The Punctured Plane

How Topology Governs Analysis

Vishwambhar Pati

All loops in R^2 can be continuously shrunk to a point but there are loops in $R^2 - (0,0)$ that cannot be; for example, any circle enclosing the origin. This difference in the 'topology' of R^2 and that of $R^2 - (0,0)$ results in significant difference in the 'analysis' on these spaces. The main theme of this article is to illustrate how topology governs analysis.

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Most of you are probably familiar with analysis, another name for calculus. At its core are the fundamental concepts of limits, differentiation and integration of functions on R, and more generally R^n. What is topology? Perhaps some of you have studied metric spaces, and continuous maps between metric spaces. You may be aware that it has to do with Möbius strips, Klein bottles, doughnuts, knots and the like.

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To make a very crude definition, the objective of topology is to study continuity in its utmost generality, and to seek the right setting for this study. The basic objects that topology studies are called *topological spaces*, i.e. sets which have some additional set theoretic structure, governed by some axioms, that enables us to define the notion of a neighbourhood. For example, all the objects listed in the first paragraph are topological spaces. Once this is done, it is easy to define continuity of a map, just as one does for metric spaces. It turns out that to study continuity, it is not really necessary to have a metric. The axioms for topological spaces are set up so that all our familiar intuitive expectations about continuity are realised. On the other hand, the general-

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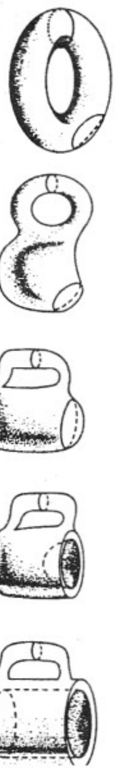

A* topologist *has been defined to be a mathematician who can't tell the difference between a doughnut and a cup of coffee!

ity achieved by ridding ourselves of a metric is so powerful that topology permeates all of mathematics. A continuous map between topological spaces which has a continuous inverse is called a *homeomorphism*. One would like to classify topological spaces upto homeomorphism (i.e., without distinguishing spaces which are homeomorphic), just as one would like to classify, say, groups upto isomorphism. For example, a doughnut is homeomorphic to a coffee cup. A property, such as connectedness, or compactness, which is preserved under homeomorphism, is called a *topological invariant*. In this article, I hope to give you a flavour of how a topological invariant called *homology*, which we shall define, governs the solvability of a problem in calculus. For starters, let us review some *several variable calculus*.

Statement of the Problem

Let $\mathbf{v}(x, y) = (p(x, y), q(x, y))$ be a smooth vector field on an open subset X of the Euclidean plane R^2, i.e. both $p(x, y)$ and $q(x, y)$ are defined on X, and infinitely differentiable as functions on X. Those with a physics background may like to think of $\mathbf{v}$ as an electric field, or the velocity field of a fluid confined to the planar region X. A natural fundamental question which arises is whether there exists a *potential function* for this vector field. In other words, does there exist a smooth function $\phi(x, y)$ on X such that

$$p = \frac{\partial \phi}{\partial x}, \quad q = \frac{\partial \phi}{\partial y} \tag{1}$$

holds identically all over X? This pair of simultaneous differential equations is often abbreviated as $\mathbf{v} = \nabla\phi$ (read gradient of ϕ, or grad ϕ).

Let me give you a quick reason as to why it is useful to have a potential function. It is easier to perform summation (and more generally integration) of potentials, which are scalar valued functions, rather than vector fields. If one has, for example, a line of charge, then to find the electric field at a point you would have to take field contributions of 'infinitesimal' bits of the line, take the components along, say the x-axis, and then integrate. For the potential, you do not need to take components, but simply integrate the potential contributions. The other reason, which is of more interest to us here, is the matter of 'work done' in moving along a smooth path γ. Let $\gamma : [0,1] \to X$, where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a smooth function of $t \in [0,1]$. γ is called a *smooth path* joining $P = \gamma(0)$ to $Q = \gamma(1)$. The work done along γ in the field $\mathbf{v}$ is the *line integral* defined by :

It is easier to perform summation of potentials, which are scalar valued functions, rather than vector fields.

