NOTES ON INTESECTION HOMOLOGY

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1. Rapid Review of Sheaf Theory

1.1. **Injective, soft, fine and flabby sheaves.** In all that follows X is a locally compact, paracompact, hausdorff topological space.

Definition 1.1.1 (Injective Sheaf)**.** A sheaf $\mathcal{I}$ is said to be *injective* if given morphisms of sheaves α and β as below:

$$\begin{array}{cccc} 0 & \longrightarrow \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} \\ & & \beta \searrow & \downarrow \exists \\ & & & \mathcal{I} \end{array}$$

the right vertical arrow exists. That is, the functor $\hom(-,\mathcal{I})$ from the category $Sh(X)$ of sheaves on X to the category of abelian groups is right exact.

Definition 1.1.2 (Flabby sheaf)**.** Say that a sheaf $\mathcal{F}$ is *flabby* if for each open subset $U \subset X$, the restriction map of sections:

$$\Gamma(X,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F})$$

is surjective.

Definition 1.1.3 (Soft sheaf)**.** Say that a sheaf $\mathcal{S}$ is *soft* (sometimes called *c-soft*) if for each compact subset $K \subset X$, the natural restriction map of sections:

$$\Gamma(X,\mathcal{S}) \to \Gamma(K,\mathcal{S})$$

is surjective. (The right hand abelian group is defined as $\lim_{\rightarrow} \Gamma(U,\mathcal{S})$, the direct limit over the directed system of all open neighbourhoods U of K).

Remark 1.1.4. If $\mathcal{S}$ is a soft sheaf, then the restriction map:

$$\Gamma_c(X,\mathcal{S}) \to \Gamma_c(Z,\mathcal{S})$$

is surjective for all *closed* sets Z. In fact, it is enough to prove that $\Gamma_c(X,\mathcal{S}) \to \Gamma(K,\mathcal{S})$ is surjective for all compact $K \subset X$. Let $s \in \Gamma(K,\mathcal{S})$. By normality, local compactness etc., without loss we may assume there is a compact neighbourhood $U \supset K$ with $s \in \Gamma(U,\mathcal{S})$. Define the new section $u \in \Gamma(K \coprod \partial U,\mathcal{S})$ by $u \equiv s$ on K and $u \equiv 0$ on ∂U. Then since $K_1 := K \coprod \partial U$ is compact, and $\mathcal{S}$ is soft, there is a section $t \in \Gamma(X,\mathcal{S})$ with $t_{|K_1} = u$. Define the section t_1 by $t_1 \equiv t$ on K, and $t_1 \equiv 0$ on $X \setminus U^\circ$. This t_1 is compactly supported (in U), and the required section extending s.

Conversely, the surjectivity of the above map for every closed subset Z implies softness trivially.

Remark 1.1.5 (Restricting injective, flabby and soft sheaves)**.** From the definition above, it is clear that restricting a flabby sheaf to an open set gives a flabby sheaf. The Remark 1.1.4 above similarly implies that restricting a soft sheaf to a closed subset again gives a soft sheaf. In fact, restricting a soft sheaf to any *locally closed set* gives a soft sheaf (Exercise). Finally, restricting an injective sheaf to an open set again gives an injective sheaf, as is not difficult to prove.

Example 1.1.6 (The Godement flabby envelope)**.** Let $\mathcal{F} \in Sh(X)$ be any sheaf. Then define the sheaf $\mathcal{E}(\mathcal{F})$ by:

$$\Gamma(U, \mathcal{E}(\mathcal{F})) := \prod_{x \in U} \mathcal{F}_x$$

with the obvious restriction maps coming from projection of a product to a smaller product. This is often called the *sheaf of discontinuous sections* of $\mathcal{F}$. It is trivial to check (using extension by zero) that this sheaf is flabby. Note that there is a natural map:

$$\begin{array}{rcl} 0 \to \Gamma(U, \mathcal{F}) & \to & \Gamma(U, \mathcal{E}(\mathcal{F})) \\ s & \mapsto & (s_x)_{x \in U} \end{array}$$

which is easily checked to define an injective sheaf map $0 \to \mathcal{F} \to \mathcal{E}(\mathcal{F})$.

Remark 1.1.7. Let X, $\mathcal{F}$ and $\mathcal{E}(\mathcal{F})$ be as above. Then for each $x \in X$, the stalk $\mathcal{F}_x$ is a direct summand of $\mathcal{E}(\mathcal{F})_x$.

Proof: Clear, since for a fixed $x \in X$, the morphism defined by:

$$\begin{array}{rcl} \mathcal{F}_x & \hookrightarrow & \prod_{y \in U} \mathcal{F}_y = \Gamma(U, \mathcal{E}(\mathcal{F})) \\ s_x & \mapsto & (s_y)_{y \in U} \end{array}$$

for some open neighbourhood U of x, is a split map whose left inverse is the projection to $\mathcal{F}_x$. Taking the direct limit of these projections as U shrinks to x gives the required left inverse of $\mathcal{F}_x \hookrightarrow \mathcal{E}(\mathcal{F})_x$. $\square$

Proposition 1.1.8. Injective $\Rightarrow$ flabby $\Rightarrow$ soft, for all sheaves on X as above.

Proof: Let $\mathcal{I}$ be flabby, and let $0 \to \mathcal{I} \to \mathcal{E}(\mathcal{I})$ be the inclusion into the Godement envelope. By the injectivity of $\mathcal{I}$, this morphism splits and $\mathcal{I}$ becomes a direct summand of $\mathcal{E}(\mathcal{I})$. This last sheaf is flabby by Example 1.1.6 above, and it is trivial to check that direct summands of flabby sheaves are flabby. Thus injective $\Rightarrow$ flabby.

Because for a flabby sheaf $\mathcal{F}$, the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is onto, and for a compact subset K of X, we have $\Gamma(K, \mathcal{F}) = \lim_{\to} \Gamma(U, \mathcal{F})$, and direct limit is an exact functor, we find that $\Gamma(X, \mathcal{F}) \to \Gamma(K, \mathcal{F})$ is surjective for all K compact. Thus flabby $\Rightarrow$ soft.

$\square$

Remark 1.1.9. It is easy to see that all the implications above are strict. For let G be an abelian group which is not injective. It is easily seen that the skyscraper sheaf $\mathcal{F}$ on X with stalk $\mathcal{F}_x = G$ at some fixed $x \in X$ and 0 elsewhere is flabby, but not injective. Similarly the sheaf of continuous $\mathbb{R}$-valued functions on say $X = [0, 1]$ is soft (by using Tietze's extension theorem, X is a metric space, so normal) is soft, but not flabby.

Example 1.1.10 (Godement injective envelope and resolution)**.** Let $\mathcal{F}$ be any sheaf of k-modules, k any commutative ring. For each $x \in X$, embed the stalk $\mathcal{F}_x$, which is a k-module, in an injective k-module $\mathcal{I}_x$ via $j_x : \mathcal{F}_x \hookrightarrow \mathcal{I}_x$. Now consider the sheaf $\mathcal{I}$ defined by

$$\Gamma(U, \mathcal{I}) := \prod_{x \in U} \mathcal{I}_x$$

It is easily verified that (a) the map $s \mapsto (j_x(s_x))_{x \in U}$ defines a k-embedding $0 \to \mathcal{F} \to \mathcal{I}$ and (b) $\mathcal{I}$ is an injective sheaf of k-modules.

Using this construction, and calling $\mathcal{I}$ as $\mathcal{I}^0$, one can now consider the k-sheaf $\mathcal{I}^0/\mathcal{F}$, embed it in an injective k-sheaf $\mathcal{I}^1$, and proceed inductively to get an *injective resolution* $\mathcal{I}^\cdot$ of $\mathcal{F}$, viz. an exact sequence of sheaves:

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 .... \to \mathcal{I}^m \to ..$$

where each $\mathcal{I}^m$ is injective.

Remark 1.1.11 (Canonical flabby resolution)**.** In conjunction with the construction in Example 1.1.6, and using the same construction as the last para, one can construct the *canonical flabby resolution* of any sheaf $\mathcal{S}$ as :

$$0 \to \mathcal{S} \to \mathcal{F}^0 \to \ldots \to \mathcal{F}^m \to \ldots$$

where $\mathcal{F}^i := \mathcal{E}(\mathcal{F}^{i-1}/\mathrm{Im}\,\mathcal{F}^{i-2})$. By definition, all the $\mathcal{F}^i$ are flabby.

Definition 1.1.12 (Sheaf cohomology)**.** If $\mathcal{F}$ is a sheaf on X, we define the *i-th sheaf cohomology of X* in $\mathcal{F}$, denoted $H^i(X, \mathcal{F})$, to be the i-th cohomology of the cochain complex:

$$0 \to \Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \ldots \to \Gamma(X, \mathcal{I}^m) \to \ldots$$

where $\mathcal{F} \to \mathcal{I}^{\cdot}$ is an injective resolution of $\mathcal{F}$. It is again checked easily that two different injective resolutions of $\mathcal{F}$ produce chain homotopically equivalent complexes, and hence the cohomology above is well-defined upto isomorphism.

Similarly, one can define the *i-th cohomology with compact supports* $H^i_c(X, \mathcal{F})$ to be the i-th cohomology of the complex $\Gamma_c(X, \mathcal{I}^{\cdot})$, where Γ_c denotes compactly supported sections.

Note that since $\Gamma(X, -)$ and $\Gamma_c(X, -)$ are left exact functors, the definitions above imply that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and $H^0_c(X, \mathcal{F}) = \Gamma_c(X, \mathcal{F})$.

More generally, if $F : Sh(X) \to A$ is any functor of sheaves to an abelian category, then the *i-th derived functor* $R^iF(\mathcal{F})$ is defined as the i-th cohomology of the complex $F(\mathcal{I}^{\cdot})$, and is an object in the category A. With this definition, $H^i(X, \mathcal{F})$ is the i-th derived functor of the global section functor $\Gamma(X, -)$, and $H^i_c(X, \mathcal{F})$ is the i-th derived functor of the global-sections-with-compact-supports functor $\Gamma_c(X, -)$.

Proposition 1.1.13. Let X be as above, and let

$$0 \to \mathcal{E} \hookrightarrow \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

be an exact sequence of sheaves on X.

(i): If $\mathcal{E}$ is flabby, then the sequence of global sections:

$$0 \to \Gamma(X, \mathcal{E}) \hookrightarrow \Gamma(X, \mathcal{F}) \xrightarrow{\beta} \Gamma(X, \mathcal{G}) \to 0$$

is exact.

(ii): If $\mathcal{E}$ is soft, then the sequence of *compactly supported* global sections:

$$0 \to \Gamma_c(X, \mathcal{E}) \hookrightarrow \Gamma_c(X, \mathcal{F}) \xrightarrow{\beta} \Gamma_c(X, \mathcal{G}) \to 0$$

is exact.

Proof: For (i), let $s \in \Gamma(X, \mathcal{G})$ be a global section. Let M be the family :

$$M := \{(t, U) : U \subset X \text{ open},\ t \in \Gamma(U, \mathcal{F}) \text{ with } \beta(t) = s_{|U}\}$$

Since by hypothesis $\beta : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$, M is non-empty. Partially order M by the relation $(t, U) \leq (r, V)$ if $U \subset V$ and $r_{|U} = t$. Clearly any chain $\{(t_i, U_i)\} \in M$ has the upper bound (t, U), where we define $U := \cup_i U_i$ and $t := \cup_i t_i$.

