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Classifying reductive groups via their root data

Existence and uniqueness theorems

B.Sury

Stat-Math Unit
Indian Statistical Institute
Bangalore, India.

§ Introduction.

Throughout our study, algebraic groups are defined over an algebraically closed field K and are usually identified with their groups of K-rational points.

In this workshop, we have already analyzed the structure of linear algebraic groups. We have seen the structure theorem for connected solvable algebraic groups and we have also seen that a connected group is a semidirect product of its unipotent radical and a reductive group (called a Levi subgroup). Thus, the study of a general connected algebraic group reduces to that of reductive groups. Further, a connected reductive group is the (almost) direct product of its derived subgroup (a semisimple group) and a central torus. The classification of semisimple groups will be accomplished by means of the root system and some modification of this makes it possible to also classify reductive groups. Somehow, more than semisimple groups, it is the reductive groups which arise naturally and we will classify reductive groups in this chapter by means of certain combinatorial data. The corresponding combinatorial data from linear algebra for reductive groups are called root data. In these 3 talks, we introduce root data for reductive groups and show that an isomorphism of two root data yields an isomorphism of the corresponding algebraic groups. In the next 3 talks, we will solve the (easier) problem of constructing the reductive algebraic group corresponding to a root datum. Here, we follow an approach due to Steinberg - from Journal of Algebra, Vol. 216, 1999 - which is simpler than the one in Springer's book which needs explicit technical computations in groups of semisimple-rank 2.

§ Examples of root data.

Even before defining what root data are, it is better to meet them for some simple examples. Recall that for any connected, reductive group G, the adjoint representation of G on $\mathcal{G} = Lie(G)$ when restricted to a maximal torus T decomposes into common eigenspaces for T as

$$\mathcal{G} = \mathcal{G}_0 \oplus \bigoplus_{\alpha} \mathcal{G}_\alpha$$

for some characters α of T, where $\mathcal{G}_0 = Lie(T)$. The characters $\alpha \in X^*(T)$ for which $\mathcal{G}_\alpha \neq 0$ are known as the *roots* of G with respect to T. In a similar, more general, situation for any representation of G, the characters of T for which the corresponding common eigenspace is non-zero, are called the weights of the representation. The finite subset of $X^*(T)$ consisting of roots is denoted usually by $\Phi(G,T)$ or simply by Φ. We define the group $X_*(T)$ of cocharacters of T as the set of algebraic group homomorphisms from $\mathbf{G_m}$; this is a free abelian group of the same rank as the group $X^*(T)$.

Root datum for GL_n for $n \geq 2$.

The diagonal torus T of $G = GL_n$ over K is a maximal torus. Now $X^*(T) = \bigoplus_{i=1}^n \mathbf{Z}\chi_i$, where $\chi_i : diag(x_1, \cdots, x_n) \mapsto x_i$. The adjoint action on $\mathcal{G} = Lie(G) = M_n$ is simply conjugation. The set Φ of roots is $\alpha_{i,j} = \chi_i - \chi_j; i \neq j$ and $\mathcal{G}_{\alpha_{i,j}} =< E_{i,j} >$, generated by the elementary matrices. If we identify $X^*(T)$ with $\mathbf{Z}^n$ by means of the basis $\{\chi_i\}$, the subset of roots is identified with $\{e_i - e_j : 1 \leq i \neq j \leq n\}$, where $\{e_i\}$ is the canonical basis of $\mathbf{Z}^n$. The group $X_*(T)$ of cocharacters is the free abelian group with basis $\lambda_i : t \mapsto diag(1, \cdots, 1, t, 1, \cdots, 1)$ where t is at the i-th place in the diagonal matrix. Let us write $\alpha_{i,j}^\vee = \lambda_i - \lambda_j$ for all $i \neq j$. It should be noted that we are writing $X^*(T)$ and $X_*(T)$ additively; this means, for example that $\alpha_{i,j}^\vee(t) = diag(1, \cdots, 1, t, 1, \cdots, 1, t^{-1}, 1, \cdots, 1)$ where t is at the i-th place and t^{-1} is at the j-th place. Now, we note that the composite

$$\alpha_{i,j} \circ \alpha_{i,j}^\vee : t \mapsto t^2$$

for each $i \neq j$. For $\chi \in X^(T), \lambda \in X_*$, one writes $(\chi \circ \lambda)(t) = t^{<\chi,\lambda>}$. Thus, $< \alpha_{i,j}, \alpha_{i,j}^\vee >= 2$. The above map $< .,. >$ is actually defined on $X^*(T) \times X_*$ and maps to $\mathbf{Z}$ as $(\chi, \lambda) \mapsto n$ where $\chi \circ \lambda : t \mapsto t^n$ for all $t \in K^*$.
This notation $< .,. >$ is deliberately chosen so as to point out that the group of characters and cocharacters are in duality by means of a pairing

$$< .,. >: (X^*(T) \otimes \mathbf{R}) \times (X_*(T) \otimes \mathbf{R}) \to \mathbf{R}.$$

For each root $\alpha_{i,j}$, we have an automorphism

$$s_{\alpha_{i,j}} : X^*(T) \to X^*(T) \ ;$$

$$x \mapsto x - < x, \alpha_{i,j}^\vee > \alpha_{i,j}.$$

The important point to note is that these automorphisms map Φ into itself; indeed, if i, j, k are distinct, then

$$s_{\alpha_{i,j}}(\alpha_{i,k}) = \alpha_{j,k}.$$

Also $s_{\alpha_{i,j}}(\alpha_{i,j}) = -\alpha_{i,j}$ and $s_{\alpha_{i,j}}$ fixes the other $\alpha_{k,l}$.
Let $X_{i,j}$ denote the permutation matrix where the i-th and the j-th rows of the identity matrix have been interchanged. Clearly, the conjugation action of $X_{i,j}$ on GL_n leaves T invariant and acts on a diagonal matrix by interchanging the i-th and the j-th entries. Observe that if g normalizes T, then the left coset of g modulo T acts on T and there is an induced action on $X^*(T)$ given as $g.\chi : x \mapsto \chi(g^{-1}xg)$. In other words, the induced action of $X_{i,j}$ on $X^*(T)$ is the map $s_{\alpha_{i,j}}$. Therefore, the Weyl group - the group of automorphisms of $X^*(T)$ which is generated by the $s_{\alpha_{i,j}}$'s - is S_n.

To summarise, we have :
• two free abelian groups $X^*(T)$ and $X_*(T)$ of finite rank in duality, by means of a pairing to $\mathbf{Z}$,
• two finite subsets $\Phi, \Phi^\vee$ of $X^*(T), X_*(T)$ respectively, which are in bijection

by a map $\alpha_{i,j} \mapsto \alpha_{i,j}^\vee$ satisfying :
(i) $< \alpha_{i,j}, \alpha_{i,j}^\vee >= 2$,
(ii) $s_{\alpha_{i,j}}(\Phi) \subset \Phi$ where the automorphism $s_{\alpha_{i,j}}$ of $X^*(T)$ is defined as

$$x \mapsto x - < x, \alpha_{i,j}^\vee > \alpha_{i,j},$$

(iii) the automorphisms $s_{\alpha_{i,j}}$ generate S_n, a finite group.
Under the isomorphisms of $X^*(T)$ and $X_*(T)$ with $\mathbf{Z}^n$ respectively, by identifying χ_i with the e_i and λ_i with the e_i, the roots are $e_i - e_j$, the coroots are $e_i - e_j$ (with the identity bijection between Φ and $\Phi^\vee$) and the pairing on $\mathbf{Z}^n$ is the standard one; that is, $< e_i, e_j >= \delta_{i,j}$.

Root datum for SL_2.

Here $T = \{diag(t, t^{-1}) : t \in K^*\}$ and $X^*(T) = \mathbf{Z}\chi$ with $\chi : diag(t, t^{-1}) \mapsto t$. Also, $X_*(T) = \mathbf{Z}\lambda$ where $\lambda : t \mapsto diag(t, t^{-1})$. The set of roots is $\Phi = \{\alpha, -\alpha\}$ where $\alpha = 2\chi$. The set of coroots is $\Phi^\vee = \{\alpha^\vee, -\alpha^\vee\}$ where $\alpha^\vee = \lambda$. Clearly $< \alpha, \alpha^\vee >= 2$ as the composite $\alpha \circ \alpha^\vee$ takes any t to t^2. The automorphism $s_\alpha = s_{-\alpha}$ interchanges α and $-\alpha$ and the Weyl group is $\{1, s_\alpha\}$. Identifying $X^*(T)$ and $X_*(T)$ with $\mathbf{Z}$ by means of $\chi \mapsto 1$ and $\lambda \mapsto 1$ respectively, the roots are $\Phi = \{2, -2\}$ and the coroots are $\Phi^\vee = \{1, -1\}$. The pairing $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is the standard one $(x, y) \mapsto xy$ and the bijection between roots and coroots is $2 \mapsto 1, -2 \mapsto -1$.

Root datum for $G = PGL_2$.

Recall that this group is the quotient group of GL_2 by its center (more precisely, $G(R) = GL_2(R)/R^*$ for all rings R).
Here T is the image of the diagonal torus of GL_2 modulo the nonzero scalar matrices over K. One has $X^*(T) = \mathbf{Z}\alpha$, where $\alpha : diag(t_1, t_2) \mapsto \frac{t_1}{t_2}$. The roots are $\Phi = \{\alpha, -\alpha\}$. Identifying $X^*(T)$ with $\mathbf{Z}$ by sending α to 1, the roots are $\{1, -1\}$. It is clear that $X_*(T) = \mathbf{Z}\lambda$ and the set of coroots is $\Phi^\vee = \{2\lambda, -2\lambda\}$, where $\lambda : t \mapsto diag(t, 1)$. Note that the diagonal matrix above is to be interpreted modulo the scalar matrices. Identifying λ with 1 for an isomorphism of X_* with $\mathbf{Z}$, the coroots become $\{2, -2\}$. The pairing is the standard one $(x, y) \mapsto xy$, as in the SL_2 case and, the bijection $\alpha \mapsto \alpha^\vee$ from Φ to $\Phi^\vee$ is the obvious one $1 \mapsto 2, -1 \mapsto -2$.

