanging e and e'. Since we have performed transposition which is equivalent to adding one 2-cycle to the permutation and hence equivalent to changing the sign of the permutation, so $\epsilon(\omega) = -\epsilon(\omega')$.

2. The theorem is true if Γ is not connected since then there are no unicursal paths at all.

3. If the theorem is true for the case $E = 2V$, it is also true for the case $E > 2V$. To see this, we modify Γ by introducing $k = E - 2V$ new vertices and edges as in *Figure* 2, getting a new graph Γ'.

   This new graph satisfies the condition $E = 2V$. There is a 1-1 correspondence between the unicursal paths from P_1 to Q on Γ' and the unicursal paths from P to Q on Γ, because for any path from P_1 to Q on Γ', the path from P_1 to P is always included without any permutations. Similarly any path from P can be extended to get a path from P_1

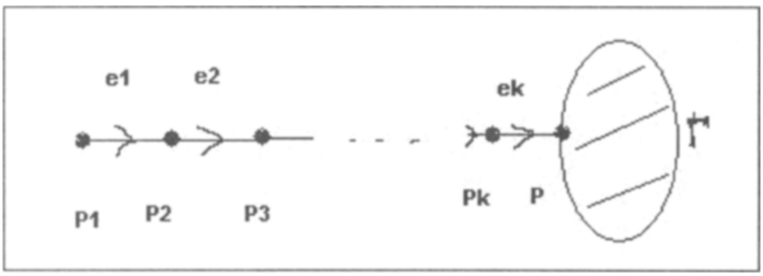


Figure 2.

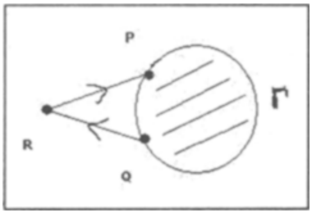


Figure 3.

which has the same number and kinds of permutations as before. This correspondence preserves the signs $\epsilon(\omega)$.

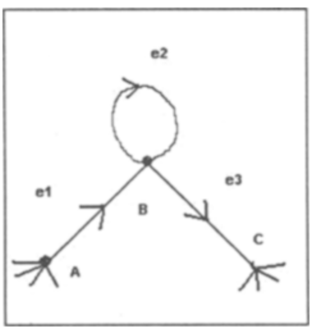
4. If the theorem is true for the case where $E = 2V$ and all vertices have flux 0, it is then true in general. To see this, we note that if not all vertices have flux 0, the only non-trivial case is that where all but P and Q have flux 0, P has flux +1 and Q has flux -1. Proposition 1 shows that there is no unicursal path from P to Q in any other case. We now define a new graph Γ' by adding two edges and a vertex as in *Figure* 3.

There is clearly a 1-1 correspondence between unicursal paths from P to Q on Γ and unicursal paths from R to R on Γ'. This can be seen in a way similar to the previous argument for correspondence between paths in the augmented and the original graph. Again $\epsilon(\omega)$ is preserved by this correspondence.

Now we proceed to prove Theorem 1.

By the points noted above, we can assume that $E = 2V$ and that all vertices have flux 0. All the transformations of Γ performed below will preserve these conditions. We now proceed by induction on V, the number of vertices. For $V = 1$, the theorem is trivial by the first observation above. Let us assume that $V > 1$. We shall show that all the possible graphs which fall under the above assumptions ($E = 2V$ and flux = 0) may be classified into three cases. Then, every graph satisfying the assumptions will be reduced to a graph which has fewer vertices and still satisfies the assumptions and thus, to a case for which the theorem holds.

Figure 4.



Case 1. Γ *contains the configuration of Figure* 4.

This is the only case in which the induction hypothesis will be used. The other cases will be treated by reducing them to this case.

If $P = B$, note that every unicursal path must begin or end with e_2. By moving e_2 from beginning to end or vice-versa, we get a correspondence $\omega \leftrightarrow \omega'$ between unicursal paths. This means that for every unicursal path with sign $\epsilon(\omega)$, there is a unicursal path with sign $-\epsilon(\omega)\{= \epsilon(\omega')\}$, so the theorem holds in this case.