Thus M has a maximal element, say (t, U). We claim that $U = X$. For let $x \notin U$ and let V be an open neighbourhood of x with the section $r \in \Gamma(V, \mathcal{F})$ satisfying $\beta(r) = s_{|V}$. Then it follows that $r_{|U\cap V} - t_{|U\cap V}$ is in the kernel of β, and hence equal to $q \in \Gamma(U \cap V, \mathcal{E})$.

Since $\mathcal{E}$ is flabby, there exists a $p \in \Gamma(U, \mathcal{E})$ such that $p_{|U\cap V} = q$. Then the sections $r \in \Gamma(V, \mathcal{F})$ and $t + p \in \Gamma(U, \mathcal{F})$ agree on $U \cap V$, to give a section $t' \in \Gamma(U \cup V, \mathcal{F})$ which lifts s, contradicting the maximality of (t, U). This proves (i).

The proof of (ii) is analogous, for let $s \in \Gamma_c(X, \mathcal{G})$, with $\mathrm{supp}\, s$ compact. Let K be a compact neighbourhood of $\mathrm{supp}\, s$. Let $s_1 \in \Gamma(K, \mathcal{G})$ such that $s_1 \equiv 0$ on ∂K and $s_1 \equiv s$ on $\mathrm{supp}\, s$.

For each $x \in K$, there is a compact neighbourhood U_x of x and $t_x \in \Gamma(U_x, \mathcal{F})$ with $\beta(t_x) = s_1$ on U_x. Since K is compact, we may assume $t_i \in \Gamma(U_i, \mathcal{F})$ for $i = 1, .., m$ satisfying the same. Note $t_i - t_j \in \Gamma(U_i \cap U_j, \mathcal{E})$, as in last para. $U_i \cap U_j$ being compact, by the softness of $\mathcal{E}$, we can find $t_{ij} \in \Gamma(X, \mathcal{E})$ with $t_{ij|U_i \cap U_j} = t_i - t_j$. Then the section t_i agrees with $t_j + t_{ij}$ on $U_i \cap U_j$ to give a section over $U_i \cup U_j$ lifting s_1.

Proceeding inductively, and letting $U := \cup_{i=1}^m U_i$, one certainly gets a section $t \in \Gamma(U, \mathcal{F})$ lifting s_1 over $\partial K \cup \operatorname{supp} s$. Now observe that $t_{|\partial K}$ maps to $s_{1|\partial K} \equiv 0$ under β, so $t_{|\partial K} \in \Gamma(\partial K, \mathcal{E})$. By using the softness of $\mathcal{E}$, there is a section $v \in \Gamma(X, \mathcal{E})$ whose restriction to ∂K is $t_{|\partial K}$. Then the difference $t_1 := t - v$ on U lifts s on supp s and vanishes on ∂K. Extending t_1 by zero on $X \setminus K^\circ$ is the required compactly supported section which lifts s. □

Corollary 1.1.14. Let X be as above, and

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of sheaves on X. If $\mathcal{E}$ and $\mathcal{F}$ are flabby (resp. soft), then $\mathcal{G}$ is also flabby (resp. soft).

Proof: By the Proposition 1.1.13 above, and the Remark 1.1.5, for $U \subset X$ open, we have the diagram:

$$\begin{array}{ccccccc} 0 \to & \Gamma(X, \mathcal{E}) \to & \Gamma(X, \mathcal{F}) \to & \Gamma(X, \mathcal{G}) \to 0 \\ & \downarrow & \downarrow & \downarrow \\ 0 \to & \Gamma(U, \mathcal{E}) \to & \Gamma(U, \mathcal{F}) \to & \Gamma(U, \mathcal{G}) \to 0 \end{array}$$

If we assume $\mathcal{E}$ and $\mathcal{F}$ are flabby, the two left vertical arrows are surjections. The snake lemma then implies that the right vertical arrow is also a surjection, and hence $\mathcal{G}$ is flabby. The argument for the case of $\mathcal{E}$, $\mathcal{F}$ soft is analogous, using a closed set Z instead of open U, Γ_c instead of Γ, and the Remark 1.1.4. □

Proposition 1.1.15 (Cohomology long exact sequences)**.** Let X be as above, and let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of sheaves on X. Then there is the long exact cohomology sequence:

$$... \to H^i(X, \mathcal{F}_1) \to H^i(X, \mathcal{F}_2) \to H^i(X, \mathcal{F}_3) \to H^{i+1}(X, \mathcal{F}_1) \to ...$$

and the corresponding compactly supported analogue:

$$... \to H^i_c(X, \mathcal{F}_1) \to H^i_c(X, \mathcal{F}_2) \to H^i_c(X, \mathcal{F}_3) \to H^{i+1}(X, \mathcal{F}_1) \to ...$$

Proof: It is not difficult to see that for a short exact sequence of sheaves:

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

one can come up with a short exact sequence of complexes of sheaves:

$$0 \to \mathcal{I}_1^{\cdot} \to \mathcal{I}_2^{\cdot} \to \mathcal{I}_3^{\cdot} \to 0$$

where $\mathcal{F}_j \to I_j^{\cdot}$ is an injective resolution for $\mathcal{F}_j$, $j = 1, 2, 3$. Then, since injective sheaves are flabby, and also soft, by Proposition 1.1.13 we will have a short exact sequences of cochain complexes:

$$0 \to \Gamma(X, \mathcal{I}_1^{\cdot}) \to \Gamma(X, \mathcal{I}_2^{\cdot}) \to \Gamma(X, \mathcal{I}_3^{\cdot}) \to 0$$

of global sections and, likewise,

$$0 \to \Gamma_c(X, \mathcal{I}_1^{\cdot}) \to \Gamma_c(X, \mathcal{I}_2^{\cdot}) \to \Gamma_c(X, \mathcal{I}_3^{\cdot}) \to 0$$

for compactly supported global sections. These will lead to the long exact cohomology sequences:

$$... \to H^i(X, \mathcal{F}_1) \to H^i(X, \mathcal{F}_2) \to H^i(X, \mathcal{F}_3) \to H^{i+1}(X, \mathcal{F}_1) \to ...$$

and the corresponding compactly supported analogue:

$$... \to H^i_c(X, \mathcal{F}_1) \to H^i_c(X, \mathcal{F}_2) \to H^i_c(X, \mathcal{F}_3) \to H^{i+1}(X, \mathcal{F}_1) \to ...$$

which proves the lemma. □

The following fact is crucial for all sheaf-theory.

Proposition 1.1.16 (Acyclicity properties of flabby and soft sheaves)**.** Let X be as above.

(i): If $\mathcal{F}$ is flabby, then $H^i(X, \mathcal{F}) = 0$ for all $i \geq 1$. That is, flabby sheaves are acyclic (=no higher derived functors) for the functor Γ.

(ii): If $\mathcal{S}$ is soft, then $H^i_c(X, \mathcal{S}) = 0$ for all $i \geq 1$. That is, soft sheaves are acyclic (=no higher derived functors) for the functor Γ_c.

Proof: Since for an injective sheaf $\mathcal{I}$ we can use the one term injective resolution $I \to I \to 0 \to 0...$, it follows that $H^i(X, \mathcal{I}) = H^i_c(X, \mathcal{I}) = 0$ for $i \geq 1$ and $\mathcal{I}$ injective.

Now let $\mathcal{F}$ be a flabby sheaf, and let:

$$0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \to ... \to \mathcal{I}^j \to ...$$

be an injective resolution of $\mathcal{F}$. We may rewrite this as a series of short exact sequences:

$$\begin{aligned} 0 \to \mathcal{F} &\xrightarrow{\epsilon} \mathcal{I}^0 \xrightarrow{d^0} \operatorname{Im} d^0 \to 0 \\ 0 \to \operatorname{Im} d^0 &\to \mathcal{I}^1 \xrightarrow{d^1} \operatorname{Im} d^1 \to 0 \\ &.... \\ 0 \to \operatorname{Im} d^{j-1} &\to \mathcal{I}^j \xrightarrow{d^j} \operatorname{Im} d^j \to 0 \end{aligned}$$

Since $\mathcal{F}$ is flabby, by Propositions 1.1.8 and Corollary 1.1.14, we have that $\operatorname{Im} d^0$ is flabby. Inductively, we find that $\operatorname{Im} d^j$ is flabby for all j.

By the Proposition 1.1.13, when $\mathcal{F}$ is flabby, the sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}^0) \to \Gamma(X, \operatorname{Im} d^0) \to 0$$

is exact, so by the first para above, since $H^1(X, \mathcal{I}^0) = 0$, the first five terms of the long exact cohomology sequence (see Proposition 1.1.15) implies that $H^1(X, \mathcal{F}) = 0$ for all flabby sheaves $\mathcal{F}$. In particular, $H^1(X, \operatorname{Im} d^j) = 0$ for all $j \geq 0$.

Now again use the first para, together with the long exact cohomology sequence (see associated to the short exact sequence of sheaves

$$0 \to \operatorname{Im} d^{j-1} \to \mathcal{I}^j \to \operatorname{Im} d^j \to 0$$

to conclude that

$$H^1(X, \operatorname{Im} d^j) \simeq H^2(X, \operatorname{Im} d^{j-1}) \simeq .... \simeq H^{j+2}(X, \mathcal{F})$$

for all $j \geq 0$. By the last line of the previous para, we get $H^{j+2}(X, \mathcal{F}) = 0$ for all $j \geq 0$.

Similarly, when $\mathcal{F}$ is soft, $\operatorname{Im} d^j$ are soft for all j, by Proposition 1.1.14, and similar use of long exact sequence for H^i_c leads to the result. $\square$

Corollary 1.1.17. Let $\mathcal{E}$ be a sheaf on X.

(i): If $\mathcal{E} \to \mathcal{F}^\cdot$ is a flabby resolution of $\mathcal{F}$, then the i-th cohomology of the global section complex $\Gamma(X, \mathcal{F}^\cdot)$ is the sheaf cohomology $H^i(X, \mathcal{E})$.

(ii): Analogously, if $\mathcal{E} \to \mathcal{S}^\cdot$ is a soft resolution of $\mathcal{E}$, then the i-th cohomology of the compactly supported global section complex $\Gamma_c(X, \mathcal{S}^\cdot)$ is the sheaf cohomology $H^i_c(X, \mathcal{E})$ with compact supports.