Root datum for $G = \mathbf{G_m}$.
This example has really been covered under the GL_n provided we read it properly. There are no roots and no coroots.

Definition of coroots for a general reductive group.

Let G be any connected reductive group and let T be a maximal torus. We start by observing that the Weyl group $W = N_G(T)/T$ acts on $X^*(T)$ and stabilizes the set Φ of roots. Indeed, if $s \in W$, $\alpha \in \Phi$, $X \in \mathcal{G}_\alpha$ and $t \in T$, then

$$Ad(t)(Ad(s)(X)) = Ad(s)(Ad(s^{-1}ts)(X)) = Ad(s)(\alpha(s^{-1}ts).X)$$

$$= \alpha(s^{-1}ts).Ad(s)(X) = (s.\alpha)(t).Ad(s)(X)$$

which means that $s.\alpha$ is again a root. We denote the codimension 1 subtorus $Ker(\alpha)^0$ of T by T_α and write $G_\alpha = C_G(T_\alpha)$. We know from earlier study that G_α is a reductive group whose derived subgroup is a semisimple group of rank 1. Clearly, of course, T is a maximal torus of G_α. In other words, $[G_\alpha, G_\alpha]$ is either isomorphic to SL_2 or to PGL_2. From the explicit discussion of these two groups, it is clear that the Weyl group of G_α has only one nontrivial element (which is s_α) and there is a unique element $\alpha^\vee \in X_*(T)$ such that

$$s_\alpha(x) = x - < x, \alpha^\vee > \alpha$$

for all $x \in X^*(T)$. Further, we have

$$< \alpha, \alpha^\vee >= 2.$$

Thus, we have defined for any connected reductive group G and any root α, a coroot $\alpha^\vee$. Therefore, we just bear in mind that for any root α, we locate a natural copy of the rank 1 semisimple group SL_2 or PGL_2 which contains $Ker(\alpha)^0$ and use it to find $\alpha^\vee$.
As a matter of fact, one can find $\alpha^\vee$ explicitly in the following fashion. Let $\chi \in X^*(T), \alpha \in \Phi$. As χ is a regular function on T, it is the restriction of a regular function f on G_α which generates a G_α-module V by means of left translations. Writing f as a sum of non-zero weight functions in V, and restricting their domains to T, we see that χ must be the weight corresponding to one of them. If U_α denotes the unipotent root subgroup corresponding to α, recall that G_α is generated by T, U_α and $U_{-\alpha}$ and there is an isomorphism of algebraic groups

$$\epsilon_\alpha : \mathbf{G_a} \to U_\alpha$$

satisfying $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x)$ for all $t \in T$. Now, if $v \in V$, then

$$\epsilon_\alpha(x)v = \sum_{i \geq 0} x^i v_i \cdots (\diamondsuit)$$

as $\epsilon_\alpha(x)$ is a polynomial in x. Suppose now that v is a weight vector with some weight μ. Then, applying an element $t \in T$ to $(\diamondsuit)$, we get

$$\epsilon_\alpha(\alpha(t)x)\mu(t)v = \mu(t)\sum_i \alpha(t)^i x^i v_i = \sum_i x^i t v_i.$$

Therefore, we have, for each i, $tv_i = \mu(t)\alpha(t)^i v_i$; that is, v_i is a weight vector with weight $\mu + i\alpha$ (in the usual additive notation). Hence, if $\{V_\mu : \mu \in X^*(T)\}$ is the set of weight spaces for V, it follows that $V_\chi \neq 0$ and the the space

$\sum_{i \in \mathbf{Z}} V_{\chi+i\alpha}$ is invariant under the whole of G_α. In other words, $s_\alpha \chi = \chi + i\alpha$ for some integer i. The coroot $\alpha^\vee$ is defined as $\chi \mapsto -i$ with i as above.

The root datum of a general reductive group.

Attached to each pair (G, T), where G is a connected reductive group and T a maximal torus, one associates the following 6-tuple which is called its root datum denoted by $\Psi(G, T)$:

$$X, X^\vee, <.,.>, \Phi, \Phi^\vee, \Phi \to \Phi^\vee.$$

Here, $X = X^*(T), X^\vee = X_*(T), <.,.>: X \times X^\vee \to \mathbf{Z}; (\chi, \lambda) \mapsto n$ where $\chi \circ \lambda : t \mapsto t^n$. Also, note that $\Phi, \Phi^\vee$ are finite sets in bijection by a map $\alpha \mapsto \alpha^\vee$ which satisfies :
(i) $< \alpha, \alpha^\vee >= 2$, and
(ii) $s_\alpha(\Phi) \subset \Phi$, where $s_\alpha \in Aut(X)$ is : $x \mapsto x- < x, \alpha^\vee > \alpha$.
Moreover, the subgroup $W(\Psi)$ of $Aut(X)$ generated by s_α as α varies over Φ, is finite.

What are root data and when are they isomorphic/isogenous ?

A general root datum is a 6-tuple

$$\Psi = (X, X^\vee, <.,.>, \Phi, \Phi^\vee, \Phi \xrightarrow{\vee} \Phi^\vee)$$

where $X, X^\vee$ are free abelian groups of finite rank which are in duality by means of a pairing $<.,.>: X \times X^\vee \to \mathbf{Z}$, $\Phi, \Phi^\vee$ are finite subsets of $X, X^\vee$ respectively, which are in bijection by a map $\alpha \leftrightarrow \alpha^\vee$ which has the following properties :
(i) $< \alpha, \alpha^\vee >= 2$, and
(ii) $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$, where $s_\alpha \in Aut(X)$ maps any x to $x- < x, \alpha^\vee > \alpha$ and $s_{\alpha^\vee} \in Aut(X^\vee)$ maps any y to $y- < \alpha, y > \alpha^\vee$.
A root datum is said to be *reduced* if the only multiples of a root α which are again roots are $\pm\alpha$. Note that this happens for a root datum corresponding to a connected reductive group. Moreover, the above properties imply that the subgroup $W(\Psi)$ of $Aut(X)$ generated by s_α as α varies over Φ, is finite.

Two root data Ψ, Ψ' are said to be isomorphic, if there is an isomorphism $\theta : X \to X'$ which maps Φ onto Φ', such that the corresponding cohomomorphism $\theta^\vee : (X')^\vee \to X^\vee$ maps $(\Phi')^\vee$ onto $\Phi^\vee$.
A more general notion is that of isogeny. An isogeny from the root datum Ψ' to the root datum Ψ is an injective homomorphism $\theta : X' \to X$ whose image has finite index in X, and satisfies the following property :
there is a bijection $\Phi \to \Phi'$; $\alpha \mapsto \alpha'$ and, for each $\alpha \in \Phi$, there is a power $q(\alpha)$ of the characteristic exponent p of K such that

$$\theta(b(\alpha)) = q(\alpha)\alpha \ , \ \theta^\vee(\alpha^\vee) = q(\alpha)(b(\alpha))^\vee.$$

Recall that the characteristic exponent of K is 1 if K has characteristic zero, and is the characteristic of K if the latter is positive. If there is an isogeny (that is, a surjective homomorphism with finite kernel) f from an algebraic group G to G', and T, T' are respective maximal tori, then f induces an injective homomorphism from $X(T')$ to $X(T)$; the point is that it turns out that the corresponding root data will be isogenous.

Root data for other classical groups.

Root datum for SL_{n+1}.
$T = \{diag(t_1, \cdots, t_{n+1} : t_1 t_2 \cdots t_{n+1} = 1\}$ is a maximal torus.

$$X^*(T) = \oplus_{i=1}^{n+1} \mathbf{Z}\chi_i / \mathbf{Z}\chi$$

where $\chi_i : diag(t_1, \cdots, t_{n+1}) \mapsto t_i$ and $\chi = \sum_i \chi_i$.

$$X_*(T) = \{\sum_{i=1}^{n+1} a_i \lambda_i \in \sum_i \mathbf{Z}\lambda_i : \sum_i a_i = 0\}$$

where $\sum_i a_i \lambda_i : t \mapsto diag(t^{a_1}, \cdots, t^{a_n}, t^{a_{n+1}})$.
The pairing is the obvious one.
Further, if $\overline{\chi_i}$ is the class of χ_i in $X^*(T)$, then the set of roots is

$$\Phi = \{\overline{\chi_i} - \overline{\chi_j} \ ; \ 1 \leq i \neq j \leq n+1\}.$$

The corresponding set of coroots is $\Phi^\vee = \{\overline{\lambda_i} - \overline{\lambda_j} \ ; \ 1 \leq i \neq j \leq n+1\}$.

Root datum for $G = SO_{2n+1}$.
Here, we assume Char $K \neq 2$.
Then $G = SO(f)$, where f is the symmetric bilinear form on K^{2n+1} given by

$$f(x, y) = 2x_0 y_0 + \sum_{i=1}^{n} (x_i y_{n+i} + x_{n+i} y_i).$$

In other words,

$$G = \{g \in SL_{2n+1} : g^t F g = F\}$$

where $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$.
Note that the Lie algebra $\mathcal{G}$ consists of $(n+1) \times (n+1)$ matrices A of trace 0 which satisfy $A^t F = -FA$.
Now $T = \{diag(1, t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1})\}$ is a maximal torus of G and $X^*(T) = \oplus_{i=1}^n \mathbf{Z}\chi_i$, where $\chi_i : diag(1, t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}) \mapsto t_i$. Also, $X_*(T) = \oplus_{i=1}^n \mathbf{Z}\lambda_i$, where λ_i maps any t to the $(2n+1) \times (2n+1)$ diagonal matrix whose entries are 1 excepting the entry t at the $(i+1)$-th and the entry t^{-1} at

the $(n+i+1)$-th place.
The set $\Phi = \{\pm\chi_i, \pm\chi_i \pm \chi_j; i \neq j\}$ and $\Phi^\vee = \{\pm 2\lambda_i, \pm\lambda_i \pm \lambda_j; i \neq j\}$.