$$\begin{aligned} \int_\gamma \mathbf{v} &= \int_0^1 \left(\mathbf{v}(\gamma(t)) \cdot \frac{d\gamma}{dt} \right) dt \\ &= \int_0^1 \left(p(\gamma(t)) \frac{d\gamma_1}{dt} + q(\gamma(t)) \frac{d\gamma_2}{dt} \right) dt \qquad (2) \end{aligned}$$

where '$\cdot$' denotes the dot product in R^2. Clearly, if a potential function ϕ exists on X, satisfying (1), then the line integral of $\mathbf{v}$ along γ becomes

$$\begin{aligned} \int_\gamma \mathbf{v} &= \int_0^1 \frac{d\phi(\gamma(t))}{dt} dt = \phi(\gamma(1)) - \phi(\gamma(0)) \\ &= \phi(Q) - \phi(P) \qquad (3) \end{aligned}$$

by the fundamental theorem of calculus. To sum up, the work done along a path is just the difference of the values of the potential function at the *end points* of the path, and *independent* of the path. Thus no line integrals need be calculated to compute the work done. Also, in particular, the work done in moving along a smooth *loop* (i.e. a path γ satisfying $\gamma(1) = \gamma(0)$) is zero !

Another obvious consequence of the existence of a solution to (1) is the following: since ϕ is to be smooth on the open set X, one must have

$$\frac{\partial q}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial p}{\partial y} \qquad (4)$$

all over X. The (also smooth) function $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$ is called the *curl* of $\mathbf{v}$, and denoted curl $\mathbf{v}$. The discussion above shows that a *necessary condition* for (1) to have a solution is that curl $\mathbf{v} = 0$ identically on X (recall the statement 'curl grad is zero', from multivariate calculus). So, for example, the vector field $\mathbf{v}(x, y) = (xy, xy)$ on $X = \mathrm{R}^2$ has no potential function since its curl is not identically zero.

It is quite natural to ask whether curl $\mathbf{v} = 0$ is a *sufficient* condition for a smooth vector field $\mathbf{v}$ on X to have a potential function satisfying (1). The rest of this note is essentially devoted to this question.

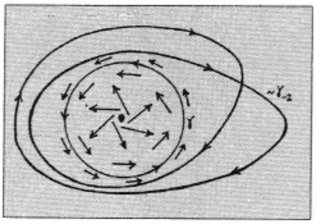

Figure 1 The whirpool vector field.

For a start, let us consider the simplest case $X = \mathrm{R}^2$. In this case, the answer turns out to be yes. Indeed, define the function ϕ by

$$\phi(x, y) = \int_0^x p(t, 0)dt + \int_0^y q(x, s)ds \qquad (5)$$

That this function satisfies (1) is an easy application of the fundamental theorems of calculus about integrals of derivatives and derivatives of integrals; we leave it as an exercise. So now we have a complete answer for a smooth vector field $\mathbf{v}$ on R^2, viz. $\mathbf{v} = \nabla\phi$ for some smooth function ϕ if and only if curl $\mathbf{v} = 0$. This is a particular instance of the Poincaré Lemma for R^2. See the book by Singer and Thorpe for the general statement.

What does topology have to do with all this? To elucidate this point, let us migrate from $X = \mathrm{R}^2$ to the punctured plane $X = \mathrm{R}^2 - (0, 0)$.

Consider the smooth vector field

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

on $\mathrm{R}^2 - (0, 0)$, which is pictured in *Figure 1.* Note that this vector field has a singularity at the origin, i.e., there is no way of extending this field to a smooth vector field on R^2. You may like to think of it as the surface of an infinite river with a whirlpool at the origin. It is easy to check that this vector field is curl free on $\mathrm{R}^2 - (0,0)$, and we may again ask whether there is a smooth potential function ϕ defined on $\mathrm{R}^2 - (0,0)$ such that $\omega = \nabla\phi$. If there were, then the earlier discussion would imply that its line integral along a loop in $\mathrm{R}^2 - (0,0)$ would have to be zero. On the other hand, a (hopefully empty!) boat drifting around the whirlpool would certainly go on gaining energy in the counterclockwise direction. Let us verify this (without getting into that boat!). Let $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$ where $t \in [0,1]$, be the loop going counter-clockwise once around the puncture (see *Figure 1*). Then $\omega(\gamma(t)) = (-\sin 2\pi t, \cos 2\pi t)$, $\frac{d\gamma}{dt} = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)$, and the line integral of ω along γ is