Proof: Let's first prove (i). One again breaks up the flabby resolution:

$$0 \to \mathcal{E} \to \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \to ...$$

into the short exact sequences:

$$\begin{array}{rll} 0 \to \mathcal{E} & \xrightarrow{\epsilon} \mathcal{F}^0 & \xrightarrow{d^0} \operatorname{Im} d^0 \to 0 \\ 0 \to \operatorname{Im} d^0 & \to \mathcal{F}^1 & \xrightarrow{d^1} \operatorname{Im} d^1 \to 0 \\ .... & & \\ 0 \to \operatorname{Im} d^{j-1} & \to \mathcal{F}^j & \xrightarrow{d^j} \operatorname{Im} d^j \to 0 \end{array}$$

Now using the Proposition 1.1.16 above, we have $H^i(X, \mathcal{F}^j) = 0$ for all $i \geq 1$ and $j \geq 0$. Thus, as in the proof of the proposition above, one finds that:

$$H^i(X, \mathcal{E}) \simeq H^{i-1}(X, \operatorname{Im} d^0) \simeq .... \simeq H^1(X, \operatorname{Im} d^{i-2})$$

Now for the short exact sequence:

$$0 \to \operatorname{Im} d^{i-2} \to \mathcal{F}^{i-1} \to \operatorname{Im} d^{i-1} \to 0$$

the initial bit of the long exact cohomology sequence reads:

$$\Gamma(X, \operatorname{Im} d^{i-2}) \to \Gamma(X, \mathcal{F}^{i-1}) \xrightarrow{d^{i-1}} \Gamma(X, \operatorname{Im} d^{i-1}) \to H^1(X, \operatorname{Im} d^{i-2}) \to 0$$

which shows that $H^1(X, \operatorname{Im} d^{i-2}) \simeq \Gamma(X, \operatorname{Im} d^{i-1})/d^{i-1}(\Gamma(X, \mathcal{F}^{i-1}))$. On the other hand the left exactness of the functor $\Gamma(X, -)$ applied to the short exact sheaf sequence $0 \to \operatorname{Im} d^{i-1} \to \mathcal{F}^i \to \operatorname{Im} d^i \to 0$ shows that

$$\Gamma(X, \operatorname{Im} d^{i-1}) = \ker(d^i : \Gamma(X, \mathcal{F}^i) \to \Gamma(X, \operatorname{Im} d^i)) = \ker(d^i : \Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{F}^{i+1}))$$

This shows that $H^i(X, \mathcal{E})$ is precisely the i-th cohomology of the global section complex $\Gamma(X, \mathcal{F}^\cdot)$. This proves (i).

The proof of the second assertion (ii) is analogous, using Γ_c in place of Γ, and using the fact that soft sheaves are Γ_c-acyclic. □

Here is a useful cohomological way of testing softness of a sheaf.

Corollary 1.1.18 (A cohomological criterion for softness)**.** Let X be as above. Then a sheaf $\mathcal{F}$ on X is soft iff $H^1_c(U, \mathcal{F}) = 0$ for all open sets $U \subset Z$.

Proof: The only if part follows immediately from the Remark 1.1.5, which says that $\mathcal{F}_{|U}$ is also soft, and the Proposition 1.1.16 above.

To see the if part, we set $Z := X \setminus U$, a closed set. For an injective resolution $\mathcal{F} \to \mathcal{I}^\cdot$, since injective sheaves are soft (by 1.1.8) the Remark 1.1.4 implies that the restriction map:

$$\Gamma_c(X, \mathcal{I}^\cdot) \to \Gamma_c(Z, \mathcal{I}^\cdot)$$

is surjective. Its kernel is clearly $\Gamma_c(U, \mathcal{F})$, so that we have a short exact sequence of complexes:

$$0 \to \Gamma_c(U, \mathcal{I}^\cdot) \to \Gamma_c(X, \mathcal{I}^\cdot) \to \Gamma_c(Z, \mathcal{I}^\cdot) \to 0$$

with $Z := X \setminus U$. This leads to the cohomology long exact sequence:

$$0 \to \Gamma_c(U, \mathcal{F}) \to \Gamma_c(X, \mathcal{F}) \to \Gamma_c(Z, \mathcal{F}) \to H^1_c(U, \mathcal{F}) \to ...$$

Since the last term is zero by hypothesis, it follows that the natural restriction map $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(Z, \mathcal{F})$ is surjective. This is equivalent to softness, by the last line of Remark 1.1.4. □

Corollary 1.1.19. Let $h : W \hookrightarrow X$ be the inclusion of a locally closed subset in X as above, and $\mathcal{S}$ a soft sheaf on X. Then the restricted sheaf $h^*\mathcal{S}$ on W is soft.

Proof: We have already noted in Remark 1.1.5 that the restriction of $\mathcal{S}$ to a closed set is soft. Since W is the intersection of a closed set and an open set, it suffices to prove that the restriction of a soft sheaf to an open set is soft. This follows immediately from the cohomological criterion for softness in Corollary 1.1.18 above. □

1.2. **Direct image with proper supports.**

Definition 1.2.1 (The functor $f_!$). Let $f : X \to Y$ be a continuous map with X, Y both locally compact, hausdorff, paracompact. Let $\mathcal{F}$ be a sheaf on X. Then define the presheaf $f_!\mathcal{F}$ by:

$$\Gamma(U, f_!\mathcal{F}) := \{s \in \Gamma(f^{-1}(U), \mathcal{F}) : \ f_{|\text{supp s}} \ \text{ is proper}\}$$

Since this is a subpresheaf of the sheaf $f_*\mathcal{F}$, it is a sheaf.

We need some facts about this functor.

Proposition 1.2.2. Let $h : W \hookrightarrow X$ be the inclusion of a locally closed subset W in X as above. If $\mathcal{S}$ is a soft sheaf on W, then $h_!\mathcal{F}$ is soft on X.

Proof: See [Iv], p. 183 □

Corollary 1.2.3. Let $h : W \hookrightarrow X$ be the inclusion of a locally closed subset, and $\mathcal{F}$ a sheaf on W. Then:

$$H^i_c(W, \mathcal{F}) = H^i_c(X, h_!\mathcal{F})$$

Proof: Let $\mathcal{F} \to \mathcal{S}^{\cdot}$ be a soft resolution of $\mathcal{F}$ on W. By (ii) of Corollary 1.1.17, $H^i_c(W, \mathcal{F}) = H^i(\Gamma_c(W, \mathcal{S}^{\cdot}))$. By the proposition above, and the (easily verified) fact that $h_!$ is exact, it follows that $h_!\mathcal{F} \to h_!\mathcal{S}^{\cdot}$ is a soft resolution of $h_!\mathcal{F}$. Thus again by 1.1.17, $H^i_c(X, h_!\mathcal{F}) = H^i(\Gamma_c(X, h_!\mathcal{S}^{\cdot}))$.

By definition it follows that $\Gamma_c(X, h_!\mathcal{S}) = \Gamma_c(W, \mathcal{S})$ for any sheaf $\mathcal{S}$ on W, so the right hand side above is just $H^i_c(W, \mathcal{F})$. □

Corollary 1.2.4. Let $h : W \hookrightarrow X$ be the inclusion of a locally closed subset. Then for a soft sheaf $\mathcal{S}$ on X, the sheaf $h_!h^*\mathcal{S}$ on X is also soft.

Proof: Immediate from Corollary 1.1.19 and Proposition 1.2.2 □

1.3. **Finite cohomological dimension.**

Definition 1.3.1. Say that X is of *finite cohomological dimension* if there exists an n such that $H^{n+i}_c(X, \mathcal{F}) = 0$ for *all* sheaves $\mathcal{F}$ on X and all $i \geq 1$. The smallest n satisfying this property is called the *cohomological dimension* of X, and denoted $\dim_c X$.

Lemma 1.3.2. Let X be locally compact hausdorff, with $\dim_c X \leq n$. Then for any locally closed subset $W \subset X$, we have $\dim_c W \leq n$.

Proof: If $\mathcal{F}$ is any sheaf on W, the Corollary 1.2.3 implies that $H^i_c(W, \mathcal{F}) = H^i_c(X, h_!\mathcal{F})$, and the right side vanishes for $i \geq n+1$ if $\dim_c X \leq n$ by definition. Thus $\dim_c W \leq n$. □

Proposition 1.3.3. Let X be paracompact and locally compact hausdorff as above, and satisfying $\dim_c X \leq n$. Let there be an exact sequence:

$$0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{S}^0 \xrightarrow{d^0} \ldots \to \mathcal{S}^{n-1} \xrightarrow{d^{n-1}} \mathcal{S}^n \to 0$$

of sheaves on X, with $\mathcal{S}^i$ soft for $0 \leq i \leq n-1$. Then $\mathcal{S}^n$ is soft.

Proof: We will use the cohomological criterion for softness of Proposition 1.1.18. We need to show that $H^1_c(U, \mathcal{S}^n) = 0$ for all open sets U. There is the chain of short exact sequences:

$$\begin{array}{lll} 0 \to \mathcal{F} & \xrightarrow{\epsilon} \mathcal{S}^0 & \xrightarrow{d^0} \operatorname{Im} d^0 \to 0 \\ 0 \to \operatorname{Im} d^0 & \to \mathcal{S}^1 & \xrightarrow{d^1} \operatorname{Im} d^1 \to 0 \\ \quad .... & & \\ 0 \to \operatorname{Im} d^{n-2} & \to \mathcal{S}^{n-1} & \xrightarrow{d^{n-1}} \mathcal{S}^n \to 0 \end{array}$$

Now use the softness ($\Rightarrow$ acyclicity) of the restricted sheaves $\mathcal{S}^i_{|U}$ for $0 \leq i \leq n-1$, and mimic the proof of Proposition 1.1.16 to conclude that:

$$H^1_c(U, \mathcal{S}^n) \simeq H^2_c(U, \operatorname{Im} d^{n-2}) \simeq .... \simeq H^{n+1}_c(U, \mathcal{F})$$

Since $\dim_c X \leq n$, it follows by Lemma 1.3.2 above that $\dim_c U \leq n$, which implies that the right hand group vanishes, which implies $H^1_c(U, \mathcal{S}^n) = 0$, and we are done. $\square$

Corollary 1.3.4. Let X be as above, with $\dim_c X \leq n$. If $\mathcal{F}$ is a sheaf on X, and $\mathcal{S}^\cdot$ is any soft resolution of $\mathcal{F}$, then the truncation $(\tau_{\leq n} S)^\cdot$ defined by:

$$\begin{aligned} (\tau_{\leq n} S)^j &= 0 \quad \text{for } j \geq n+1 \\ &= \operatorname{Im} d^{n-1} \quad \text{for } j = n \\ &= S^j \quad \text{for } j \leq n-1 \end{aligned}$$

is a soft resolution of $\mathcal{F}$ of length $(n+1)$.

Proof: By definition, we only need to check the softness of the last term Im d^{n-1}, which follows from the Proposition above. $\square$.

Proposition 1.3.5. Let X be as above, with $\dim_c X \leq n$, and let k be a commutative Noetherian ring with 1. $\mathcal{F}$ be a flat k- sheaf on X. Then $\mathcal{F}$ has a resolution:

$$0 \to \mathcal{F} \to \mathcal{S}^0 \to ...\mathcal{S}^n \to 0$$

of length $n+1$ such that each $\mathcal{S}^i$ is *soft and flat* (Recall a k-sheaf $\mathcal{S}$ is said to be *flat* if $(-) \otimes_k \mathcal{S}$ is an exact functor on the category $Sh_k(X)$ of k-sheaves on X).

Proof: Define $\mathcal{S}^i := \mathcal{F}^i$ for $i \geq 0$, where $\mathcal{F} \to \mathcal{F}^\cdot$ is the canonical flabby resolution defined in Remark 1.1.11. Since $\mathcal{F}^i$ are flabby, by Proposition 1.1.8, they are soft.

$\mathcal{F}$ flat $\Rightarrow \mathcal{F}_x$ is flat $\Rightarrow$ the sections over U of the Godement flabby envelope (see Definition 1.1.6) $\Gamma(U, \mathcal{E}(\mathcal{F}))$ is flat, for all $U \subset X$ open. This implies that $\mathcal{E}(\mathcal{F})_x$, being a direct limit of flats, is flat. This implies that the first term $\mathcal{S}^0$ of the Godement flabby resolution above is flat.