Root datum for $G = Sp_{2n}$.
We have the skew-symmetric bilinear form f on K^{2n} given by

$$f(x,y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i).$$

Then,

$$G = \{g \in SL_{2n} : g^t \Omega g = \Omega\}$$

where $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
The Lie algebra consists of all $(2n) \times (2n)$ matrices A such that $A^t\Omega = -\Omega A$.
Now $T = \{diag(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1})\}$ is a maximal torus and $X^*(T) = \oplus_{i=1}^n \mathbf{Z}\chi_i$, where $\chi_i : diag(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}) \mapsto t_i$. Also, $X_*(T) = \oplus_{i=1}^n \mathbf{Z}\lambda_i$, where λ_i maps any t to the $(2n) \times (2n)$ diagonal matrix whose entries are 1 excepting the entry t at the i-th and the entry t^{-1} at the $(n+i)$-th place.
The set of roots is

$$\Phi = \{\pm 2\chi_i, \pm\chi_i \pm \chi_j; i \neq j\}$$

and the set of coroots is

$$\Phi^\vee = \{\pm\lambda_i, \pm\lambda_i \pm \lambda_j; i \neq j\}.$$

Root datum for $G = SO_{2n}$.
Once again, we assume that Char $K \neq 2$. Then $G = SO(f)$, where f is the symmetric bilinear form on K^{2n} given by

$$f(x,y) = \sum_{i=1}^{n} (x_i y_{n+i} + x_{n+i} y_i).$$

Then,

$$G = \{g \in SL_{2n} : g^t F g = F\}$$

where $F = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.
The Lie algebra consists of all $(2n) \times (2n)$ matrices A of trace 0 such that $A^t F = -FA$.
Also $T = \{diag(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1})\}$ is a maximal torus and $X^*(T) = \oplus_{i=1}^n \mathbf{Z}\chi_i$, where $\chi_i : diag(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}) \mapsto t_i$.
The set of roots is

$$\Phi = \{\pm\chi_i \pm \chi_j; i \neq j\}$$

and the corresponding set of coroots is

$$\Phi^\vee = \{\pm\lambda_i \pm \lambda_j; i \neq j\}.$$

§ The isomorphism theorem - the statement.

The main theorem of this chapter is due to Chevalley. Though he proved the theorem only in the semisimple case (where $X^\vee$, $\Phi^\vee$ are unnecessary) but it is quite easy to prove for tori any way. The main theorem is :

Chevalley's Isomorphism Theorem.
Let G, G' be connected reductive algebraic groups and T, T' their respective maximal tori. Let $\Psi = \Psi(G,T)$ and $\Psi' = \Psi(G',T')$ denote their root data. Let $\theta : X^*(T) \to X^*(T')$ be an isomorphism of the two root data. Then, there exists an isomorphism $f : G' \to G$ mapping T onto T' and inducing θ.

As a matter of fact, almost the same labour going into the proof of this, proves the :-

Chevalley's isogeny theorem.
Let G, G' be connected reductive algebraic groups and T, T' be their maximal tori respectively. Let $\Psi = \Psi(G,T)$ and $\Psi' = \Psi(G',T')$ denote their root data. Let $\theta : X^*(T') \to X^*(T)$ give an isogeny of the two root data. Then, there exists an isogeny f of G onto G', mapping T onto T' and inducing θ. This isogeny is uniquely determined upto composition with the inner automorphism of G effected by an(y) element of T.

Isogeny theorem implies isomorphism theorem.

Indeed, if θ is an isomorphism, then the isogeny theorem produces isogenies $f : G \to G'$ and $g : G' \to G$ corresponding, respectively, to θ and θ^{-1}. Then $g \circ f$ corresponds to the identity isogeny of the root datum of (G,T). Hence $g \circ f = Int(t)$ for some $t \in T$. This shows that f is actually an isomorphism whose inverse is $g' = Int(t^{-1}) \circ g$ (note that $g' \circ f = Id$ is clear, and using it, we get $f \circ g' \circ f = f$ which implies by surjectivity of f that $f \circ g' = Id$).

§ Isogeny theorem for tori.

The isogeny theorem will be proved first for groups of semisimple-rank ≤ 1 and then the result will be deduced for the general case. In this section, we start with the semisimple rank 0 case; that is, the case of algebraic tori. Recall that we now have tori T, T' and an injective homomorphism $\theta : X^*(T') \to X^*(T)$. Now, we know that the characters of a torus form a basis of its co-ordinate ring and, thus, there is a bijective correspondence between the group $Hom(T,T')$ of algebraic group homomorphisms and the group $Hom(X^*(T'), X^*(T))$ of abstract group homomorphisms. Thus, θ corresponds to an algebraic group homomorphism $f_T : T \to T'$ which is clearly an isogeny as θ and $\theta^\vee$ are.

§ Reformulation of isogeny theorem.

Having proved/observed the isogeny theorem in the case of tori, it is convenient to reformulate the main isogeny theorem as follows :

Let G, G', T, T', θ be as in the statement of the isogeny theorem. Let $f_T : T \to T'$ be the isogeny as above. Then, f_T extends to a homomorphism $f : G \to G'$.

Let us see why it suffices to show that f_T extends to an algebraic group homomorphism f from G to G'. Once this is shown, it automatically follows that f is an isogeny because f is surjective (as it maps U_α onto $U_{\alpha'}$) and $dimG = dimT + |\Phi| = dimT' + |\Phi'| = dimG'$. Indeed, the matching of dimensions shows that $Ker(f)$ is finite; of course, connectedness shows that f is surjective.

§ A lemma on transitivity of isogenies.

As mentioned earlier, the proof of the general case will be deduced from that for the groups of semisimple rank 1. The latter are upto isomorphism either SL_2 or PGL_2. To use these groups, we need the following transitivity result (again due to Chevalley !) :

Transitivity lemma.
(i) *Let $f_1 : G \to G_1$ and $f_2 : G \to G_2$ be isogenies of reductive algebraic groups, and suppose θ_1, θ_2 are the corresponding isogenies of the root data relative to maximal tori $T, f_1(T), f_2(T)$ of G, G_1, G_2 respectively. Suppose that θ is an isogeny of the root data of G_1 and G_2 satisfying $\theta_1 \circ \theta = \theta_2$. Then, there exists an isogeny $f : G_1 \to G_2$ corresponding to which the isogeny of root data is θ.*
(ii) *The same conclusion holds if we are given instead isogenies $g_1 : G_1 \to G$ and $g_2 : G_2 \to G$ such that $\theta_1 = \theta \circ \theta_2$ relative to the maximal tori $T, g_1^{-1}(T)$ and $g_2^{-1}(T)$ of G, G_1, G_2 respectively.*
Proof.
Now $Kerf_1$ being a finite, normal subgroup, must be central; in particular, it must be contained in (any, and therefore,) the maximal torus T. Clearly $Kerf_1 \subset Kerf_2$ since $\theta_1 \circ \theta = \theta_2$. Thus, we may write $f_2 = g \circ f_1$ for some (abstract group) isogeny $g : G_1 \to G_2$.
Let us write $T_1 = f_1(T), T_2 = f_2(T)$ for simplicity of notation. Consider the (algebraic group) homomorphism $g' : T_1 \to T_2$ whose cohomomorphism is the given θ. So, $g \circ f_1 = f_2 = g' \circ f_1$; this implies by surjectivity of f_1 that $g = g'$ on T_1. In other words, g is an algebraic group homomorphism on T_1 and θ is its corresponding cohomomorphism.
We will show that g is a morphism on the whole of G_1.
Under the given isogenies of the root data, suppose the roots $\alpha, \alpha_1, \alpha_2$ on T, T_1, T_2 correspond. In other words, $f_1(U_\alpha) = U_{\alpha_1}$ and $f_2(U_\alpha) = U_{\alpha_2}$; so $g(U_{\alpha_1}) = U_{\alpha_2}$. Now g is a morphism on U_{α_1} because, if $\theta(\alpha_2) = q(\alpha_1)\alpha_1$, then $g(X_{\alpha_1}(x)) = X_{\alpha_2}(c_{\alpha_1} x^{q(\alpha_1)})$ for each $x \in K^*$.

Towards showing that g is a morphism on G, we recall what the big cell is. Relative to some ordering of roots of Φ_1, the groups $U_1 := \prod_{\alpha_1>0} U_{\alpha_1}$ and $V_1 := \prod_{\alpha_1>0} U_{-\alpha_1}$ are maximal connected unipotent subgroups of G_1 which do not depend on the order in which the factors appear. They are isomorphic as varieties to the direct products of their respective factors. The groups T_1U_1 and T_1V_1 are opposite Borel subgroups intersecting in T_1; the variety $V_1T_1U_1$ is an open dense set in G_1 and is called the big cell. The big cell is also isomorphic as a variety to the direct product of its factors.
In our situation, the map g is, therefore, a morphism on the big cell of G_1. Hence, it is also a morphism on the union of the translates of the big cell - which is the whole of G_1. This proves version (i) of the lemma.
Let us consider the version (ii) where the isogenies go in the opposite direction. Look at the fibre product $G_3 = \{(x_1, x_2) \in G_1 \times G_2 : g_1(x_1) = g_2(x_2)\}^\circ$. As the projections p_1, p_2 from G_3 onto G_1, G_2 respectively, are isgenies, the group G_3 is reductive. Denote by π_1, π_2 the corresponding isogenies of root data with respect to the maximal tori T_1, T_2 and $p_1^{-1}g_1^{-1}(T) = p_2^{-1}g_2^{-1}(T)$ of G_1, G_2, G_3 respectively. Clearly, $g_1 \circ p_1 = g_2 \circ p_2$ which gives $\pi_2 \circ \theta_2 = \pi_1 \circ \theta_1 = \pi_1 \circ \theta \circ \theta_2$. Thus, injectivity of θ_2 implies $\pi_2 = \pi_1 \circ \theta$. We may now apply version (i) to the situation $p_1 : G_3 \to G_1, p_2 : G_3 \to G_2$. We obtain an isogeny $g : G_1 \to G_2$ whose corresponding isogeny of root data is θ. This finishes the proof.