$$\int_\gamma \omega = \int_0^1 ((-\sin 2\pi t)(-2\pi \sin 2\pi t) + (\cos 2\pi t)(2\pi \cos 2\pi t))\, dt$$
$$= 2\pi \int_0^1 dt = 2\pi \qquad (6)$$

which is certainly non-zero. Hence we have a curl free smooth vector field ω on $\mathrm{R}^2 - (0,0)$ which is *not* the gradient of any potential function ! Making just one puncture in R^2 has completely changed its analytical nature.

We can have a curl free smooth vector field ω on $\mathrm{R}^2 - (0,0)$ which is not the gradient of any potential function! Making just one puncture in R^2 can completely change its analytical structure.

Now I would like to dwell upon the topological characteristic of $\mathrm{R}^2 - (0,0)$ which 'causes' this. It is well known that all loops in R^2 can be continuously shrunk to a point, but there are loops in $\mathrm{R}^2 - (0,0)$ that cannot be continuously shrunk to a point. To make all this precise, we need a little bit of 'technology'.

Some Planar Topology

Let X denote an open subset of R^2. A *piecewise smooth path* in X is a map $\gamma : [0,1] \to X$ such that (i) γ is continuous, and (ii) there is a subdivision $0 = a_0 < a_1 < \ldots < a_k = 1$ of $[0,1]$ such that γ is smooth on each of the sub-intervals $I_j = [a_j, a_{j+1}]$. $\gamma(0)$ is called the *initial point* and $\gamma(1)$ the *end point* of γ. The *inverse path* to γ is the path γ^{-1} defined by $\gamma^{-1}(t) = \gamma(1-t)$. We shall now refer to piecewise smooth paths simply as paths, for brevity. As before, a *loop* will mean a path γ whose initial and end points are the same point x. In this case we say the loop γ is *based* at x.

If γ and τ are two paths such that the end point $\gamma(1)$ of γ is the initial point $\tau(0)$ of τ, then one can form the *composite path* $\gamma * \tau$ defined by $\gamma * \tau(t) = \gamma(2t)$ for $0 \le t \le \frac{1}{2}$ and $= \tau(2t-1)$ for $\frac{1}{2} \le t \le 1$. (This is the reason for introducing piecewise smooth paths, because the composite of smooth paths need not be a smooth path, but the composite of piecewise smooth paths is piecewise smooth.) In particular, we can compose two loops based at the same point.

The *constant path* c_x at a point $x \in X$ is defined by $c_x(t) = x$ for all $t \in [0,1]$. Henceforth, we shall always assume that X is a *path connected* open subset of $\mathbf{R}^2$, i.e. given any two points P and Q in X, there is a path γ in X with P as its initial and Q as its end point.

Given a smooth vector field $\mathbf{v} = (p,q)$ on X, and a piecewise smooth path γ in X, we can define the line integral

$$\int_\gamma \mathbf{v} = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} \left(p(\gamma(t)) \frac{d\gamma_1}{dt} + q(\gamma(t)) \frac{d\gamma_2}{dt} \right) dt$$

With this definition, and the standard facts about change of variables in integration, it is easy to see that $\int_{\gamma * \tau} \mathbf{v} = \int_\gamma \mathbf{v} + \int_\tau \mathbf{v}$ and $\int_{\gamma^{-1}} \mathbf{v} = -\int_\gamma \mathbf{v}$. Also, for the constant path c_x at x, we have $\int_{c_x} \mathbf{v} = 0$.

One is now equipped to do some algebra with (piecewise smooth) loops. Let X be a path-connected open subset of

R^2, as before. I would like to define an equivalence relation on loops in X as follows. Say that the loops γ and τ are *equivalent* (or *homologous*, or *freely homotopic*) if there exists a piecewise smooth map $F : [0,1] \times [0,1] \to X$ such that (i) $F(t,0) = \gamma(t)$, $F(t,1) = \tau(t)$ for all $t \in [0,1]$, (ii) $F(0,s) = F(1,s)$ for all $s \in [0,1]$. Such a map is called a (free) *homotopy*. We write $\gamma \sim \tau$ to denote that γ is equivalent to τ. You should verify that this is an equivalence relation.