By the Remark 1.1.7, the short exact sequence

$$0 \to \mathcal{F}_x \xrightarrow{\epsilon} \mathcal{E}(\mathcal{F})_x \to (\operatorname{Im} \epsilon)_x \to 0$$

is split, so the right hand term is a summand of the middle term, which is flat. So it is flat, and thus the sheaf $\operatorname{Im} \epsilon$ is flat. Now $\mathcal{S}^1$ is defined as $\mathcal{E}(\operatorname{Im} \epsilon)$ (see Remark 1.1.11), and by the above argument repeated for $\operatorname{Im} \epsilon$ in place of $\mathcal{F}$, we see that $\mathcal{S}^1$, and inductively, all $\mathcal{S}^i$ are flat.

Now truncate to get the complex $\tau_{\leq n} \mathcal{S}^\cdot$ as in Corollary 1.3.4 to get the required soft flat resolution of length $n+1$. $\square$

It is quite impractical to do anything with the Godement flabby resolution. Fortunately, soft flat resolutions of well-known constant sheaves are abundant in nature. Here are a couple of examples.

Example 1.3.6 (Singular cochain complex of sheaves)**.** Let X be as above, and assume in addition that it *locally contractible* (i.e. every point $x \in X$ has a fundamental system of contractible neighbourhoods). Then the constant sheaf $\underline{Z}_X$ has a soft flat resolution given by:

$$0 \to \underline{Z}_X \to \mathcal{C}^0 \to ... \to \mathcal{C}^i \to ...$$

where $\mathcal{C}^i$ is the sheaf of *singular i-cochains*. It is defined as follows.

For an open cover $\mathfrak{U}$ of X, say a singular i-simplex $\sigma : \Delta^i \to X$ is $\mathfrak{U}$-*small* if $\mathrm{Im}\,\sigma \subset U$ for some $U \in \mathfrak{U}$. Notice that if an open covering $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, then a $\mathfrak{V}$-small simplex is $\mathfrak{U}$-small. For U open in X and $\mathfrak{U}$ an open cover of U, let $\Sigma^i_{\mathfrak{U}}(U)$ denote the set of all $\mathfrak{U}$-small singular i-simplices in U and set $C^{\mathfrak{U}}_i(U, \mathbb{Z})$ to be the free-abelian group on $\Sigma^i_{\mathfrak{U}}(U)$ (called the *group of* $\mathfrak{U}$*-small singular i-chains*).

Finally define the abelian group:

$$C^i_{\mathfrak{U}}(U, \mathbb{Z}) := \hom_{\mathbb{Z}}(C^{\mathfrak{U}}_i(U, \mathbb{Z}), \mathbb{Z})$$

Note that these fit into a cochain complex with coboundary operator δ defined as usual by $(\delta\, f)(\sigma) := f(\partial\sigma)$ An element of $\mathcal{C}^i_{\mathfrak{U}}(U, \mathbb{Z})$ is called a $\mathfrak{U}$*-small i-cochain in U with values in* $\mathbb{Z}$. Since $\Sigma^i_{\mathfrak{V}} \subset \Sigma^i_{\mathfrak{U}}$ for $\mathfrak{V} \geq \mathfrak{U}$, we get natural chain maps:

$$\rho_{\mathfrak{U}\mathfrak{V}} : C^{\cdot}_{\mathfrak{U}}(U, \mathbb{Z}) \to C^{\cdot}_{\mathfrak{V}}(U, \mathbb{Z})$$

We finally define the complex of sheaves of singular i-cochains (with integer coefficients) by:

$$\Gamma(U, \mathcal{C}^{\cdot}) = \lim_{\mathfrak{U}} C^{\cdot}_{\mathfrak{U}}(U, \mathbb{Z})$$

We omit the verification that these are actually sheaves (only barycentric subdivision is used to prove this). That $0 \to \underline{\mathbb{Z}}_X \to \mathcal{C}^{\cdot}$ is a resolution of the constant sheaf follows from local contractibility of X.

To see flatness of the $\mathcal{C}^i$, note that by definition above, the abelian groups $C^i_{\mathfrak{U}}(U, \mathbb{Z})$ are torsion-free, and hence flat. Then their direct limit over $\mathfrak{U}$ is also flat, i.e. $\Gamma(U, \mathcal{C}^i)$ is flat for each open subset U, which implies the sheaves $\mathcal{C}^i$ are flat.

Finally, note that $\mathcal{C}^0$ is the sheaf of *all* integer-valued functions on X, i.e. it is the Godement flabby envelope $\mathcal{E}(\underline{Z}_X)$ (see 1.1.6 for definition) of the constant sheaf $\underline{Z}_X$. In particular it is flabby, and hence by Proposition 1.1.8, it is a soft *sheaf of rings*. Now it is easily checked that $\mathcal{C}^i$ is a module over $\mathcal{C}^0$ (by "pointwise multiplication"), and that a sheaf of modules over a soft sheaf of rings is itself soft. (For a compact subset $K \subset X$, use the softness of $\mathcal{C}^0$ to construct Urysohn cut-off functions in $\Gamma(X, \mathcal{C}^0)$ which are identically 1 on K and vanish outside a compact neighbourhood U of K, as was done in the proof of 1.1.13. These can be scalar multiplied with sections of $\Gamma(K, \mathcal{C}^i)$ to extend them to global sections. The details are omitted).

By the Corollary 1.1.17, the sheaf cohomology $H^i(X, \underline{\mathbb{Z}}_X)$ is the i-th cohomology of the complex of global sections $\Gamma(X, \mathcal{C}^{\cdot})$, i.e. the integral singular cohomology of X. By repeating the above for the constant sheaf $\underline{G}_X$, (G any abelian group) and setting $\mathbb{C}^i_{\mathfrak{U}}(U, G) := \hom(C^{\mathfrak{U}}_i(X, \mathbb{Z}), G)$ one shows that the sheaf cohomology $H^i(X, \underline{G}_X)$ is the i-th singular cohomology with coefficients in G.

Example 1.3.7 (de Rham complex of sheaves)**.** Let X be as above, and further assume that X is a smooth manifold of dimension n. The *de-Rham resolution*:

$$0 \to \underline{\mathbb{R}}_X \to \underline{\Lambda}^0 \to ... \to \underline{\Lambda}^i \xrightarrow{d} \underline{\Lambda}^{i+1} ... \to \underline{\Lambda}^n \to 0$$

is a soft flat resolution of the constant sheaf $\underline{\mathbb{R}}_X$, of length $n+1$. The exactness of this complex follows from the fact that X is a manifold, and the de-Rham cohomology of $\mathbb{R}^n$ is zero in dimensions ≥ 1 (the Poincare Lemma). The flatness is clear since everything is a real vector space, and softness again follows by the fact that smooth Urysohn functions exist in $\Gamma(X, \underline{\Lambda}^0)$ and $\underline{\Lambda}^i$ are modules over $\underline{\Lambda}^0$.

By the Corollary 1.1.17, the sheaf cohomology $H^i(X, \underline{\mathbb{R}}_X)$ is isomorphic to the i-th cohomology of the de-Rham complex $\Lambda^{\cdot}(X) := \Gamma(X, \underline{\Lambda}^i)$ (called the *i-th de-Rham cohomology*). By combining this with the last para of the previous example, one sees the *de Rham Theorem*, viz. that the de-Rham cohomology is isomorphic to the singular cohomology with coefficients in $\mathbb{R}$, both being isomorphic to the sheaf cohomology $H^*(X, \underline{\mathbb{R}}_X)$.

Example 1.3.8 (Dolbeault complex of sheaves)**.** Let X be as above, and further assume that X is a complex manifold of dimension $\dim_{\mathbb{C}} X = n$. The *Dolbeault* complex of sheaves:

$$0 \to \underline{\Omega}^p \to \underline{\Lambda}^{p,0} \to \ldots \to \underline{\Lambda}^{p,q} \xrightarrow{\overline{\partial}} \underline{\Lambda}^{p,q+1} \ldots \to \underline{\Lambda}^{p,n} \to 0$$

is a soft flat resolution of the sheaf $\underline{\Omega}^p$ of holomorphic p-forms on X of length $n+1$. The exactness of this complex follows from the Dolbeault-Grothendieck lemma. The flatness is clear since everything is a complex vector space, and softness again follows by the fact that smooth Urysohn functions exist in $\Gamma(X, \underline{\Lambda}^{0,0})$ and $\underline{\Lambda}^{p,q}$ are modules over $\underline{\Lambda}^{0,0}$.

By appealing to the Corollary 1.1.17, one again finds that the sheaf cohomology $H^q(X, \underline{\Omega}^p)$ is the q-th cohomology of the Dolbeault complex of global sections $\Lambda^{p,\cdot}(X) := \Gamma(X, \underline{\Lambda}^{p,\cdot})$, called the (p,q)-Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(X)$.

Proposition 1.3.9. Let k be a ring as above, and X a topological space as above. Let $\mathcal{F}$ be any k-sheaf on X. Then there exists an exact-sequence ("left-resolution"):

$$\ldots \to \mathcal{P}_i \to \mathcal{P}_{i-1} \to \ldots \to \mathcal{F} \to 0$$

such that all the $\mathcal{P}^i$ are flat k-sheaves.

Proof: For an open set U, and $j : U \hookrightarrow X$ the inclusion, denote the sheaf $j_! j^* \underline{k}_X = j_! \underline{k}_U$ on X by k_U, for notational simplicity. This is the sheaf whose stalk is k at all $x \in U$ and 0 for all $x \notin U$. Clearly $\Gamma(U, \mathcal{F}) = \hom_k(k_U, \mathcal{F})$, and each $s \in \Gamma(U, \mathcal{F})$ may be written as $s_*(1)$, where $s_* : k_U \to \mathcal{F}$ is the sheaf morphism taking the constant section 1 to s. Thus taking a direct sum $\mathcal{P}^0 := \oplus_{s,U} k_U$, one gets a surjection:

$$\mathcal{P}^0 \xrightarrow{\epsilon} \mathcal{F} \to 0$$

Note that since its stalks are k or 0, k_U is flat for all U, and hence $\mathcal{P}^0$ is a direct sum of flat sheaves, thus flat.

Now apply the same construction to $\ker \epsilon$ to get the flat sheaf $\mathcal{P}^1$ surjecting onto $\ker \epsilon$, and inductively go on to define the flat sheaf $\mathcal{P}^i$. $\square$

Corollary 1.3.10. Let X and k be as above. Assume X is of finite cohomological dimension $\dim_c X = n$. Let $\mathcal{S}$ be a *soft and flat* k-sheaf, and $\mathcal{F}$ be *any* k-sheaf on X. Then the tensor product sheaf $\mathcal{S} \otimes_k \mathcal{F}$ is soft.

Proof: Let $j : U \hookrightarrow X$ be the inclusion, and as in the proof of the Proposition 1.3.9 above, denote by $k_U := j_! j^* \underline{k}_X$. Since $\mathcal{S}$ is a k- sheaf, one checks that scalar multiplication gives a morphism of sheaves:

$$\mathcal{S} \otimes k_U \to j_! j^* \mathcal{S}$$

By comparing stalks inside and outside U, one sees that the map above is an isomorphism. By Remark 1.1.5 and Proposition 1.2.2, the sheaf $j_! j^* \mathcal{S}$ is soft since $\mathcal{S}$ is soft. Thus $\mathcal{S} \otimes k_U$ is soft.

Direct sums of soft sheaves are soft, so $\mathcal{S} \otimes_k (\oplus_{s,U} k_U)$ is soft, where the direct sum is the one in the proof of the Proposition 1.3.9 above. Consequently, in the notation of that proof, $\mathcal{S} \otimes_k \mathcal{P}^0$ is soft.