§ **Isogeny theorem for semisimple-rank 1 groups.**

Proof for semisimple groups of rank 1.
We prove the theorem when G, G' are actually semisimple and have rank 1. We divide it into two cases :
Case I - $G' = PGL_2$:
Now, recall that *any* semisimple group H of rank 1 admits an isogeny onto PGL_2 (this basically comes from the fact that H/B is isomorphic to $\mathbf{P}^1$; so the action of H on H/B gives a homomorphism from H to Aut $(\mathbf{P}^1) = PGL_2$).
Let $f_1 : G \to PGL_2$ be such an isogeny, and write θ_1 for the corresponding isogeny of the root data. We need to find the value $q_1(\pm\alpha)$ where the roots of (G, T) are $\pm\alpha$. We claim that $f_1 : U_{-\alpha} \to f_1(U_{-\alpha})$ is an isomorphism (and so $q_1(\pm\alpha) = 1$). To see this, we show how to go back to v from the value $f_1(v)$ for $v \in U_{-\alpha}$. Writing $B = TU_\alpha$, the action of $f_1(v)$ on G/B can be restricted to $U_{-\alpha}B/B$ and can be evaluated at B/B to get vB/B. Finally, as $U_{-\alpha}B = U_{-\alpha}TU_\alpha$ is isomorphic to the direct product of its factors, there is an isomorphism $U_{-\alpha}B/B \to U_{-\alpha}$. Thus, we have $q_1(\alpha) = q_1(-\alpha) = 1$.
We may take $f_1(T) = T'$, the diagonal matrices of PGL_2. The isogeny θ_1 of root data corresponding to f_1 satisfies $\theta_1(\alpha') = \alpha$ and $\theta(\alpha') = q(\alpha)\alpha$. Thus, $\theta = q(\alpha)\theta_1$ from which we observe that the map $Fr_{q(\alpha)} \circ f_1$ is an isogeny realizing θ - here, Fr_d denotes the Frobenius map which raises each element to its d-th power.
Case II - $G' \neq PGL_2$.
Write an isogeny $g' : G' \to PGL_2$ and denote by ϕ the corresponding isogeny

of the root data. Considering the isogeny $\theta \circ \phi$ of root data, the previous case produces an isogeny $h : G \to PGL_2$ realizing it. Applying version (ii) of the transitivity lemma to h and g', we obtain an isogeny $f : G \to G'$ with θ as its corresponding isogeny of root data. This completes the proof.

Remark.
Looking at the possible root data arising from semisimple groups of rank 1, the above result also implies that the only possible semisimple groups of rank 1 are SL_2 and PGL_2.

Proof for reductive, semisimple-rank 1 groups.
Now, G, G' are reductive of semisimple-rank 1. Write $G = [G,G]T_0$ with T_0 its radical - a central torus. Then, the maximal torus T of G decomposes as $T = (T \cap [G,G])T_0$ and $T \cap [G,G]$ is a maximal torus of the semisimple group $[G,G]$; let us denote it by T_s. In general, G is only an almost direct product of $[G,G]$ and T_0 but, to begin with, let us tackle the case when it is actually a direct product. Likewise, we assume that G' is a direct product of its semisimple part and the radical. Then, $T = T_s \times T_0$ and, as well, for T'. Thus, $X^*(T), X_*(T), X^*(T'), X_*(T')$ and the isogeny θ of the root data are all direct products as well. To see the last-mentioned assertion, note first that $\theta(\alpha') = q(\alpha)\alpha$ gives $\theta(X^*(T'_s)) \subset X^*(T_s)$ as both lattices here have rank 1. Similarly, $\theta^\vee(\alpha^\vee) = q(\alpha)(\alpha')^\vee$ gives $\theta(X^*(T'_0)) \subset X^*(T_0)$ on using the observation that $X^*(T_0)$ and $X^*(T'_0)$ are the annihilators of $\alpha^\vee$ and $(\alpha')^\vee$ respectively in $X^*(T)$ and $X^*(T')$. This justifies our claim that θ is itself a direct product. Similarly, $\theta^\vee$ is also a direct product. Thus, the above two cases of tori and semisimple groups of rank 1 imply the result in our case.
Now, we no longer assume that G, G' are direct products. Therefore, we have isogenies

$$[G,G] \times T_0 \to G$$

and

$$G' \to G'/T'_0 \times G'/[G',G'].$$

Write ϕ, ϕ' for the corresponding isogenies of the root data. The above treatment of direct product implies that there is an isogeny

$$[G,G] \times T_0 \to G'/T'_0 \times G'/[G',G']$$

whose corresponding isogeny of root data is $\phi \circ \theta \circ \phi'$. The second version (ii) of the transitivity lemma now yields an isogeny $[G,G] \times T_0 \to G'$ corresponding to the isogeny $\phi \circ \theta$ of root data. Finally, the first version (i) of the transitivity lemma implies theexistence of an isogeny from G onto G' corresponding to θ. This completes the proof of the isogeny theorem for reductive groups of semisimple-rank 1.

§ **Deduction of isogeny theorem in general.**

The proof of the isogeny theorem will now be deduced from the case of semisimple-rank 1 groups. The notations to be used in the proof are as in the statement of the isogeny theorem. We break up the proof into some small simple steps.

Observation 1.
If Δ is a basis for Φ, then $\Delta' = \{\alpha' : \alpha \in \Delta\}$ is a basis of Φ'.
Indeed, the isogeny θ of the root data implies that each root in Φ' is a unique linear combination of the elements of Δ' with all coefficients rational and of the same sign. Forming the corresponding Φ'_+, we note that since no other positive multiple of any root in Φ' can be a root, no element α' of Δ' is expressible as a sum $\beta' + \delta'$ for some $\beta', \delta' \in \Phi'_+$. This proves that Δ' is indeed a basis.

Observation 2.
Let $f_T : T \to T'$ be the isogeny corresponding to $\theta : X^*(T') \to X^*(T)$. Then, f_T extends to an isogeny from G_α onto $G_{\alpha'}$ for each $\alpha \in \Delta$, where G_α is the group of semisimple-rank 1 generated by T, U_α, and $U_{-\alpha}$.
This simply follows from the semisimple-rank 1 case of the isogeny theorem (as reformulated later).

Observation 3.
Let $G'' \subset G \times G'$ be the subgroup generated by the graph $\{(x, f(x) : x \in \bigcup_{\alpha\in\Delta} G_\alpha\}$ (here, f is given on the union by the above observation). Then, the isogeny theorem follows if we show that the first projection π_1 maps G'' isomorphically onto G.
If the isomorphism of π_1 on G'' onto G is proved, then clearly $\pi_2 \circ \pi_1^{-1} : G \to G'$ gives a homomorphism which extends the map f on $\bigcup_{\alpha\in\Delta} G_\alpha$ and, therefore, extends f_T thereby proving the reformulated version of the isogeny theorem.

Observation 4.
Both projections of G'' to G and to G' are surjective and, hence, the group G'' is reductive.
The surjectivity is a simple consequence of the fact that the image contains a generating set $\bigcup_{\alpha\in\Delta} G_\alpha$ in case of G and similarly, for G'. Note that we have used the first observation here. The second statement follows since G, G' are reductive and, therefore, their radicals are tori; consequently the first assertion implies that the radical of G'' is also a torus. This means that G'' is reductive.

Notation :
For $\alpha \in \Delta$, consider the closed, connected, unipotent subgroup U''_α of G'' given by the graph of $f|U_\alpha$; that is, $\{(x, f(x) : x \in U_\alpha\}$. Similarly, the torus T'' in G'' is defined; it is a torus isomorphic to T via π_1. The groups U'', V'' are defined, respectively, to be the subgroups of G'' generated by U''_α's and $U''_{-\alpha}$'s as α runs over Δ. The groups U'', V'' are closed, connected and unipotent and are normalized by T''.

Observation 5.
If $\alpha \neq \beta$ are in Δ, then U''_β and $U''_{-\alpha}$ commute elementwise.
This is obvious from the corresponding result for G, G'.