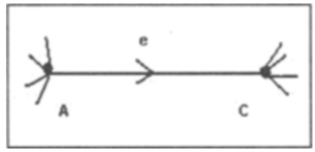


Figure 5.

If $P \neq B$, we replace the configuration made up of e_1, e_2, e_3, and B by a single edge e as in *Figure* 5, getting a new graph Γ'. There is a 1-1 correspondence between unicursal paths from P on Γ and Γ' because any path from A to C must pass through e_1, e_2, e_3 in that order only, thus eliminating the need for vertex B.

This correspondence also preserves $\epsilon(\omega)$ since the part between A to C has no permutations in either case. Now Γ' has fewer vertices than Γ so the induction hypothesis applies.

Case 2. Γ *has vertex of order* 2.

Since $E > V$, not all vertices have order 2. By the second observation, we may assume Γ is connected. Therefore Γ contains a configuration as in *Figure* 6 where A has order greater than 2.

For each edge $e_i, i = 1, \ldots, k$ terminating at A, define a new graph Γ_i by making the transformation indicated in *Figure* 7. The part of Γ not shown is left unaltered.

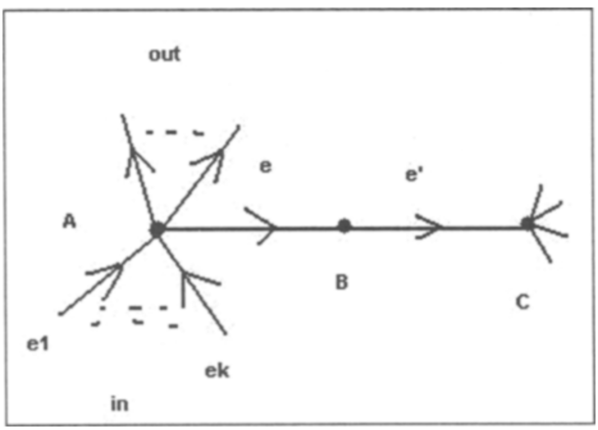
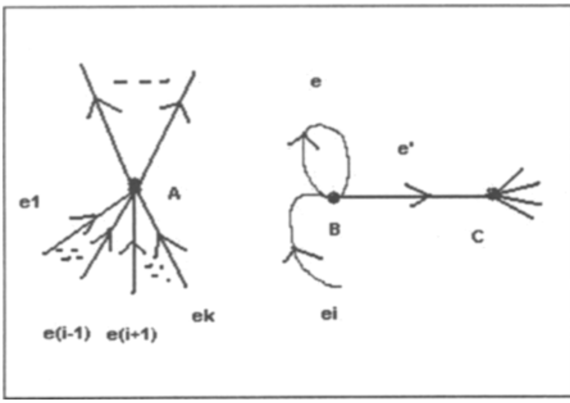


Figure 6.



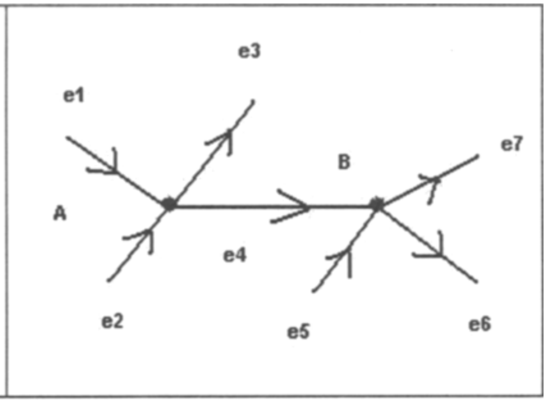


Figure 7 (left).
Figure 8 (right).