Likewise each $\mathcal{S} \otimes_k \mathcal{P}^i$ is soft. Let us take the flat left-resolution $\mathcal{P}^\cdot \to \mathcal{F}$ of 1.3.9 above and truncate it to get:

$$0 \to \ker d^n \to \mathcal{P}^{n-1} \to \ldots \to \mathcal{P}^0 \to \mathcal{F} \to 0$$

$\mathcal{S}$ flat implies $0 \to \mathcal{S} \otimes \ker d^n \to \mathcal{S} \otimes \mathcal{P}^{n-1} \to \ldots \to \mathcal{S} \otimes \mathcal{P}^0 \to \mathcal{S} \otimes \mathcal{F} \to 0$ is also exact. By the previous para $\mathcal{S} \otimes_k \mathcal{P}^i$ is soft for $i = n-1, n-2, .., 0$. Because $\dim_c X = n$, it follows by Proposition 1.3.3 that $\mathcal{S} \otimes_k \mathcal{F}$ is soft. The proposition is proved. $\square$

2. Dualising Complex and Homology

As always, X will be a locally compact hausdorff space. k will always be a Noetherian commutative ring with 1.

2.1. The Dualising Sheaf.

Definition 2.1.1 (Dualising presheaf)**.** Let $\mathcal{S}$ be a *soft and flat* k-sheaf. Let G be any k-module. For an open subset $U \subset X$ define *the dualising presheaf* $D(\mathcal{S}, G)$ *of* $\mathcal{S}$ *with coefficients in* G by the sections:

$$\Gamma(U, D(\mathcal{S}, G)) := \hom_k(\Gamma_c(U, \mathcal{S}), G)$$

For open sets $V \subset U$ of X, there is the k-module homomorphism $\Gamma_c(V, \mathcal{S}) \to \Gamma_c(U, \mathcal{S})$ given by extension by zero. The restriction map ρ_{UV} of the presheaf $D(\mathcal{S}, G)$ is defined by dualising this map.

Proposition 2.1.2. Let $\mathcal{S}$ and G be as in the definition above, with $\mathcal{S}$ assumed to be soft. Then:

(i): The presheaf $D(\mathcal{S}, G)$ is a sheaf, called the *dualising sheaf of* $\mathcal{S}$ (with coefficients in G).

(ii): If G is an injective k-module, then $D(\mathcal{S}, G)$ is an injective sheaf.

Proof: Let U, V be open sets in X. Then $\mathcal{S}$ soft implies that $H^1_c(U \cap V, \mathcal{S}) = 0$, by Proposition 1.1.18. The first four terms of the Mayer-Vietoris sequence for compactly supported cohomology read as:

$$\begin{array}{ccccccccc} 0 \to & \Gamma_c(U \cap V, \mathcal{S}) \to & \Gamma_c(U, \mathcal{S}) & \oplus & \Gamma_c(V, \mathcal{S}) \to & \Gamma_c(U \cup V, \mathcal{S}) \to & H^1_c(U \cap V, \mathcal{S}) \to .. \\ & & & & & & \| \\ & & & & & & 0 \end{array}$$

Taking $\hom_k(-, G)$, which is a left exact functor, leads to:

$$0 \to \Gamma(U \cup V, D(\mathcal{S}, G)) \to \Gamma(U, D(\mathcal{S}, G)) \oplus \Gamma(V, D(\mathcal{S}, G)) \to \Gamma(U \cap V, D(\mathcal{S}, G))$$

which is precisely the sheaf condition for the pair $\{U, V\}$. To generalise to arbitrary collections of open sets use Zorn's lemma (or transfinite induction). This proves (i)

To see (ii), first make the

Claim 1: Let $\mathcal{S}$ be a *soft and flat* k-sheaf on X. Then there is an isomorphism:

$$\hom_k(\mathcal{F}, D(\mathcal{S}, G)) \simeq \hom_k(\Gamma_c(X, \mathcal{F} \otimes_k \mathcal{S}, G) \tag{1}$$

for all k-sheaves $\mathcal{F}$ on X.

Proof of Claim 1: Note the left side of (1) is an abelian group of morphisms between two k-sheaves, whereas the right side is an abelian group of morphisms between two k-modules. First of all there is the map of abelian groups

$$\begin{array}{rcl} \theta : \hom_k(\mathcal{F}, D(\mathcal{S}, G)) & \longrightarrow & \hom_k(\Gamma_c(X, \mathcal{F} \otimes_k \mathcal{S}, G) \\ \alpha & \mapsto & \alpha_X \end{array}$$

where α_X is the morphism on global sections of the two sheaves $\mathcal{F}$ and $D(\mathcal{S}, G)$ induced by the sheaf morphism α. We claim that this is the map effecting the isomorphism claimed.

We first take the special case of a sheaf $\mathcal{F} = k_U := j_! j^* \underline{k}_X$ which was introduced in the proof of the Proposition 1.3.9, where U is an open set. In this case, using the remark made in the proof of Corollary 1.3.10, we have $\mathcal{F} \otimes_k \mathcal{S} = k_U \otimes_k \mathcal{S} = j_! j^* \mathcal{S}$. Thus

$$\Gamma_c(X, \mathcal{F} \otimes_k \mathcal{S}) = \Gamma_c(X, j_! j^* \mathcal{S}) = \Gamma_c(U, \mathcal{S})$$

so the right side of (1) is equal to $\hom_k(\Gamma_c(U, \mathcal{S}), G)$. The left side is

$$\hom_k(k_U, D(\mathcal{S}, G)) = \Gamma(U, D(\mathcal{S}, G)) = \hom_k(\Gamma_c(U, \mathcal{S}), G)$$

where the first equality follows by seeing the image of the section "1" and the second one is from the definition. Thus Claim 1 follows for $\mathcal{F} = k_U$.

For a general sheaf $\mathcal{F}$, write the two step flat left-resolution described in Proposition 1.3.9:

$$\mathcal{P}^1 \to \mathcal{P}^0 \to \mathcal{F} \to 0$$

where both $\mathcal{P}^0$ and $\mathcal{P}^1$ are direct sums of sheaves of the type k_U considered in the last para. Thus (1) holds for both $\mathcal{P}^0$ and $\mathcal{P}^1$.

Since $\mathcal{S}$ is assumed to be flat, we have exactness of:

$$\mathcal{P}^1 \otimes_k \mathcal{S} \to \mathcal{P}^0 \otimes_k \mathcal{S} \to \mathcal{F} \otimes_k \mathcal{S} \to 0$$

By Corollary 1.3.10, since $\mathcal{S}$ is flat and soft, all the three sheaves above are soft. Thus $\Gamma_c(X, -)$ preserves exactness, and we have an exact sequence:

$$\Gamma_c(X, \mathcal{P}^1 \otimes_k \mathcal{S}) \to \Gamma_c(X, \mathcal{P}^0 \otimes_k \mathcal{S}) \to \Gamma_c(X, \mathcal{F} \otimes_k \mathcal{S}) \to 0$$

Since $\hom_k(-, G)$ is a left exact functor on k-modules, and $\hom_k(-, D(\mathcal{S}, G))$ is a left exact functor of sheaves we have exactness of the rows in the diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \hom(\mathcal{F}, D(\mathcal{S}, G)) \longrightarrow & & \hom(\mathcal{P}^0, D(\mathcal{S}, G)) \longrightarrow & & \hom(\mathcal{P}^1, D(\mathcal{S}, G)) \\
& \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\
0 \to & \hom_k(\Gamma_c(X, \mathcal{F} \otimes_k \mathcal{S}), G) \to & & \hom_k(\Gamma_c(X, \mathcal{P}^0 \otimes_k \mathcal{S}), G) \to & & \hom_k(\Gamma_c(X, \mathcal{P}^1 \otimes_k \mathcal{S}), G)
\end{array}$$

where the right two vertical arrows are isomorphisms by the last para. Hence the first vertical arrow is also an isomorphism. This proves Claim 1 $\square$

To return to the proof of the proposition, note that $\mathcal{S}$ soft and flat implies (by Corollary 1.3.10) that for a general short exact sequence of sheaves on X:

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

the tensored sequence

$$0 \to \mathcal{F}_1 \otimes \mathcal{S} \to \mathcal{F}_2 \otimes \mathcal{S} \to \mathcal{F}_3 \otimes \mathcal{S} \to 0$$

is an exact sequence of soft sheaves. Since Γ_c is acyclic for soft sheaves, (see (ii) of Lemma 1.1.13), we have exactness of:

$$0 \to \Gamma_c(X, \mathcal{F}_1 \otimes \mathcal{S}) \to \Gamma_c(X, \mathcal{F}_2 \otimes \mathcal{S}) \to \Gamma(X, \mathcal{F}_3 \otimes \mathcal{S}) \to 0$$

Now G injective implies that the sequence:

$$0 \to \hom_k(\Gamma_c(X, \mathcal{F}_1 \otimes \mathcal{S}), G) \to \hom_k(\Gamma_c(X, \mathcal{F}_2 \otimes \mathcal{S}), G) \to \hom_k(\Gamma(X, \mathcal{F}_3 \otimes \mathcal{S}), G) \to 0$$

is exact. Now appealing to Claim 1 shows that:

$$0 \to \hom(\mathcal{F}_1, D(\mathcal{S}, G)) \to \hom(\mathcal{F}_2, D(\mathcal{S}, G)) \to \hom(\mathcal{F}_3, D(\mathcal{S}, G)) \to 0$$

is exact. That is, $\hom_k(-, D(\mathcal{S}, G))$ is an exact functor, so $D(\mathcal{S}, G)$ is an injective sheaf and (ii) is proved. $\square$

2.2. **The Dualising Complex of Sheaves and Borel-Moore Homology.**

Definition 2.2.1 (Dualising complex)**.** Let X, k be as above, and let $\mathcal{S}^\cdot$ be a complex of *soft* and *flat* k-sheaves on X. Let $G^\cdot$ be a complex of injective k-modules. Define a complex of presheaves by setting its sections over $U \subset X$ open as:

$$\Gamma(U, D^\cdot(\mathcal{S}^\cdot, G^\cdot)) := \hom_k^\cdot(\Gamma_c(U, \mathcal{S}^\cdot), G^\cdot)$$

where $\hom^\cdot$ denotes the hom-complex. It is called the *dualising complex of* $\mathcal{S}^\cdot$ *with coefficients in* $G^\cdot$.

Since each term of the dualising complex above is a direct sum of $D(\mathcal{S}^j, G^i)$, the following proposition is an immediate corollary of Proposition 2.1.2:

Proposition 2.2.2. Let $\mathcal{S}^\cdot$ be a complex of *flat* and *soft* k-sheaves on X and $G^\cdot$ be a complex of injective k-modules. Then we have:

(i): The dualising complex $D^\cdot(\mathcal{S}^\cdot, G^\cdot)$ is a complex of k-sheaves on X.

(ii): The complex $D^\cdot(\mathcal{S}^\cdot, G^\cdot)$ is an injective complex.