Observation 6.
The set $C := V''T''U''$ is an open, dense subset of G''.
First, we note that C is an orbit in G'' for the action of $V'' \times T''U''$ by $(v, tu)g = vgtu$; so, C is open in its closure. We will show that this closure is the whole of G''. In fact, we will prove by induction on $n \geq 0$ that, for arbitrary roots $\alpha, \alpha_1, \cdots, \alpha_n \in \Delta$ that

$$U''_\alpha U''_{-\alpha_1} U''_{-\alpha_2} \cdots U''_{-\alpha_n} \subset \overline{C} \cdots\cdots (\spadesuit)$$

Suppose we have proved this; then clearly $U''_\alpha V'' \subset \overline{C}$ for all $\alpha \in \Delta$. Thus, keeping in mind that $C = V''T''U''$, we have for any $\alpha \in \Delta$,

$$U''_\alpha \overline{C} \subset \overline{C}.$$

Of course, evidently $U''_{-\alpha}\overline{C} \subset \overline{C}$ for any $\alpha \in \Delta$ and $T''\overline{C} \subset \overline{C}$. Since these subgroups generate the whole of G'', it follows that $G''\overline{C} \subset \overline{C}$; that is, $G'' = \overline{C}$. Now, we are left with proving ($\spadesuit$).
Then ($\spadesuit$) is obvious if $n = 0$. Assume that $n > 0$ and that we have proved it for all $m < n$. Now, if $\alpha \neq \alpha_1$, then we know that U''_α and $U''_{-\alpha_1}$ commute elementwise, and we can pass the first factor in ($\spadesuit$) once to the right. Repeating this, let us suppose that there is some $i < n$ so that $\alpha_i = \alpha$ - otherwise, we are done. Now,

$$U''_\alpha U''_{-\alpha_i} = U''_\alpha U''_{-\alpha} \subset G''_\alpha = \overline{U''_{-\alpha} T'' U''_\alpha}.$$

Thus, we get ($\spadesuit$) by the induction hypothesis. Thus the proof of observation 6 is complete.

Observation 7.
$C_{G''}(T'') = T''$; in particular, the torus T'' is maximal.
The centralizer of T'' in C is T'' itself, since this is true over G and G'. Hence the centralizer $C_{G''}(T'')$, being a connected (and hence irreducible) group, which contains T'' as an open, dense subset, must equal it (as T'' is closed).

Observation 8.
The projection $\pi_1 : G'' \to G$ is injective as well; so it is bijective.
Now, $(Ker\pi_1)^0$ is a normal subgroup which is disjoint from T''; hence it consists of unipotent elements only. By Lie-Kolchin theorem, it can be identified with a group of matrices in superdiagonal form and, therefore, is solvable and equals its radical. But, being a connected normal subgroup of the reductive group G'', it must consist of semisimple elements only. This forces it to be trivial. But then, $Ker\pi_1$ is a finite, normal subgroup and, hence, central; this means it is contained in T''. But T'' maps isomorphically (onto T) under π_1; thus π_1 is injective.

Conclusion of proof of isogeny theorem :
We claim : *The projection π_1 is an isomorphism of algebraic groups.*
As we observed (in observation 3), this claim will prove the isogeny theorem. To prove the claim, note that π_1 certainly maps the root subgroups U''_α isomorphically onto U_α for $\alpha \in \Delta$. Therefore, it maps all the root subgroups of G'' isomorphically onto the corresponding root subgroups of G; this is because these subgroups are (respective) Weyl group-conjugates of the above root subgroups. Consequently, π_1 induces an isomorphism (of varieties) between the corresponding big cells. As the translates of the big cell in any group form an open covering, it follows that π_1 is indeed an isomorphism on the whole of G''. The proof of the isogeny theorem is complete (at last !).

Remarks.
It should be noted that only the last observation 8 on bijectivity of π_1 which is an isomorphism at the level of maximal tori is not sufficient to imply it is an isomorphism. Here is a counter-example in characteristic 2. Consider $G = SO_{2n+1}$ which is with respect to the quadratic form $x_0^2 + \sum_{i=1}^n x_i x_{n+i}$ on K^{2n+1}. As Char $K = 2$, G fixes the basis vector v_0; hence it acts on $K^{2n+1}/Kv_0 \cong K^{2n}$. Look at the skew-symmetric form $\sum_{i=1}^n (x_i y_{n+i} - y_i x_{n+i})$ on K^{2n}. Then, the above action of G on K^{2n} gives an isogeny from G to the symplectic group $G' = Sp_{2n}$. This isogeny is not an isomorphism but gives an isomorphism at the level of the diagonal maximal tori in these groups.

§ **A presentation for G.**

Remarks.
As mentioned in the introduction, the proof of the isogeny theorem we have given here is due to Steinberg and is simpler than the proof in Springer's book. However, the proof in Springer's book has the advantage that it gives an explicit presentation of the algebraic group as an abstract group. Interestingly, though Steinberg's proof above does not give a presentation, the first presentation was given by Steinberg himself! Let us just recall now the presentation in Springer's book. The notation is as follows. Let G be a connected reductive group and $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be its root datum with respect to a maximal torus T. Fix a system Φ^+ of positive roots, a total ordering on Φ and fix also for each $\alpha \in \Phi$, an algebraic group homomorphism

$$\epsilon_\alpha : \mathbf{G}_a \to G$$

such that $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x)$ for all $t \in T, x \in K$. Thus, we have fixed a realization of the root system in G. Recall that if α, β are independent roots, then for any $i, j > 0$ with $i\alpha + j\beta$ a root, there is a constant $c_{\alpha,\beta,i,j}$ in K associated to it which satisfies

$$\epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_\alpha(x)^{-1} = \epsilon_\beta(y) \prod_{i,j>0; i\alpha+j\beta\in\Phi} \epsilon_{i\alpha+j\beta}(c_{\alpha,\beta,i,j} x^i y^j)$$

where the product is taken with respect to the total ordering. Recall that if $c'_{\alpha,\beta,i,j}$ is another set of structure constants (for the same ordering), these are related to the above constants as

$$c'_{\alpha,\beta,i,j} = \frac{c_{\alpha,\beta,i,j} c_{i\alpha+j\beta}}{c_\alpha^i c_\beta^j}$$

for some constants c_α satisfying $c_\alpha c_{-\alpha} = 1$.
Now, let Δ be the basis of Φ corresponding to the choice of Φ^+. If α, β are simple roots (that is, are in Δ), then the set of roots which are linear combinations of α, β form a root system $\Phi_{\alpha,\beta}$ and its intersection with Φ^+ forms a positive system for it.
Let us put $\Phi_2 = \cup\{\Phi_{\alpha,\beta} : \alpha \neq \beta \in \Delta\}$.
Let us also denote by π the canonical isomorphism

$$\pi : Hom(X, K^*) \to T$$

which satisfies $\theta(\chi) = \chi(\pi(\theta))$ for $\theta \in Hom(X, K^*), \chi \in X$.
So, we have for each $\chi \in X$, a homomorphism

$$\bar{\chi} : Hom(X, K^*) \to K^* \ ; \ \theta \mapsto \theta(\chi).$$

Similarly, for $\lambda \in X^\vee$, there is a homomorphism

$$\bar{\lambda} : K^* \to Hom(X, K^*) \ ; \ x \mapsto (\chi \mapsto x^{<\chi,\lambda>}).$$

Then, we have :

Presentation theorem :
Let $\mathbf{T}$ denote the abstract group $Hom(X, K^*)$ and, let $\mathbf{G}$ denote the (abstract) group generated by the group $\mathbf{T}$, and symbols $\eta_\gamma(x); x \in K, \gamma \in \Phi_2$ subject to the following relations :

$$\eta_\gamma(x+y) = \eta_\gamma(x)\eta_\gamma(y) \cdots\cdots\cdots\cdots\cdots\cdots (R_1)$$

$$\eta_\gamma(x)\eta_\delta(y)\eta_\gamma(x)^{-1} = \eta_\delta(y) \prod_{i,j>0; i\gamma+j\delta\in\Phi} \eta_{i\gamma+j\delta}(c_{\gamma,\delta,i,j}x^i y^j) \cdots (R_2)$$

whenever $\gamma, \delta \in \Phi_{\alpha,\beta}$ for some $\alpha \neq \beta \in \Delta$.

$$\theta\eta_\gamma(x)\theta^{-1} = \eta_\gamma(\bar{\gamma}(\theta)x) \cdots\cdots\cdots\cdots (R_3)$$

for $\theta \in \mathbf{T}$.
For any $\gamma \in \Phi_2$ and $n_\gamma := \eta_\gamma(1)\eta_{-\gamma}(-1)\eta_\gamma(1)$,

$$n_\gamma \eta_\gamma(x) n_\gamma^{-1} = \eta_{-\gamma}(-x) \cdots\cdots\cdots\cdots (R_4)$$

$$n_\gamma^2(x) = (-1)^{<\chi,\gamma^\vee>} \ \forall \ \chi \in X \cdots\cdots\cdots (R_5)$$

$$\eta_\gamma(x)\eta_{-\gamma}(-x^{-1})\eta_\gamma(x) = \overline{\gamma^\vee}(x) n_\gamma \ \forall \ x \in K^* \cdots (R_6)$$

$$n_\alpha n_\beta n_\alpha \cdots = n_\beta n_\alpha n_\beta \cdots\cdots\cdots\cdots\cdots\cdots \;(R_7)$$

for all $\alpha \neq \beta \in \Delta$ where the number of factors is the order $m(\alpha, \beta)$ of $s_\alpha s_\beta$. Then, the isomorphism $\pi : Hom(X, K^*) \to T$ extends to an isomorphism of abstract groups $\pi : \mathbf{G} \to G$ such that $\pi(\eta_\gamma(x)) = \epsilon_\gamma(x)$ for all $\gamma \in \Phi_2, x \in K$.