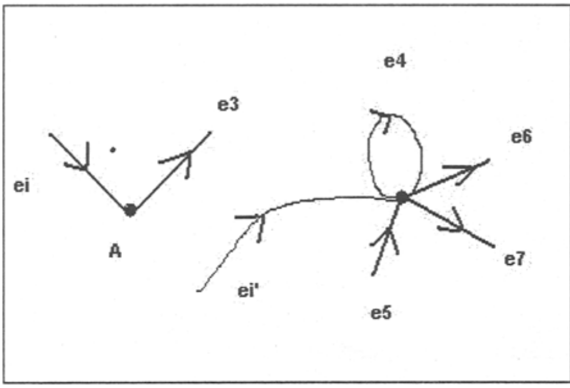
In any unicursal path on Γ, one of the e_i's must immediately precede e. Consequently, this path is also unicursal on Γ_i; moreover, this path is not unicursal for any other $\Gamma_j, j \neq i$ since, on any other Γ_j, it is e_j which precedes e. Conversely, we claim that any unicursal on any Γ_i, is also unicursal on Γ. To see this, we break up vertex B of Γ_i into two vertices, A' and B, where A' and B are joined by e, and we fuse vertex A' with A giving us Γ. Because of this correspondence, it is sufficient to prove the theorem for each Γ_i. But, each Γ_i satisfies the condition of Case 1.

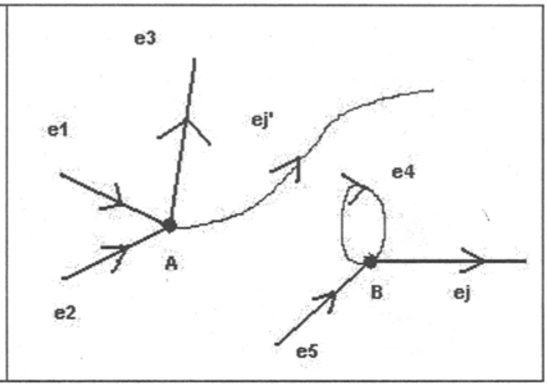
Case 3. *Cases* 1 *and* 2 *do not apply.*

Since each vertex has flux 0, it must have an even order. Consequently, each has order ≥ 4 since Case 2 does not apply. Now, the average order of the vertices is $2(E/V) = 4$, since each edge has 2 endpoints. This implies that each vertex has order exactly 4. Therefore, Γ contains the configuration of *Figure* 8.

Let us now try the construction used in Case 2. This gives us the graphs $\Gamma_i, i = 1, 2$ of *Figure* 9. Here we let $i' = 2$ if i=1 and $i' = 1$ if $i = 2$.

As in Case 2, every unicursal path on Γ is unicursal on exactly one of Γ_1 and Γ_2. However, we note that there are paths on Γ_i which enter B by e_i, but then leave by e_6 or e_7 without going around e_4. Such paths are not





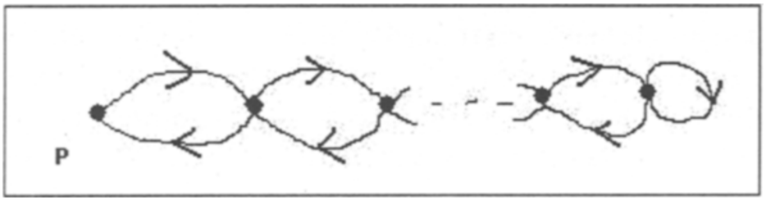


Figure 9 (left).
Figure 10 (right).

Figure 11 (center)

unicursal on Γ. However, they all contain a subpath $e_5e_4e_j$, where $j = 6$ or 7. Consequently, they are exactly those paths which are unicursal on one of the graphs $\Gamma'_j, j = 6, 7$ of *Figure* 10. Here, as before, $j' = 7$ if $j = 6$, and $j' = 6$ if $j = 7$.

Now, the unicursal paths on the $\Gamma_i, i = 1, 2$ are exactly those unicursal on either Γ or on some $\Gamma'_j, j = 6, 7$. Since Case 2 applies to the Γ_i and Case 1 applies to the Γ'_j, it follows that the theorem holds for Γ.

Finally, we show that the condition $E \geq 2V$ is really needed in Theorem 1. Consider the graph Γ in *Figure* 11. Here $E = 2V - 1$ and there is evidently only one unicursal path from P to P.

Suggested Reading

[1] S A Amitsur and J Levitzki, Minimal Identities for Algebras, *Proc. of Amer. Math. Soc.*, Vol.1, pp.449-463, 1950.

[2] R G Swan, An Application of Graph Theory to Algebra, *Proc. of Amer. Math. Soc.*, pp.367-373, 1963.

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