One intuitively thinks of $\gamma_s = F(\ ,s)$ as a continuous one parameter family of loops evolving from γ at $s = 0$ to τ at $s = 1$. Finally, given a loop γ in X, and an arbitrary point $x \in X$, there is a loop $\tilde{\gamma}$ which is equivalent to γ, and which is based at x. For, take a fixed path σ joining x to $y = \gamma(0) = \gamma(1)$, which is possible by the path connectedness of X. Define $\tilde{\gamma} = \sigma * \gamma * \sigma^{-1}$. *Figure 2* should enable you to construct a homotopy. Because of this, one can compose equivalence classes of loops. If γ and τ represent two equivalence classes, the above remark allows us to assume without loss of generality that γ and τ are based at the same point, and we may define $\gamma + \tau$ to be the equivalence class of the loop $\gamma * \tau$. We will omit the proof that this operation is well-defined, i.e. that $\gamma \sim \gamma'$, $\tau \sim \tau'$ implies $\gamma * \tau \sim \gamma' * \tau'$, though it isn't difficult to prove this, by 'pasting homotopies'. The notation '+' is meant to indicate that the operation is abelian, and it is not difficult to show that $\gamma * \tau$ and $\tau * \gamma$ are equivalent. If γ is a loop with $\gamma(0) = \gamma(1) = x$, you may verify, for example, that $\gamma * c_x \sim \gamma \sim c_x * \gamma$. Also, $\gamma * \gamma^{-1} \sim c_x \sim \gamma^{-1} * \gamma$. Thus, the equivalence class of the constant loop (at any point) is the identity element, and the inverse of (the equivalence class of) γ is (the equivalence class of) the loop γ^{-1}. For notational simplicity, we shall denote a loop and its equivalence class by the same letter, say, γ, τ, etc.

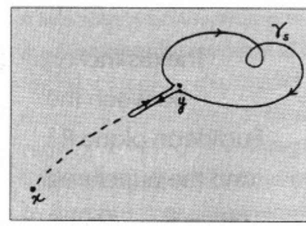

Figure 2 Moving loops around.

The abelian group of these equivalence classes of loops in

X is a very important one, and is called the *first homology group* of X, and denoted $H_1(X)$. It was essentially invented by Poincaré and Riemann for their study of Riemann surfaces.

I claim that the distinction between the Euclidean plane R^2 and the punctured plane $\mathrm{R}^2 - (0,0)$ is detected by the first homology group. First, let us see that $H_1(\mathrm{R}^2)$ is the trivial group $\{0\}$. This is because the homotopy

$$F(t,s) = (1-s)\gamma(t)$$

makes any loop γ equivalent to the trivial loop, so there is only the equivalence class of the constant loop in $H_1(\mathrm{R}^2)$, which therefore is the trivial group ! In fact, this argument shows that the first homology group of any convex (in fact any starlike) open subset of R^2 is trivial. For more on H_1, see the Greenberg lectures on algebraic topology or the book by Bott and Tu.

The distinction between the Euclidean plane R^2 and the punctured plane R^2 - (0,0) is detected by the first homology group.

Of course, $\mathrm{R}^2 - (0,0)$ is not convex, or starlike, and one would like to compute its first homology group. First, let me try to convince you that it is non-trivial. For this, we will need the following lemma, which is the crucial bridge between topology and calculus.

Let X be a path connected open subset of R^2, and $\mathbf{v}$ be a curl free smooth vector field on it. Then, for two loops γ and τ in X such that $\gamma \sim \tau$, we have $\int_\gamma \mathbf{v} = \int_\tau \mathbf{v}$.