§ **Steinberg's presentation.**

For any simply-connected algebraic group G, Steinberg gave a presentation of $G(K)$ (this works for $G(k)$ for *any* field k provided G is 'k-split' and $|k| > 3$). We recall this interesting presentation without proof. Steinberg's presentation for $G(K)$ is the following :
$G(K)$ is generated by $x_\alpha(t)$, as α varies over roots (for some maximal torus) and t varies over K. The defining relations can be taken to be :

$$x_\alpha(s+t) = x_\alpha(s)x_\alpha(t) \ ,$$

$$[x_\alpha(s), x_\beta(t)] = \prod_{i\alpha + j\beta \in \Phi} x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j} s^i t^j) \ , and$$

$$h_\alpha(st) = h_\alpha(s)h_\alpha(t)$$

where the constants $c_{\alpha,\beta,i,j}$ are as before and $h_\alpha(t) := w_\alpha(t)w_\alpha(1)^{-1}$ with $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ for $t \in K^*$. The generators $x_\alpha(t)$ can be thought as $\epsilon_\alpha(t)$.
For $SL_2(K)$, the above commutator relations become vacuous and Steinberg's presentation is the following :
$SL_2(K)$ is generated by $x(a), y(a)$ (for $a \in K$) with the defining relations

$$x(a+b) = x(a)x(b) \ , \ y(a+b) = y(a)y(b),$$
$$w^+(a)x(b)w^+(a)^{-1} = y(-ba^{-2}),$$
$$w^-(a)y(b)w^-(a)^{-1} = x(-ba^{-2}),$$
$$h^+(ab) = h^+(a)h^+(b),$$
$$h^-(ab) = h^-(a)h^-(b)$$

where

$$w^+(a) = x(a)y(-a^{-1})x(a),$$
$$w^-(a) = y(a)x(-a^{-1})y(a),$$
$$h^+(a) = w^+(a)w^+(1)^{-1},$$
$$h^-(a) = w^-(a)w^-(1)^{-1}.$$

The generators $x(a), y(a)$ can be thought of as

$$x(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \ , \ y(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

Each root datum comes from a reductive group

§ Introduction.

As before, our algebraic groups are defined over an algebraically closed field K and are usually identified with their groups of K-rational points. The existence theorem for a group for a given root system is an easier problem in comparison with the uniqueness problem we discussed prior to this. Firstly, we just outline the basic philosophy for the proof.
To start with, any abstract root system Φ with a basis $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ enables one to define a finite-dimensional Lie algebra in characteristic zero. Recalling briefly, one considers the free Lie algebra generated by $3l$ symbols $x_i, y_i, h_i (1 \leq i \leq l)$. On this free Lie algebra, if we force the 'Serre' relations

$$[x_i, y_i] = h_i \ , \ [x_i, y_j] = 0 \ \forall \ i \neq j$$

$$[h_i, h_j] = 0 \ \forall \ i, j$$

$$[h_i, x_j] = < \alpha_j, \alpha_i > x_j \ \forall \ i, j$$

$$[h_i, y_j] = - < \alpha_j, \alpha_i > y_j \ \forall \ i, j$$

$$(ad \ x_i)^{-<\alpha_j, \alpha_i>+1}(x_j) = 0 \ \forall \ i \neq j$$

$$(ad \ y_i)^{-<\alpha_j, \alpha_i>+1}(y_j) = 0 \ \forall \ i \neq j$$

where $< \alpha_j, \alpha_i >$ are the corresponding Cartan integers, one obtains a finite-dimensional semisimple Lie algebra $\mathcal{G}$ as Serre proved. Then, the next step is to show that this Lie algebra has a certain basis $\{x_\alpha; \alpha \in \Phi\} \cup \{h_i; i \leq l\}$ with certain special properties like the structure constants being in $\mathbf{Z}$ (what is known as a Chevalley basis). All this depends only on the root system really. The $\mathbf{Z}$-span $\mathcal{G}(\mathbf{Z})$ of a Chevalley basis is then a lattice in $\mathcal{G}$. For any $m \geq 0$ and any $\alpha \in \Phi$, the operators $(ad \ x_\alpha)^m/m!$ leave $\mathcal{G}(\mathbf{Z})$ invariant; hence the operator $exp(ad \ x_\alpha)$ itself leaves this lattice invariant (as $ad \ x_\alpha$ is nilpotent). Then, the group A of inner derivations of $\mathcal{G}$ is a matrix group which has the subgroup G generated by all $exp(ad \ cx_\alpha)$ as c varies in $\mathbf{Z}$. This G gives the algebraic group sought for, as it leaves the lattice $\mathcal{G}(\mathbf{Z})$ invariant, and hence consists of some integral matrices of determinant 1. Indeed, if T is a general indeterminate, then the matrix group generated by all $exp(ad \ Tx_\alpha)$ has entries from $\mathbf{Z}[T]$ and determinant 1. Specializing T to elements of any field (including finite fields), we get an algebraic group G. This is the group of adjoint type. Similarly, using other faithful representations of the Lie algebra (the adjoint representation was used in the above construction), one can construct other 'covers'. The main point to note for a general faithful representation, one needs to choose an 'admissible' lattice - a lattice which is invariant under $U(\mathcal{G})_{\mathbf{Z}}$.
We shall discuss the approach in Springer's book and it should be noted that

a simple proof for the semisimple case appears in Steinberg's famous 'Lectures on Chevalley groups' as well.

§ Statement of existence theorem and reductions.

The existence theorem asserts :
Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be a root datum. Then, there is a connected reductive (linear algebraic) group G defined over K and a maximal torus T in it such that the corresponding root datum $\Psi(G,T) = \Psi$.

We break up the existence problem into one for 'smaller' root data. We start by observing what may be considered reasonable 'pieces'.
For a connected, reductive group G, we know that the radical $R(G)$ is a central torus and there is an isogeny

$$[G,G] \times R(G) \to G.$$

The group $[G,G]$ is semisimple, and the root datum of (G,T) for any maximal torus T is determined in terms of those of $[G,G]$ and $R(G)$. More precisely, look at the root datum $(X, \Phi, X^\vee, \Phi^\vee)$ of (G,T). If Q and $Q^\vee$ are the subgroups of X and $X^\vee$ generated, respectively, by the roots Φ and the coroots $\Phi^\vee$, one considers the subgroups

$$Q^\perp = \{y \in X^\vee : (Q, y) = 0\}$$

and

$$\tilde{Q} = \{x \in X : \mathbf{Z}x \cap Q \neq 0\}$$

of $X^\vee$ and X respectively.

We have (Springer, P.135) that :
(i) over $\mathbf{R}$, Q generates the orthogonal complement of $(Q^\vee)^\perp$ and Φ maps injectively into $X/(Q^\vee)^\perp$;
(ii) $(X/(Q^\vee)^\perp, \Phi, ((Q^\vee)^\perp)^\perp, \Phi^\vee)$ is the root datum of $([G,G], T_1)$ where T_1 is the subtorus of T generated by $Im(\alpha^\vee); \alpha \in \Phi$; and
(iii) $(X/(Q^\vee)^\perp \oplus X/\tilde{Q}, \Phi \oplus (0), ((Q^\vee)^\perp)^\perp \oplus Q^\perp, \Phi^\vee \oplus (0))$ is the root datum of $[G,G] \times R(G)$ with respect to the maximal torus $T_1 \times R(G)$.

The product map $[G,G] \times R(G) \to G$ gives an injection of the corresponding character groups; that is,

$$i : X \to X/(Q^\vee)^\perp \oplus X/\tilde{Q}.$$

Finally, the image of this injection is seen to be (Springer, 8.1.10)

$$\{(y + (Q^\vee)^\perp, z + \tilde{Q} : y - z \in (Q^\vee)^\perp \oplus \tilde{Q}\}.$$

We shall not need the precise information we have recalled above but it is motivating and would be useful if one starts discussing structure theory over general fields. Now, for the root datum corresponding to a semisimple group, the roots generate a subgroup of finite index in the character group (Springer, 8.1.11). In other words, the existence theorem would reduce to the case of a root datum with Φ of finite index in X (that is, we must produce a connected, semisimple group and a maximal torus in it giving such a root datum. Such an existence theorem was proved by Chevalley.

§ Producing groups for isogenic root data

Proposition 1.
Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ and $\Psi_0 = (X_0, \Phi, X_0^\vee, \Phi^\vee)$ be root data where $X \supset X_0 \supset \Phi$ with $[X : X_0] < \infty$. If there is a connected reductive group G_0 and a maximal torus T_0 realizing Ψ_0 as its root datum, then there is a connected reductive group G and a maximal torus T realizing Ψ as its root datum.

Before going through the proof in detail, we indicate how it can be proved which will also show why it should be true. As usual, in proving an existence result, the job becomes easier if we have located the candidate - this is all the more emphasised by noting that we have already proved a uniqueness theorem which would imply that a candidate which works would be *the* correct candidate. Choosing T to be a torus whose character group is X, the assumption that $\Phi \subset X_0$ implies that the Weyl group of Φ acts on T. Of course, T would be the maximal torus of the candidate G that we would like to choose. The inclusion $X_0 \subset X$ gives a homomorphism $\theta : T \to T_0$. Now, relative to the maximal torus T_0 of G_0, one has a system Φ^+ of positive roots, a system Δ of simple roots, a Borel subgroup B_0 and a realization of Φ in G_0 - that is, suitable homomorphisms $\epsilon_\alpha : \mathbf{G}_a \to G_0$ for $\alpha \in \Phi$. If w_0 is the longest element of W (with respect to Φ^+), the big cell $B_0 w_0 B_0$ of G_0 is an open dense subvariety. Now $B_0 w_0 B_0 \cong T_0 \times U_0 w_0 U_0$, where U_0 is the unipotent radical of B_0. With these choices, one can identify the function field $K(G_0)$ with $K_0 := K(T_0 \times U_0 w_0 U_0)$ and the function field $K(G) = K(T \times U_0 w_0 U_0)$ (of the group G yet to be constructed !) with $K_0(\chi_1, \cdots, \chi_m)$, where $\chi_1, \cdots, \chi_m \in X$ generate X modulo X_0. Thus, we have got the candidate of the function field of the G which was to be constructed. Now, the left action of G_0 on $K[G_0]$ defines an action as automorphisms of the field $K(G_0)$. Let us bear in mind that G will be generated by T and the various root subgroups. Thus, we need to make the root subgroups of G_0 act on $T \times U_0 w_0 U_0$. If we have a suitable action, then we will define G to be the group of automorphisms of the field $\tilde{K} := K(T \times U_0 w_0 U_0) = K_0(\chi_1, \cdots, \chi_m)$ generated by T and the root subgroups. Finally, G will be shown to be an algebraic group using induced representations. That also makes it possible to note that G is connected and reductive and that θ is a central isogeny.
To make the root subgroups of G_0 act, we need to recall some things proved in earlier lectures (see P. 151). Recall the commutator formulae for root subgroups