Example 2.2.3 (Fundamental sheaf for integral Borel-Moore homology)**.** It is instructive to work out the special example of the dualising complex for the case of $\mathcal{S}^\cdot = \mathcal{C}^\cdot$, the singular cochain complex of sheaves, which was seen to be a soft flat complex in the Example 1.3.6. For simplicity, take $k = \mathbb{Z}$, and set $G^\cdot$ to be the 2-term injective resolution

$$\mathbb{Q} \xrightarrow{\alpha} \mathbb{Q}/\mathbb{Z} \to 0$$

of $\mathbb{Z}$. In this case the dualising complex is a *cochain complex* concentrated in negative degrees, so for convenience we define the *chain complex* by the usual trick of flipping the sign of the grading. More precisely, we define the chain complex of sheaves $\mathcal{D}_.$ by:

$$\Gamma(U, \mathcal{D}_i) := \Gamma(U, D^{-i}(\mathcal{C}^\cdot, G^\cdot)) = \hom^{-i}(\Gamma_c(U, \mathcal{C}^\cdot), G^\cdot) = \hom(\Gamma_c(U, \mathcal{C}^i), \mathbb{Q}) \oplus \hom(\Gamma_c(U, \mathcal{C}^{i+1}), \mathbb{Q}/\mathbb{Z})$$

By the Proposition above, this is an injective chain complex of sheaves called the *fundamental homology sheaf.* The i-th *integral Borel-Moore homology of* X is defined as:

$$H_i^{BM}(X, \mathbb{Z}) := H_i(\Gamma(X, \mathcal{D}_.))$$

the homology of the global section complex of $\mathcal{D}_.$. The i-th *integral homology of* X is defined as

$$H_i(X, \mathbb{Z}) := H_i(\Gamma_c(X, \mathcal{D}_.))$$

The following lemmas and propositions will hopefully clarify the relation with our familiar geometric notions of homology.

First we state a universal coefficient theorem for cell-complexes X.

Lemma 2.2.4 (Universal coefficients)**.** Let X be a CW-complex with finitely many cells in each dimension. Then there is a functorial, but non-functorially split short exact sequence:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H^{n+1}(X), \mathbb{Z}) \to H_n(X) \to \hom_{\mathbb{Z}}(H^n(X), \mathbb{Z}) \to 0 \tag{2}$$

where $H^\cdot(X)$ (resp. $H_.(X)$) denote singular cohomology (resp. singular homology) with integer coefficients.

Proof: We let $C^\cdot$ denote the cellular cochain complex of X, defined by $C^i = H^i(X^i, X^{i-1})$. Each C^i is free and of finite rank by hypothesis. Thus the short exact sequence:

$$0 \to Z^\cdot \to C^\cdot \xrightarrow{d} B^\cdot[1] \to 0$$

is a short exact sequence of free cochain complexes. Consequently, by taking $\hom(-, \mathbb{Z})$, we get the short exact sequence of chain complexes:

$$0 \to \hom(B^\cdot[1], \mathbb{Z}) \to C_. \to \hom(Z^\cdot, \mathbb{Z}) \to 0$$

where $C_i := \hom(C^i, \mathbb{Z}) = H_i(X^i, X^{i-1})$ defines the cellular chain complex of X. Thus, by the associated long-exact homology sequence:

$$... \to \hom(B^{n+1}, \mathbb{Z}) \to H_n(X) \to \hom(Z^n, \mathbb{Z}) \xrightarrow{j^{n*}} \hom(B^n, \mathbb{Z}) \to ...$$

where $j^n : B^n \to Z^n$ is the inclusion (since $H_n(C_.)$ is well-known to be the integral singular homology $H_n(X)$). Thus we have short exact sequences:

$$0 \to \mathrm{Coker}\, j^{n+1*} \to H_n(X) \to \ker j^{n*} \to 0 \tag{3}$$

which are (non-functorially) split since the right hand group $\ker j^{n*}$ is a subgroup of $\hom(Z^n, \mathbb{Z})$, and hence finitely generated torsion free, so free.

Now the short exact sequence:

$$0 \to B^n \xrightarrow{j^n} Z^n \to H^n(X) \to 0$$

is a free resolution of the integral cohomology $H^n(X) = H^n(C^\cdot)$, so we have exact sequences:

$$0 \to \hom_{\mathbb{Z}}(H^n(X), \mathbb{Z}) \to \hom_{\mathbb{Z}}(Z^n, \mathbb{Z}) \xrightarrow{j^{n*}} \hom_{\mathbb{Z}}(B^n, \mathbb{Z}) \to \mathrm{Ext}_{\mathbb{Z}}(H^n(X), \mathbb{Z}) \to 0$$

which shows that $\ker j^{n*} \simeq \hom_{\mathbb{Z}}(H^n(X), \mathbb{Z})$ and $\mathrm{Coker}\, j^{n*} \simeq \mathrm{Ext}_{\mathbb{Z}}(H^n(X), \mathbb{Z})$. Substituting these groups in (3) above yields the result. $\square$

Remark 2.2.5 (Algebraic duality)**.** The result above shows that for X a CW-complex with finitely many cells in each dimension, the integral singular homology $H_i(X)$ is *algebraically dual* to integral singular cohomology in the sense of the exact sequence (2).

Lemma 2.2.6. Let $M^\cdot$ be a cochain complex of abelian groups, and define the *chain complex*

$$D(M)_i = \hom(M^i, \mathbb{Q}) \oplus \hom(M^{i+1}, \mathbb{Q}/\mathbb{Z}) = \hom^{-i}(M^\cdot, G^\cdot)$$

where $G^\cdot : \mathbb{Q} \xrightarrow{\alpha} \mathbb{Q}/\mathbb{Z}$ is the standard injective resolution of $\mathbb{Z}$. Then we have a functorial short exact sequence:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H^{n+1}(M^\cdot), \mathbb{Z}) \to H_n(D(M)_\cdot) \to \hom_{\mathbb{Z}}(H^n(M^\cdot), \mathbb{Z}) \to 0$$

If we further assume $H^\cdot(M^\cdot)$ are finitely generated, then the exact sequence above is non-functorially split.

Proof: We denote by α_r the map:

$$\alpha_r := \hom(1, \alpha) : \hom(M^r, \mathbb{Q}) \to \hom(M^r, \mathbb{Q}/\mathbb{Z})$$

Then we see that $\alpha_\cdot : \hom(M^\cdot, \mathbb{Q}) \to \hom(M^\cdot, \mathbb{Q}/\mathbb{Z})$ is a chain map of chain complexes, and the complex $D(M)_\cdot$ defined above identifies as:

$$D(M)_\cdot = C(\alpha)_\cdot$$

the mapping cone of $\alpha_\cdot$. The usual Puppe long exact sequence of the mapping cone thus reads as:

$$\to H_{n+1}(\hom(M^\cdot, \mathbb{Q}/\mathbb{Z})) \to H_n(D(M)_\cdot) \to H_n(\hom(M^\cdot, \mathbb{Q})) \xrightarrow{\alpha_{n*}} H_n(\hom(M^\cdot, \mathbb{Q}/\mathbb{Z})) \to ..$$

Noting that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, it follows that $\hom(-, \mathbb{Q})$ and $\hom(-, \mathbb{Q}/\mathbb{Z})$ commute with taking cohomology, so $H_n(\hom(M^\cdot, \mathbb{Q}/\mathbb{Z})) = \hom(H^n(M^\cdot), \mathbb{Q}/\mathbb{Z})$ and likewise $H_n(\hom(M^\cdot, \mathbb{Q})) = \hom(H^n(M^\cdot), \mathbb{Q})$. Thus we have short exact sequences:

$$0 \to \mathrm{Coker}\, \alpha_{n+1\#} \to H_n(D(M)_\cdot) \to \ker \alpha_{n\#} \to 0 \tag{4}$$

where $\alpha_{n\#} := \hom(1, \alpha) : \hom(H^n(M^\cdot), \mathbb{Q}) \to \hom(H^n(M^\cdot), \mathbb{Q}/\mathbb{Z})$

Applying $\hom(H^n(M^\cdot), -)$ to the short exact sequence:

$$0 \to Z \to \mathbb{Q} \xrightarrow{\alpha} \mathbb{Q}/\mathbb{Z} \to 0$$

leads (by the Snake lemma)to the four term sequence:

$$0 \to \hom(H^n(M^\cdot), \mathbb{Z}) \to \hom(H^n(M^\cdot), \mathbb{Q}) \xrightarrow{\alpha_{n\#}} \hom(H^n(M^\cdot), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}_{\mathbb{Z}}(H^n(M^\cdot, \mathbb{Z}) \to 0$$

since $\mathrm{Ext}_{\mathbb{Z}}(H^n(M^\cdot), \mathbb{Q}) = \mathrm{Ext}_Z(H^n(M^\cdot), \mathbb{Q}/\mathbb{Z}) = 0$ by the injectivity of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Thus we get:

$$\mathrm{Coker}\, \alpha_{n\#} = \mathrm{Ext}_Z(H^n(M^\cdot), \mathbb{Z}); \;\; \ker \alpha_{n\#} = \hom(H^n(M^\cdot), \mathbb{Z})$$

Plugging this in (4) yields the result. If $H^n(M^\cdot)$ is finitely generated, then $\hom(H^n(M^\cdot), \mathbb{Z})$ is finitely generated torsion free, so free. Hence the sequence above splits (non-functorially). $\square$

The following proposition shows that Borel-Moore homology as defined in Example 2.2.3 is algebraically dual to sheaf-cohomology with compact supports in the constant sheaf $\underline{\mathbb{Z}}_X$, in the sense of Remark 2.2.5.

Proposition 2.2.7 (Borel-Moore homology)**.** For X as above, and $\mathcal{D}_\cdot$ the chain complex of sheaves defined in Example 2.2.3, we have a functorial short exact sequence:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_c^{n+1}(X, \underline{\mathbb{Z}}_X), \mathbb{Z}) \to H_n^{BM}(X, \mathbb{Z}) \to \hom_{\mathbb{Z}}(H_c^n(X, \underline{\mathbb{Z}}_X), \mathbb{Z}) \to 0$$

where $H_c^\cdot(X, \underline{\mathbb{Z}}_X)$ is the integer (sheaf) cohomology with compact supports. If these last cohomologies are all finitely generated, then the sequence above splits non-functorially.

Proof: Follows immediately from Lemma 2.2.6 by noting that for the cochain complex of $\mathbb{Z}$-modules $M^j := \Gamma_c(X, \mathcal{S}^\cdot)$, where $\mathcal{S}^\cdot$ is a soft flat resolution of $\underline{\mathbb{Z}}_X$, the chain complex $D(M)_\cdot$ is nothing but the global section chain complex $\Gamma(X, \mathcal{D}_\cdot)$ of the fundamental homology complex of sheaves. $\square$

Proposition 2.2.8. Let X be as above. Arguments similar (and simpler) to the above will show that if we define a chain complex of sheaves by:

$$\Gamma(U,\mathcal{D}_i^{\mathbb{R}}) := \Gamma(U, D^{-i}(\mathcal{S}^\cdot,\mathbb{R})) = \hom_{\mathbb{Z}}(\Gamma_c(U,\mathcal{S}^i),\mathbb{R})$$

where $\mathcal{S}^\cdot$ is the soft flat resolution of the constant sheaf $\underline{\mathbb{Z}}_X$ by $\mathbb{Z}$-sheaves, the then we have a similar algebraic duality:

$$H^n(\Gamma(X,\mathcal{D}_\cdot^{\mathbb{R}})) = \hom_{\mathbb{R}}(H_c^n(X,\mathbb{Z}),\mathbb{R})$$

where no ext term appears since $\mathbb{R}$ is injective as an abelian group. The left hand side is Borel-Moore homology with $\mathbb{R}$-coefficients.