To see this, first let us make the simplifying assumption that both γ and τ are smooth, and that the homotopy $F : [0,1] \times [0,1] \to X$ between them, satisfying $F(t,0) = \gamma(t)$, $F(t,1) = \tau(t)$, is also smooth. We let σ denote the path defined by $\sigma(s) = F(0,s) = F(1,s)$. Write $F(t,s) = (F_1(t,s), F_2(t,s))$ in terms of its component functions. Write $\mathbf{v}(x,y) = (p(x,y), q(x,y))$, and for brevity let us denote partial differentiation by subscripts e.g. $p_y = \frac{\partial p}{\partial y}$, $F_{1,s} = \frac{\partial F_1}{\partial s}$

etc. We will use the smooth homotopy F to 'pullback' the vector field $\mathbf{v}$ from X to a smooth vector field $\mathbf{w}$ on the square $[0,1]\times[0,1]$. More precisely, $\mathbf{w}(t,s) = (\tilde{p}(t,s), \tilde{q}(t,s))$ where :

$$\begin{aligned}\tilde{p}(t,s) &= p(F(s,t))F_{1,t} + q(F(s,t))F_{2,t}\\ \tilde{q}(t,s) &= p(F(s,t))F_{1,s} + q(F(s,t))F_{2,s}\end{aligned}$$

This seems a bit concocted, but is got from substituting $F_1(t,s)$ for x, $F_2(t,s)$ for y in the 'differential' $p\ dx + q\ dy$, and reading the coefficients of dt and ds in the resulting differential. Using the chain rule, one directly computes

$$\begin{aligned}\text{curl}\,\mathbf{w}(t,s) &= (\tilde{q}_t(t,s) - \tilde{p}_s(t,s))\\ &= (q_x(F(t,s)) - p_y(F(t,s)))\,(F_{1,t}F_{2,s} - F_{1,s}F_{2,t})\\ &= 0\end{aligned}$$

A two-dimensional analogue of the fundamental theorem of integral calculus expresses a double integral over a planar region, R, as a line integral taken along a path determined by the boundary of R whenever the integrand is a partial derivative. This result is usually attributed to G Green and is known as Green's theorem. In fact it appeared earlier in the work of Lagrange and Gauss.

since $\mathbf{v}$ curl free implies $q_x - p_y = 0$. Thus this new vector field $\mathbf{w}$ on $[0,1]\times[0,1]$ is also curl free. Now, by Green's Theorem, (see page 134 of Spivak's book), we have

$$\begin{aligned}0 &= \int_{[0,1]\times[0,1]} (\tilde{q}_t - \tilde{p}_s)dtds\\ &= \int_0^1 \mathbf{w}(t,0)dt + \int_0^1 \mathbf{w}(1,s)ds + \int_1^0 \mathbf{w}(t,1)dt + \int_1^0 \mathbf{w}(0,s)ds\\ &= \int_\gamma \mathbf{v} + \int_\sigma \mathbf{v} - \int_\tau \mathbf{v} - \int_\sigma \mathbf{v}\\ &= \int_\gamma \mathbf{v} - \int_\tau \mathbf{v}\end{aligned}$$

where the second line is the line integral of $\mathbf{w}$ along the boundary of the square, and the third line follows by substituting the definition of $\mathbf{w}$ and change of variables. (If you want to avoid Green's theorem, use the fact that $\mathbf{w}$ is curl free on $[0,1]\times[0,1]$ implies the existence of a potential function ψ constructed exactly as we did for R^2 in (5). Then the second equation above is true since the line integral of $\mathbf{w}$ on the closed loop defined by the (counterclockwise) boundary of $[0,1]\times[0,1]$, will have to be zero, by (3).)

For the more general piecewise smooth situation, one subdivides the square $[0,1] \times [0,1]$ into small subsquares on each of which the homotopy is smooth, and replaces the integral of curl $\mathbf{w}$ on each subsquare by a line integral of $\mathbf{w}$ on its boundary. Adding up for all these subsquares, the line integrals on all the internal edges cancel pairwise, and what remains in the end, as before, is the line integral of $\mathbf{w}$ on the boundary of $[0,1] \times [0,1]$. This proves the lemma.