(with $\alpha \neq \pm\beta$) :

$$[\epsilon_\alpha(x), \epsilon_\beta(y)] = \prod_{i,j>0; i\alpha+j\beta\in\Phi} \epsilon_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}x^i y^j).$$

From this, it follows that for any *simple* root α, the α-component x_α of any element $u = \prod_{\beta>0} \epsilon_\beta(x_\beta$ of U_0 is uniquely defined; it is denoted by $\xi_\alpha(u)$. Thus, for any element $x = t_0 u w_0 u'$ of the big cell $\Omega = T_0 U_0 w_0 U_0$, one may define $\xi_\alpha(x) = \xi_\alpha(u)$ for any simple α. For any simple α, one denotes by Y_α, the elements $y \in U_0 w_0 U_0$ for which $\xi_\alpha(y) \neq 0$. As usual, n_α denotes (see 8.1.4(i)) the lift $\epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$ of s_α in $N_{G_0}(T_0)$. Then, we already know the following result (lemma 8.5.9 in Springer) which is basically a consequence of computations in the corresponding SL_2 and the normal form of Bruhat decomposition.
:

there is a morphism $\phi_\alpha : Y_\alpha \to Y_\alpha$ such that for $t \in T_0, y \in Y_\alpha$

$$n_\alpha(ty) = (s_\alpha t)(\alpha^\vee.(-\xi_\alpha(y))^{-1}.\phi_\alpha(y) \cdots\cdots (\bullet)$$

We also have $\xi_\alpha(\phi_\alpha y) = -\xi_\alpha(y)$ and ϕ_α^2 =Identity. Finally, $\Omega \cap n_\alpha\Omega = T_0 Y_\alpha$.

We shall use the above notations of $Y_\alpha, n_\alpha, \xi_\alpha$ etc. in the proof of the proposition now.

Proof of proposition 1.
Inductively, it suffices to consider the case when $lX \subset X_0$ for some prime l. In such a case, one may choose 'characters' $\chi_1, \cdots, \chi_m$ in X which generate the quotient X/X_0 as an $\mathbf{F}_l$-vector space. In a short while, we will see how to use this. We shall use the notations put down just above. For any $\alpha > 0$ and $x \in K$, the action of ϵ_α as automorphisms of the above field is defined as :

$$u_\alpha(x)(t, y) := (t, \epsilon_\alpha(\alpha(t)^{-1}x)y).$$

There is an automorphism corresponding to each of the elements n_α of G_0 for simple roots; it is defined by the formula $(\bullet)$ above. Note that G is the group defined as the automorphisms of the field $\tilde{K} = K(T \times U_0 w_0 U_0)$ generated by T, u_α(x)'s for $x \in K$ and positive α and, n_β's for simple β. We observe that all these automorphisms stabilize the function field $K(T_0 \times U_0 w_0 U_0)$ of G_0, and induce automorphisms of this latter field which come from left translation action on G_0. This means that the restriction of the G-action to $K(G_0)$ is induced by the left translation by an element of G_0. One has then a surjective homomorphism $\theta : G \to G_0$ which extends the homomorphism from T to T_0. The surjectivity actually uses the presentation of G_0 as recalled (without proof) in the lectures on uniqueness. Specifically, one needs relations R_1 and R_2 there to be satisfied by the $u_\alpha(x)$ ($\alpha > 0, x \in K$) and the relations

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x) \ \forall \ t \in T,$$

and for β simple,

$$n_\beta t n_\beta^{-1} = s_\beta(t) \;\forall\; t \in T.$$

These are all verified directly by computation.

To compute the kernel, it is useful to realize that $K(G)$ as $K_0(\chi_1, \cdots, \chi_m)$ and realize that elements of the kernel of θ act as K_0-automorphisms of the latter field. If $g \in Ker(\theta)$, then $g\chi_h = \zeta_h \chi_h$ for some l-th root of unity ζ_h. For example, the elements $g \in T$ such that $\chi_h(g) = \zeta_h$ and $\psi(g) = 1$ for all $\psi \in X_0$ are in the Kernel. If p is the characteristic exponent of K, then θ must be surjective if $p = l$ and must have kernel of order l^m if $p \neq l$.

Now, we verify that G is an algebraic group with $\Psi(G,T) = \Psi$. Roughly, the idea is to use induced representations.
Choose $a \in K[G_0]$ such that $(a\chi_h)^l \in K[G_0]$. Then, there is a finite-dimensional subspace of $K[G_0]$ containing all $(g.(a\chi_h))^l = \theta(g).(a\chi_h)^l$ with $g \in G$. Then, $a\chi_h (1 \leq h \leq m)$ lie in

$$K[T \times U_0 w_0 U_0] \cong R := K[T_1, \cdots, T_m, U_1, \cdots, U_n, U_1^{-1}, \cdots, U_n^{-1}].$$

By considering the degrees, one sees that the set of $r \in R$ such that r^l lies in a fixed finite-dimensional subspace of R itself spans a finite-dimensional subspace. Hence, $\tilde{K}$ contains a finite-dimensional subspace V_1 which is generated by all translates $g.(a\chi_h)$ for $g \in G, a \in K[G_0], h \leq m$. Let V_0 be a finite-dimensional subspace of $K[G_0]$ which is stable under the left translations of G_0; that is, G_0 is isomorphic to the group given by the restriction of the G_0-action to V_0. Then $V := V_0 + V_1$ generates $\tilde{K} = K(T \times U_0 w_0 U_0)$ and G is isomorphic to the group given by the restriction of the G-action to V.
For $\alpha > 0$, the maps u_α give rational representations of $\mathbf{G}_a$ in $K[T \times U_0 w_0 U_0]$, and the restrictions to V give finite-dimensional representations. For $-\beta$ with β simple, if we put $u_{-\beta}(x) = n_\beta u_\beta(-x) n_\beta^{-1}$, we have by restricting to V, actions of all u_α with $\alpha \in \Phi^+ \cup (-\Delta)$. Also, T acts. Thus, G – which is generated by T and all u_α with $\alpha \in \Phi^+ \cup (-\Delta)$ – is a connected algebraic subgroup of $GL(V)$. Note that the homomorphism $\theta : G(\subset GL(V)) \to G_0(\subset GL(V_0))$ (that is, when G, G_0 are identified with their restrictions to the finite-dimensional spaces V, V_0) is nothing but the restriction to V_0. As G_0 is reductive, G must be as well. Since T must be a maximal torus of G, θ is a central isogeny. As the weights of T in V_0 span X_0 and the T-translates of $a\chi_h$ contain a weight $\chi + \chi_h$, it follows that the weights of T in V span X. Therefore, Ψ is the root datum of (G,T).

§ Reducing to the adjoint, simple case.

Let Ψ be a root datum as before. If Q is the subgroup generated by Φ, and X' is defined to be the subgroup of X orthogonal to $Q^\vee$, then we have a lattice $X_0 := Q \oplus X'$ of finite index, and proposition 1 shows that it suffices to produce a group corresponding to X_0. In turn, therefore, it suffices to separately produce groups corresponding to character groups Q and X'. The latter case is trivial because we may take a torus. Thus, the existence theorem reduces to the case when $X =< \Phi >$ (that is the group to be produced is adjoint, semisimple). We may assume that Φ is irreducible. Thus, the existence problem reduces to the case when the group to be produced is an adjoint, simple algebraic group. The proof will turn out to be easier when the root system Φ (in $\mathbf{R} \otimes X$) is simply-laced; that is, there are no multiple bonds in the Dynkin diagram. The others (non-simply laced cases) are deduced in proposition 10.3.5 of Springer from the simply-laced ones by diagram folding using some automorphism. We discuss only the simply-laced case in detail here.

§ **Proof of existence theorem for simply-laced case.**

We assume that $X =< \Phi >$, where Φ is an irreducible, simply-laced root system in $V = \mathbf{R} \otimes X$. This means that $< \alpha, \beta^\vee >= 0$ or ± 1 for any two linearly independent roots α, β. Recall that here $(.,.)$ is a positive-definite, symmetric, W-invariant bilinear form on V and $< \alpha, \beta^\vee >:= 2\frac{(\alpha,\beta)}{(\beta,\beta)}$. With the above assumption on Φ, we have :

Theorem 2.
Consider a K-vector space with a basis $\{e_\alpha : \alpha \in \Phi\}$. Define

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi} K e_\alpha$$

where $\mathcal{H} = K \otimes X^\vee$. There exist constants $c_{\alpha,\beta} = 0, 1$ or -1 so that the requirements

$$[H, H'] = 0, [H, e_\alpha] =< \alpha, H > e_\alpha, [e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}, [e_\alpha, e_{-\alpha}] = 1 \otimes \alpha^\vee$$

define a Lie algebra structure.

The choice of the structure constants is possible because of some special properties of simply-laced root systems which we recall now.