There is an analogous result for the de-Rham complex of a smooth manifold X. More precisely:

Proposition 2.2.9. Let X be a smooth manifold of dimension n. Arguments similar (and simpler) to the above will show that if we define a chain complex of sheaves by:

$$\Gamma(U,\mathcal{D}_i^{dR}) := \Gamma(U, D^{-i}(\underline{\Lambda}^\cdot,\mathbb{R})) = \hom_{\mathbb{R}}(\Gamma_c(U,\underline{\Lambda}^i),\mathbb{R})$$

where $\underline{\Lambda}^\cdot$ is the soft, flat de-Rham complex of $\mathbb{R}$-sheaves resolving the constant sheaf $\underline{\mathbb{R}}_X$, then we have an algebraic duality:

$$H^n(\Gamma(X,\mathcal{D}_\cdot^{dR})) = \hom_{\mathbb{R}}(H_{c,dR}^n(X),\mathbb{R})$$

where again no exts appear since everything is an $\mathbb{R}$-module. The left hand side is Borel-Moore homology of X with $\mathbb{R}$-coefficients, and $H_{c,dR}^\cdot$ denotes de-Rham cohomology with compact supports.

Remark 2.2.10. Note that the sheaf $\Gamma(X,\mathcal{D}_i^{dR}) = \hom_{\mathbb{R}}(\Gamma_c(X,\underline{\Lambda}^i),\mathbb{R})$ defined in Proposition 2.2.9 is a very large vector space, even larger than the space $\mathcal{E}_i(X)$ of i-currents, which contains only the *continuous elements* of $\hom_{\mathbb{R}}(\Gamma_c(U,\underline{\Lambda}^i),\mathbb{R})$. It is therefore quite miraculous that both these complexes compute the i-th Borel-Moore homology with $\mathbb{R}$-coefficients (which is *algebraically dual* to i-th *compactly supported* de Rham cohomology by 2.2.9, *and* which Poincare duality will later show to be *Poincare dual* to $(n-i)$-th de Rham cohomology when X is orientable.

3. Verdier Duality and Poincare Duality

3.1. **Verdier Duality.** The dualising complex construction above will enable us to clarify the extent to which Poincare Duality is a local cohomological result. In articular, one will be able to conclude Poincare Duality for cohomology manifolds. First we state the formal algebraic result which is the key to Poincare duality at the local level.

Theorem 3.1.1 (Verdier Duality)**.** Let X be as above, k a Noetherian commutative ring with 1. Let $\mathcal{S}^\cdot$ be a *soft and flat* resolution of the constant sheaf $\underline{k}_X$ by k-sheaves, and let $G^\cdot$ be an injective complex of k-modules. Let $\mathcal{I}^\cdot$ be an injective complex of k-sheaves on X. Then, for the dualising complex $D^\cdot(\mathcal{S}^\cdot, G^\cdot)$ (see Definition 2.2.1) we have a quasi-isomorphism of complexes of abelian groups:

$$\hom^\cdot(\mathcal{I}^\cdot, D(\mathcal{S}^\cdot,G^\cdot)) \simeq \hom^\cdot(\Gamma_c(X,\mathcal{I}^\cdot),G^\cdot) \tag{5}$$

On taking 0-th cohomology of both complexes above, we have a functorial isomorphism of abelian groups:

$$[\mathcal{I}^\cdot, D^\cdot(\mathcal{S}^\cdot,G^\cdot)] \simeq [\Gamma_c(X,\mathcal{I}^\cdot),G^\cdot] \tag{6}$$

where $[-,-]$ denotes chain-homotopy classes of chain maps in the appropriate category.

Proof: Tensoring the resolution $\underline{k}_X \to \mathcal{S}^\cdot$ with $\mathcal{I}^\cdot$ gives a morphism:

$$\theta : \mathcal{I}^\cdot \otimes \underline{k}_X \to \mathcal{I}^\cdot \otimes_k \mathcal{S}^\cdot$$

We claim θ is a quasi-isomorphism (of complexes of sheaves). Enough to check that θ induces isomorphism of derived sheaves (stalk cohomologies). There are two Kunneth spectral sequences associated to tensor product complexes of stalks, abutting to the cohomologies of the tensor product stalk complexes on either side above. The sheaf morphism θ induces the following morphism of E_2-terms at the stalk level:

$$\theta : E_2^{pq} = H^p(\mathcal{I}_x^\cdot)\otimes H^q(k) = E_2^{p0} = H^p(\mathcal{I}_x^\cdot)\otimes k \longrightarrow \widetilde{E}_2^{pq} = H^p(\mathcal{I}_x^\cdot)\otimes H^q(\mathcal{S}_x^\cdot) = \widetilde{E}_2^{p0} = H^p(\mathcal{I}_x^\cdot)\otimes k$$

which is clearly an isomorphism at E_2, and both spectral sequences collapsing shows that θ is a q.i. as asserted.

Now $\mathcal{I}^\cdot$ is injective, thus soft (see Proposition 1.1.8), $\mathcal{I}^\cdot \otimes_k \mathcal{S}^\cdot$ is soft by Corollary 1.3.10 (since $\mathcal{S}^\cdot$ are soft and flat). The functor Γ_c is acyclic for soft sheaves (see (ii) of Proposition 1.1.16), so the map induced by θ (also denoted θ)

$$\theta : \Gamma_c(X, \mathcal{I}^\cdot) \to \Gamma_c(X, \mathcal{I}^\cdot \otimes_k \mathcal{S}^\cdot)$$

is a quasi-isomorphism as well. Since $G^\cdot$ is injective, $\hom_k(-, G^\cdot)$ is an exact functor and we have a quasi-isomorphism:

$$\hom^\cdot(\Gamma_c(X, \mathcal{I}^\cdot \otimes_k \mathcal{S}^\cdot), G^\cdot) \to \hom^\cdot(\Gamma_c(X, \mathcal{I}^\cdot), G^\cdot)$$

Now we go back to the identity (1) of Claim 1 in the proof of Proposition 2.1.2. By rewriting that identity for complexes (and veifying that the differentials agree), we see that the left hand side of the last relation can be identified as:

$$\hom^\cdot(\Gamma_c(X, \mathcal{I}^\cdot \otimes_k \mathcal{S}^\cdot), G^\cdot) \simeq \hom^\cdot(\mathcal{I}^\cdot, D(\mathcal{S}^\cdot, G^\cdot))$$

which proves the q.i. asserted in (5). Finally, as mentioned in the statement of the theorem, (6) follows from (5). □

Corollary 3.1.2 (Verdier Duality 2)**.** Let $\mathcal{T}^\cdot$ be a complex of *soft* k-sheaves on X, and $\mathcal{S}^\cdot$ and $G^\cdot$ as in the Theorem 3.1.1 above. Then again we have a quasi-isomorphism:

$$\hom^\cdot(\mathcal{T}^\cdot, D(\mathcal{S}^\cdot, G^\cdot)) \simeq \hom^\cdot(\Gamma_c(X, \mathcal{T}^\cdot), G^\cdot) \tag{7}$$

On taking 0-th cohomology of both complexes above, we have a functorial isomorphism of abelian groups:

$$[\mathcal{T}^\cdot, D^\cdot(\mathcal{S}^\cdot, G^\cdot)] \simeq [\Gamma_c(X, \mathcal{T}^\cdot), G^\cdot] \tag{8}$$

Proof: Let $\mathcal{I}^\cdot$ be an injective resolution for $\mathcal{T}^\cdot$, so we have a quasi-isomorphism $\mathcal{T}^\cdot \to \mathcal{I}^\cdot$. Since $D(\mathcal{S}^\cdot, G^\cdot)$ is an injective complex of sheaves (by (ii) of Proposition 2.2.2), we have $\hom^\cdot(-, D(\mathcal{S}^\cdot, G^\cdot))$ is an exact functor on the category of complexes of k-sheaves, and hence we have a quasi-isomorphism:

$$\hom_k^\cdot(\mathcal{I}^\cdot, D(\mathcal{S}^\cdot, G^\cdot)) \to \hom_k^\cdot(\mathcal{T}^\cdot, D(\mathcal{S}^\cdot, G^\cdot))$$

Since Γ_c is acyclic for soft sheaves and injective sheaves (which are soft), and $G^\cdot$ injective implies $\hom_k(-, G^\cdot)$ is exact, we have a quasi-isomorphism of complexes of k-modules

$$\hom_k^\cdot(\Gamma_c(X, \mathcal{I}^\cdot), G^\cdot) \to \hom_k^\cdot(\Gamma_c(X, \mathcal{T}^\cdot), G^\cdot)$$

By the Verdier Duality Theorem 3.1.1 above, the left hand sides of the two relations above are quasi-isomorphic. Hence so are the right-hand sides, proving the corollary. The relation (8), follows from (7) by taking H^0. □

3.2. **Poincare Duality.**

Proposition 3.2.1 (Derived sheaf of the dualising sheaf)**.** Let X, k be as above, with $\dim_c X = n$. Let let $\underline{k}_X \to \mathcal{S}^\cdot$ be a *soft and flat* resolution of the constant sheaf, and $k \to K^\cdot$ be an injective resolution of k by k-modules. Then the dualising complex $D^\cdot(\mathcal{S}^\cdot, K^\cdot)$ is quasi-isomorphic to a complex $\mathcal{D}^\cdot$ of sheaves satisfying:

$$\begin{aligned} \mathcal{D}^p &= 0 \quad \text{for } p < -n \\ H^{-n}(\Gamma(U, \mathcal{D}^\cdot)) &= \hom_k(H^n_c(U, \underline{k}_U), k) \quad \text{for } U \text{ any open subset of } X \end{aligned}$$

Proof: For simplicity of notation, write $\mathcal{E}^\cdot := D^\cdot(\mathcal{S}^\cdot, K^\cdot)$.

Since $\dim_c X = n$, we can (by Proposition 1.3.5) find a soft flat resolution of length $(n+1)$:

$$0 \to \underline{k}_X \to \mathcal{S}^0 \to \ldots \to \mathcal{S}^n \to 0$$

for the constant sheaf $\underline{k}_X$. For an open subset U, letting $j : U \hookrightarrow X$ denote the inclusion map, we see that (by Corollary 1.1.19 and Proposition 1.2.2) the complexes $j_! j^* \mathcal{S}^\cdot[p]$ are soft complexes for all p.