Solution of the Problem

To get back to our story now, let us review the opening discussion about solving (1) for R^2 in the light of homology. Note that since every loop in R^2 is equivalent to the *constant loop* or *trivial loop*, the lemma above implies that the line integral of a curl free vector field along any loop is equal to the line integral around the constant loop, which is zero. Furthermore, if γ_1 and γ_2 are two paths joining the point P to the point Q in R^2, $\gamma = \gamma_1 * \gamma_2^{-1}$ is a loop based at P, so $0 = \int_\gamma \mathbf{v} = \int_{\gamma_1} \mathbf{v} - \int_{\gamma_2} \mathbf{v}$ for a curl free field $\mathbf{v}$, which implies that the work done along a path in a curl free field depends *only on the end-points* of the path. This is precisely the statement (3). Thus for a curl free $\mathbf{v}$ on R^2, we may define the potential function $\phi(x,y) = \int_\gamma \mathbf{v}$ where γ is *any path* joining a predetermined point P to the moving point $Q = (x,y)$. We chose one such path in (5), but could have chosen any other.

On the other hand, for $\mathrm{R}^2 - (0,0)$, we have the curl free (whirlpool) vector field ω introduced earlier, whose integral along the loop $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$ is non-zero. So the lemma above implies, in particular, that *this loop cannot be equivalent to the trivial loop !*

In other words, the homology group $H_1(\mathrm{R}^2 - (0,0))$ is non-trivial.

In fact, if we consider the loops $\gamma_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$ for $n \in \mathbf{Z}$, we see that $\int_{\gamma_n} \omega = 2\pi n$, so that γ_n cannot be equivalent to γ_m for $n \neq m$. Thus the first homology group of $\mathrm{R}^2 - (0,0)$ is at least as large as the group of integers.

In fact, it turns out (though the proof is quite non-trivial) that the first homology of the punctured plane $H_1(\mathrm{R}^2 - (0,0)) \simeq \mathbf{Z}$, with the loop γ_n introduced above representing the integer n. So every loop γ in $\mathrm{R}^2 - (0,0)$ is equivalent to some γ_n, and in view of the preceding lemma, this integer n is determined by the relation $2\pi n = \int_\gamma \omega$. The integer n is called the *winding number* of γ about $(0,0)$. For the snaky loop in *Figure 1*, for example, the winding number is -2. For more on this fascinating topic, and the connections with complex analysis, see chapter 4 of the book by Ahlfors.

To tie up this discussion, it would be very pleasing if instead of throwing up one's hands about the insolubility of (1), one could use the fact that $H_1(\mathrm{R}^2 - (0,0)) \simeq \mathbf{Z}$ to give a quantitative answer regarding (1). For this we will use a very beautiful theorem, which is due to Georges de Rham. Note that our foregoing lemma says that for X a path connected open subset of R^2, the line integral $\int_\gamma \mathbf{v} = 0$ for a curl free field $\mathbf{v}$ on X if γ is equivalent to a constant loop in X. (In fact, this is a reformulation of the lemma).

The de Rham theorem (in this particular situation) asserts the following: If for a curl free vector field $\mathbf{v}$ we have that $\int_\gamma \mathbf{v} = 0$ for *all loops* γ in X, then $\mathbf{v} = \nabla\phi$ for some smooth function ϕ on X.

Now let $\mathbf{v}$ be a curl free vector field on $\mathrm{R}^2 - (0,0)$. Compute the line integral $\int_{\gamma_1} \mathbf{v}$, where $\gamma_1(t) = (\cos 2\pi t, \sin 2\pi t)$ is the generating loop for $H_1(\mathrm{R}^2 - (0,0))$. This will be some real number α, say. Since $\int_{\gamma_1} \omega = 2\pi$, where ω is the whirlpool vector field, it follows that

$$\int_{\gamma_1}(\mathbf{v} - \frac{\alpha}{2\pi}\omega) = 0$$

Since γ_1 is a generator for $H_1(\mathrm{R}^2 - (0,0))$, and the line integral over a sum of loops is the sum of the line integrals over those loops, it follows that

$$\int_{n\gamma_1}(\mathbf{v} - \frac{\alpha}{2\pi}\omega) = 0$$

for all $n \in \mathbf{Z}$. Since every loop γ in $\mathrm{R}^2 - (0,0)$ is equivalent to $\gamma_n \sim n\gamma_1$ for some integer n, and $\mathbf{v} - \frac{\alpha}{2\pi}\omega$ is curl free, it follows that $\int_\gamma(\mathbf{v} - \frac{\alpha}{2\pi}\omega) = 0$ for *every loop* γ in $\mathrm{R}^2 - (0,0)$. Thus, by de Rham's theorem, we have $\mathbf{v} - \frac{\alpha}{2\pi}\omega = \nabla\phi$ for some smooth function ϕ on $\mathrm{R}^2 - (0,0)$.