Lemma 3.
Let α, β be independent roots. Then
(i) $\alpha \pm \beta$ is a root if and only if $< \alpha, \beta^\vee >= \mp 1$,
(ii) $\exists w \in W$ such that $\beta = w\alpha$, and
(iii) if $\alpha + \beta$ is a root, we have $\alpha^\vee + \beta^\vee = (\alpha + \beta)^\vee$.
Proof.
(i) We know that the β-string through α is $\alpha - c\beta, \cdots, \alpha, \cdots, \alpha + b\beta$ with $b - c = - < \alpha, \beta^\vee >$. By the hypothesis, all string lengths are at most 1. This clearly implies (i).
(ii) Since each root is in the W-orbit of some simple root, we may assume that α, β are simple. In this case, we know that there is a chain of simple roots $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_h = \beta$ with each $< \alpha_i, \alpha_{i+1}^\vee >= -1$. Therefore, we may take

$< \alpha, \beta^\vee >$ itself to be -1. In this case, we may take $w = s_\alpha s_\beta$.
(iii) This simply follows from the identity

$$< \alpha, \beta^\vee > = 2\frac{(\alpha,\beta)}{(\beta,\beta)}.$$

Defining signs/structure constants.
Assume that $(\alpha,\alpha) = 2$ for all roots - this is not a loss of generality as (ii) above shows. Let $\{e_1, \cdots, e_n\}$ be a basis of X and define $f(e_i, e_j) = (e_i, e_j)$ or 0 or $\frac{(e_i,e_i)}{2}$ according as to whether $i < j$ or $i > j$ or $i = j$. Then, we note that f is a $\mathbf{Z}$-valued bi-additive function on $X \times X$ such that (x, y) and $f(x, y)+f(y, x)$ have the same parity and $\frac{(x,x)}{2}$ and $f(x, x)$ have the same parity. With a choice of f with the just-mentioned properties, one can associate signs $c_{\alpha,\beta}$ for independent roots α and β, in the following manner. First, for any root γ, put $\epsilon(\gamma) = 1$ or -1 according as to whether γ is positive or negative.
If $\alpha + \beta$ is not a root, define $c_{\alpha,\beta}$ to be 0. If $\alpha + \beta$ is a root, define $c_{\alpha,\beta}$ to be $\epsilon(\alpha)\epsilon(\beta)\epsilon(\alpha + \beta)(-1)^{f(\alpha,\beta)}$.

Lemma 4.
(i) $c_{\alpha,\beta} = -c_{\beta,\alpha}$.
(ii) $c_{-\alpha,\beta}c_{\alpha,-\alpha+\beta} + c_{\beta,\alpha}c_{-\alpha,\alpha+\beta} =< \beta, \alpha^\vee >$.
(iii) If α, β, γ are linearly independent, then

$$c_{\alpha,\beta}c_{\alpha+\beta,\gamma} + c_{\beta,\gamma}c_{\beta+\gamma,\alpha} + c_{\gamma,\alpha}c_{\gamma+\alpha,\beta} = 0.$$

Proof.
(i) This follows from the property that (x, y) and $f(x, y)+f(y, x)$ have the same parity because $(\alpha, \beta) = \pm 1$.
(ii) This follows from considering the three cases $< \beta, \alpha^\vee >= 0, -1, 1$ separately. The first is easy. The second one needs the observation that $\alpha + \beta$ is a root but $-\alpha + \beta$ is not (from (i) of previous lemma). The third case is similar.
(iii) Without loss of generality, we may assume that $c_{\alpha,\beta}c_{\alpha+\beta,\gamma} \neq 0$. This means by (i) of previous lemma that $(\alpha, \beta) = -1 = (\alpha+\beta, \gamma)$. Therefore, either $(\alpha, \gamma) = 0$ and $(\beta, \gamma) = -1$ or the other way. We may assume the first possiblity. Then, the relation to be proved reduces to proving

$$f(\alpha, \beta) + f(\alpha + \beta, \gamma) + f(\beta, \gamma) + f(\beta + \gamma, \alpha) \equiv \; 1 \;\; mod \;\; 2.$$

This follows because $(\alpha, \beta) + (\alpha, \gamma) + 2f(\beta, \gamma)$ is odd.

The proof of the theorem follows from this lemma.

§ **Defining the Chevalley group.**

With the notations as above, we shall define a closed subgroup of $GL(\mathcal{G})$. Let T be a torus with $X(T) = X$. Now T acts on $\mathcal{G}$ by $t.e_\alpha = \alpha(t)e_\alpha$ and with trivial action on $\mathcal{H}$. Clearly, this defines T as a closed subgroup of $GL(\mathcal{G})$. Put $X_\alpha = ad(e_\alpha)$ for every root α. Then, X_α^2 is the linear map of $\mathcal{G}$ which sends $e_{-\alpha}$ to $-2e_\alpha$ and the rest of the generators e_β as well as the whole of $\mathcal{H}$ to 0. Write this linear map as $2X_\alpha^{(2)}$. Then, we see that

$$X_\alpha X_\alpha^{(2)} = 0 = (X_\alpha^{(2)})^2 \cdots\cdots\cdots\cdots (1)$$

$$X_\alpha^{(2)}[a,b] = [X_\alpha^{(2)}a, b] + [a, X_\alpha^{(2)}b] + [X_\alpha a, X_\alpha b] \cdots (2)$$

for any $a, b \in \mathcal{G}$. Then, an immediate consequence of the first identity is that the linear maps $u_\alpha(x)$ defined for each $x \in K$ by

$$u_\alpha(x) = 1 + xX_\alpha + x^2 X_\alpha^{(2)}$$

satisfy

$$u_\alpha(x+y) = u_\alpha(x)u_\alpha(y).$$

The second identity (2) shows that each $u_\alpha(x)$ is a Lie algebra automorphism. In other words, the images U_α of $u_\alpha : \mathbf{G}_a \to GL(\mathcal{G})$ are closed subgroups. Then, we define G to be the subgroup of $GL(\mathcal{G})$ generated by T and the U_α's. Then, it is a closed, connected subgroup. As we observed from the identity (2), the elements of U_α's are Lie algebra automorphisms of $\mathcal{G}$. Thus, G itself consists of Lie algebra automorphisms. If $q : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ is the map which goes down to $\Lambda^2\mathcal{G}$ to give the Lie algebra product, it commutes with the G-action; that is, we have $q \circ (g \otimes g) = g \circ q$ for all $g \in G$. Therefore, the Lie algebra of G (as a subalgebra of $gl(\mathcal{G})$) are derivations of $\mathcal{G}$. But, all derivations of $\mathcal{G}$ are inner; indeed, each derivation is of the form $ad(a)$ for a unique element a of $\mathcal{G}$ by the following lemma (note that the uniqueness assertion on a is trivial because $\mathcal{G}$ has center zero) :

Lemma 5.
With the above notations, every derivation δ on $\mathcal{G}$ is inner.
Proof.
If $H \in \mathcal{H}$, let us write

$$\delta(H) = d(H) + \sum_\alpha l_\alpha(H)e_\alpha$$

where d is an endomorphism of $\mathcal{H}$ and l_α's are linear functions on it. Now, if $H, H' \in \mathcal{H}$, then the rule $[\delta(H), H'] + [H, \delta(H')] = 0$ gives us for each $\alpha \in \Phi$ that

$$< \alpha, H > l_\alpha(H') =< \alpha, H' > l_\alpha(H).$$

Therefore, there exists constants $c_\alpha \in K$ with $l_\alpha(H) = c_\alpha < \alpha, H >$ for all H. Thus, the derivation $\delta - ad(\sum_\alpha c_\alpha e_\alpha)$ maps $\mathcal{H}$ into itself. Replacing δ by this element, we may assume that δ itself stabilises $\mathcal{H}$. The derivation property of δ applied to the pairs (a, e_α) with $a \in \mathcal{H}$ shows the existence of constants d_α's in K such that $\delta(e_\alpha) = d_\alpha e_\alpha$. Similarly, the derivation property of δ applied to the pairs (e_α, e_β) forces $d_{\alpha+\beta} = d_\alpha + d_\beta$ if $\alpha + \beta \in \Phi$. Further, $d_{-\alpha} = -d_\alpha$.

These relations imply that it must be represented by some $H_0 \in \mathcal{H}$; that is, $d_\alpha =< \alpha, H_0 >$ for all $\alpha \in \Phi$. In other words, $\delta = ad(H_0)$. Note that since we replaced δ earlier, the original δ is $ad(a)$ for some $a \in \mathcal{H} + \sum_\alpha Ke_\alpha = \mathcal{G}$.

The above assertion on derivations implies that $dim(G) = dimLie(G) \leq dim\mathcal{G}$. However, clearly $Lie(G)$ contains $Lie(T) \oplus \bigoplus_\alpha KX_\alpha$ as $Lie(U_\alpha = KX_\alpha$. In other words, $dim(G) \geq dim(T) + |\Phi| = dim(\mathcal{G})$ so that $Lie(G) = \mathcal{G}$. We have that T is a maximal torus of G (we know that the Lie algebra of the centralizers of tori in a group is its Lie algebra centralizer). We claim :

Proposition 6.
G is reductive and the root datum of (G,T) is isomorphic to the root datum $(X, \Phi, X^\vee, \Phi^\vee)$ we started with.
Proof.
Consider the centralizers G_α of the singular tori $Ker(\alpha)^0$ of codimension 1. Then, clearly G_α contains $U_\alpha, U_{-\alpha}$. Now the groups $U_\alpha, U_{-\alpha}$ stabilize the 3-dimensional subspace $K(1 \otimes \alpha^\vee) \oplus Ke_\alpha \oplus Ke_{-\alpha}$, and their action on this subspace clearly generate a non-solvable group - as a matter of fact, this latter group will be isomorphic to PSL_2. Thus, each root α is actually a root of the pair (G,T). Therefore, $dimR_u(G) = dim(G) - dim(T) - |\Phi| = 0$. Hence G is reductive.
Finally, via the form $(.,.)$ identify V with its dual. Then, $X^\vee$ is identified with a lattice in V and $\alpha^\vee = \alpha$ for each root. Also, for each root α, the cocharacter $\alpha^\vee$ takes, by definition, each $x \in K^*$ to the map which sends e_β to $x^{(\alpha,\beta)} e_\beta$ and fixes $\mathcal{H}$. In other words, the root datum of (G,T) is indeed $(X, \Phi, X^\vee, \Phi^\vee)$.