So the final answer is : If $\mathbf{v}$ is a vector field on $\mathrm{R}^2 - (0,0)$ such that curl $\mathbf{v} = 0$, then there exists a real number λ and a smooth function ϕ on $\mathrm{R}^2 - (0,0)$ such that $\mathbf{v} = \lambda\omega + \nabla\phi$, where $\lambda = 1/2\pi\int_{\gamma_1}\mathbf{v}$ and ω is the whirlpool vector field. So we have 'measured' exactly how far we are from the solvability of (1).

Another algebraic way of saying the same thing is as follows. Denote, for X as above, the R-vector space of curl free vector fields on X by $Z^1(X)$. In this vector space, there sits the vector subspace of all vector fields which are gradients of potential functions, and this subspace is denoted $B^1(X)$. The quotient space $Z^1(X)/B^1(X)$, which is called the *first de Rham cohomology* of X and denoted $H^1(X)$ is therefore a real vector space which measures how much curl free fields depart from being gradients of functions. For example, the opening discussion showed that $H^1(\mathrm{R}^2) = 0$. What we have seen as the outcome of the entire discussion for $\mathrm{R}^2 - (0,0)$ is that $H^1(\mathrm{R}^2 - (0,0))$ is isomorphic to the one dimensional real vector space R, and a basis element is, for example, the 'whirlpool' vector field ω.

Suggested Reading

- **M Spivak. Calculus on Manifolds. Benjamin. 1965.**
- **L Ahlfors. Complex Analysis. McGraw Hill. 1966.**
- **M J Greenberg. Lectures on Algebraic Topology. Benjamin. 1967.**
- **I M Singer. J A Thorpe. Lecture Notes in Elementary Topology and Geometry. Springer UTM. 1967.**
- **R Bott. L Tu. Differential Forms in Algebraic Topology. Springer GTM 82. 1982.**

More generally, for X as above, given a de Rham cohomology class represented by a curl free field $\mathbf{v}$, we get a natural abelian group homomorphism $\theta_{\mathbf{v}} : H_1(X) \rightarrow \mathrm{R}$, which takes the homology class of a loop γ to $\int_\gamma \mathbf{v}$. That this map $\mathbf{v} \rightarrow \theta_{\mathbf{v}}$ is well defined follows from the foregoing discussion. The full force of the de Rham theorem is: This map $\theta : H^1(X) \rightarrow \hom_{\mathbf{Z}}(H_1(X), \mathrm{R})$ is an isomorphism of R-vector spaces. The symbol on the right side denotes abelian group homomorphisms, and it is an R vector space via pointwise scalar multiplication. For a proof, see the book by Bott and Tu or the book by Singer and Thorpe.

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Finally, the first homology group $H_1(X)$ can be defined using *continuous* loops and homotopies, instead of piecewise smooth loops and homotopies. Certain approximation theorems say that the homology remains unchanged. This clearly makes $H_1(X)$ a topological invariant. The de Rham theorem therefore asserts that the vector space $H^1(X)$, which is a purely analytical object governing the solvability of a system of first order partial differential equations, is in fact a topological invariant. So, for example, if you took any open starlike subset of R^2, and punched out a closed disc contained in its interior, the space you'd get has the same first de Rham cohomology as the punctured plane ! In particular, the above analysis of curl free fields on $\mathrm{R}^2 - (0, 0)$ applies to such a space.

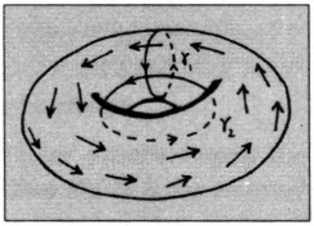

Figure 3 The torus

The reader may want to guess what happens to homology and de Rham cohomology for $X = \mathrm{R}^2 - F$, where F is a finite set of points. I urge you to try. I also leave you with a drawing (*Figure 3*) of the torus that you may want to analyse.