By the relation (8) of Corollary 3.1.2 applied to $\mathcal{T}^{\cdot} = j_! j^* \mathcal{S}$, we therefore have the isomorphism:

$$[j_! j^* \mathcal{S}^{\cdot}[p], \mathcal{E}^{\cdot}] \simeq [\Gamma_c(X, j_! j^* \mathcal{S}^{\cdot}[p]), K^{\cdot}] = [\Gamma_c(U, \mathcal{S}^{\cdot}[p]), K^{\cdot}] \quad \text{for all } p \tag{9}$$

Now $j_! j^* \mathcal{S}^{\cdot}$ is a sheaf that is zero outside U, and hence the left side can be rewritten:

$$[j_! j^* \mathcal{S}^{\cdot}[p], \mathcal{E}^{\cdot}] = [[j_! j^* \mathcal{S}^{\cdot}, \mathcal{E}^{\cdot}[-p]] = [\mathcal{S}^{\cdot}_{|U}, \mathcal{E}^{\cdot}[-p]_{|U}] \quad \text{for all } p \tag{10}$$

Since $\mathcal{E}^{\cdot}_{|U}[-p]$ is an injective complex of sheaves (by Remark 1.1.5), and $\underline{k}_U \to \mathcal{S}^{\cdot}_{|U}$ continues to be a soft resolution (by Corollary 1.1.19), we have:

$$[\mathcal{S}^{\cdot}_{|U}, \mathcal{E}^{\cdot}[-p]_{|U}] \simeq [\underline{k}_U, \mathcal{E}^{\cdot}_{|U}[-p]] = H^{-p}(\hom(\underline{k}_U, \mathcal{E}^{\cdot}_{|U})) = H^{-p}(\Gamma(U, \mathcal{E}^{\cdot})) \quad \text{for all } \ p$$

Combining this relation with (9) above, we have:

$$[\Gamma_c(U, \mathcal{S}^{\cdot}[p]), K^{\cdot}] = H^{-p}(\Gamma(U, \mathcal{E}^{\cdot})) \quad \text{for all } U \text{ open and all } \ p \tag{11}$$

Now if $p > n$, then $\mathcal{S}^{\cdot}[p]$ will be non-zero only in strictly negative degrees, whereas $K^{\cdot}$ lives only in non-negative degrees. Thus the left side of relation (11) vanishes. Thus it follows that

$$H^{-p}(\Gamma(U, \mathcal{E}^{\cdot})) = 0 \quad \text{for } p > n, \text{ and all open } U$$

By taking direct limits over neighbourhoods U of a fixed point x, we get $H^{-p}(\mathcal{E}^{\cdot}_x) = 0$ for $p > n$. Thus the derived sheaf $\mathcal{H}^{-p}(\mathcal{E}^{\cdot}) = 0$ for $p > n$. Hence we can replace the complex $\mathcal{E}^{\cdot}$ with the complex $\mathcal{D}^{\cdot} := \tau_{\geq -n}\mathcal{E}^{\cdot}$.

By the relation (11) above,

$$H^{-n}(\Gamma(U, \mathcal{D}^{\cdot}) = H^{-n}(\Gamma(U, \mathcal{E}^{\cdot})) = [\Gamma_c(U, \mathcal{S}^{\cdot}[n]), K^{\cdot}]$$

where the last but one equality follows from the quasi-isomorphism of $\mathcal{E}^{\cdot}$ and $\mathcal{D}^{\cdot}$ and the last equality follows from the relation (11). Now $\mathcal{S}^{\cdot}[n]$ is concentrated in ≤ 0 degrees, and $K^{\cdot}$ in ≥ 0 degrees, so the right hand group above is nothing but

$$[\Gamma_c(U, \mathcal{S}^{\cdot}[n]), K^{\cdot}] = \hom(H^0(\Gamma_c(U, \mathcal{S}^{\cdot}[n]), H^0(K^{\cdot})) = \hom(H^n_c(U, \underline{k}_U), k)$$

which proves the proposition. □

Definition 3.2.2. Let X be a k-cohomology n-manifold of $\dim_c X = n$. That is, each point $x \in X$ has a fundamental system of neighbourhoods U satisfying $H^i_c(U, \underline{k}_U) = k$ for $i = n$ and zero for $i \neq n$. Then consider the presheaf Or_X defined by:

$$\Gamma(U, Or_X) = \hom_k(H^n_c(U, \underline{k}_U), k)$$

where the restriction maps come from dualising the natural forward map $H^n_c(V, \underline{k}_U) \to H^n_c(U, \underline{k}_U)$ coming from extension of sections by zero.

That Or_X is a sheaf follows from the Mayer-Vietoris sequence for cohomology with compact supports (see the Proof of (i) of Proposition 2.1.2). It is called the *k-orientation sheaf* of X.

Finally say that X is *k-orientable* if Or_X turns out to be the constant sheaf $\underline{k}$.

Proposition 3.2.3. Let X be a k-cohomology n-manifold. Then the dualising complex $\mathcal{E}^{\cdot} = D^{\cdot}(\mathcal{S}^{\cdot}, K^{\cdot})$ of Proposition 3.2.1 above is quasi-isomorphic to the single term complex $Or_X[n]$. (That is, the sheaf Or_X placed at the $(-n)$-th spot.

Proof: Recall the relation (11) from the previous proposition:

$$H^{-p}(\Gamma(U,\mathcal{E}^{\cdot}))\simeq[\Gamma_c(U,\mathcal{S}^{\cdot}[p]),K^{\cdot}]$$

with all the notation of that proposition. Now if an open set U is cohomologically equivalent to $\mathbb{R}^n$, then the complex of k-modules $\Gamma_c(U,\mathcal{S})^{\cdot}$ is quasi-isomorphic to $H^n_c(U,\underline{k}_U)[-n]$ (since soft resolutions also compute compactly supported cohomology). Thus we have a quasi-isomorphism

$$\Gamma_c(U,\mathcal{S}^{\cdot}[p])\xrightarrow{q.i.}H^n_c(U,\underline{k}_U)[p-n]$$

Since $K^{\cdot}$ is an injective complex of k-modules, we have:

$$[\Gamma_c(U,\mathcal{S}^{\cdot}[p]),K^{\cdot}]\xrightarrow{q.i.}[H^n_c(U,\underline{k}_U)[p-n],K^{\cdot}]\simeq[[H^n_c(U,\underline{k}_U),K^{\cdot}[n-p]]=\mathrm{Ext}^{n-p}_k([H^n_c(U,\underline{k}_U),k)$$

But since $H^n_c(U,\underline{k}_U)=k$, it follows that the right hand side of the above relation is 0 for $n\neq p$, and $=\hom(H^n_c(U,\underline{k}_U),k)=\Gamma(U,Or_X)$ for $p=n$.

Thus we have for a cohomologically trivial neighbourhood U of any $x\in X$:

$$\begin{aligned} H^{-p}(\Gamma(U,\mathcal{E}^{\cdot})) &= \Gamma(U,Or_X) \text{ for } p=n \qquad (12)\\ &= 0 \text{ for } p\neq n \qquad (13)\end{aligned}$$

from which it follows (by taking direct limit over U a fundamental system of cohomologically trivial neighbourhoods U of x) that the derived sheaf $\mathcal{H}^{-p}(\mathcal{E}^{\cdot})$ coincides with that of $Or_X[n]$ via the map above. The proposition follows. □.

Corollary 3.2.4 (Poincare Duality)**.** Let X be a cohomology n- manifold. Let $\mathcal{D}_{\cdot}=\mathcal{E}^{-\cdot}$ denote the dualising complex, of the previous proposition (written as a chain complex, with ≥ 0 grading). Then we have isomorphisms:

$$H^i(X,Or_X)\simeq H_{n-i}(\Gamma(X,\mathcal{D}_{\cdot})$$

and analogously:

$$H^i_c(X,Or_X)\simeq H_{n-i}(\Gamma_c(X,\mathcal{D}_{\cdot})$$

Note that by the discussion in Example 2.2.3, for $k=\mathbb{Z}$, the right hand sides of the relations above maybe interpreted as integral Borel-Moore homology (resp. integral homology) of X.

Proof: The complex of sheaves $\mathcal{E}[-n]^{\cdot}$ is quasi-isomorphic to the sheaf $Or_X[n]$. Hence we have an injective resolution:

$$Or_X\to\mathcal{E}^{\cdot}[-n]$$

of the orientation sheaf Or_X. Hence the sheaf cohomology $H^i(X,Or_X)=H^i(\Gamma(X,\mathcal{E}[-n]_{\cdot})=H^{i-n}(\Gamma(X,\mathcal{E}^{\cdot}))=H_{n-i}(\Gamma(X,\mathcal{D}_{\cdot}))$. Likewise for cohomology with compact supports. □

4. Appendix

4.1. **PL-spaces, geometric chains and cochains.** Rather like the definition of a C^∞-structure on a smooth manifold as an *equivalence class* of C^∞-atlases on it, one can define a PL-structure on a polyhedron as an equivalence class. *All triangulations considered below are locally finite by assumption.*

Definition 4.1.1 (PL-structure and PL-space)**.** Let X be a topological space which is a simplicical complex (=polyhedron). Say a triangulation T' of X is *equivalent* to a triangulation T of X if there exists a triangulation T'' of X which is a rectilinear subdivision of both T and T'. Note that by this definition, each rectilinear subdivision T' of a fixed triangulation T of X is equivalent to T. A *PL-structure* on X is an *equivalence class of triangulations* of X. A triangulation arising out of a fixed PL-structure (i.e. in the given equivalence class) is called an *admissible triangulation.* Every triangulation of X uniquely determines a PL structure on it. A space with a PL-structure is called a *PL-space.* A closed sub-complex of X (with respect to an admissible triangulation T) automatically inherits an equivalence class of triangulations, i.e. a PL-structure, and is called a *closed PL- subspace.*

Let T be a triangulation of a topological space X, and k a Noetherian ring with 1. Denote by Σ^i_T the set of oriented i-simplices of T. We recall the k-module $C_i^{T,BM}(X,k)$ of *simplicial, or Borel-Moore i-chains with k coefficients* with respect to T. It consists of formal sums (possibly infinite) $\xi = \sum_{\sigma\in\Sigma^i_T}\xi(\sigma)\sigma$.

These k-modules clearly fit into the *chain complex of locally finite simplicial chains* by the obvious boundary operator ∂ which makes sense because of local finiteness of the triangulation T, i.e. each point of X has a neighbourhood meeting only finitely many simplices. It follows that each point can be a vertex of at most finitiely many simplices, which, in turn, implies that each $(i-1)$-simplex can be a face of at most finitely many i-simplices.

There is also the k-submodule $C_i^T(X,k)\subset C_i^{T,BM}(X,k)$ of *finite simplicial i-chains*, which defines a subcomplex of $C_{\cdot}^{T,BM}(X,k)$. Clearly:

$$\begin{aligned} C_i^{T,BM}(X,k) &\simeq \prod_{\sigma\in\Sigma^i_T} k_\sigma \\ C_i^T(X,k) &\simeq \oplus_{\sigma\in\Sigma^i_T} k_\sigma \end{aligned}$$

and the inclusion $C_i^T\subset C_i^{T,BM}$ is the obvious inclusion of the direct sum in the direct product. Finally, X is *compact* iff Σ^0_T is a *finite set* iff Σ^i_T is a *finite set for all i* iff $C_i^T(X,k)=C_i^{T,BM}(X,k)$.

Likewise, we have the k-module of *simplicial k-valued i-cochains* with respect to T, defined as $C^i_T(X,k) := \hom_{\mathbb{Z}}(C_i^T(X,\mathbb{Z}),k)$ (which is also $= \hom_k(C_i^T(X,k),k)$). Note that $C^{\cdot}_T$ is a cochain-complex of k-modules, the coboundary operator being defined by the adjointness formula $\langle\delta\, f,\sigma\rangle := \langle f,\delta\sigma\rangle$. It contains the cochain subcomplex $C^i_{T,c}(X,k)$ of *compactly suppported k-valued i-cochains* with respect to T. Again it is easy to see that in the notation of the last paragraph, we have:

$$\begin{aligned} C^i_T(X,k) &\simeq \prod_{\sigma\in\Sigma^i_T} k_\sigma \\ C^i_{T,c}(X,k) &\simeq \oplus_{\sigma\in\Sigma^i_T} k_\sigma \end{aligned}$$

with the natural inclusion $C^i_{T,c}\subset C^i_T$ being the obvious one.

Remark 4.1.2. The identifications above clearly show that for a fixed triangulation T of X,

$$C_i^{T,BM}(X,k) = \hom_k(C^i_{T,c}(X,k),k)$$

For a PL-space X with some fixed PL-structure, there is a way of defining the simplicial-chain (or cochain) complex of X with respect to *all* the admissible triangulations simultaneously. This eliminates the need for constantly subdividing a given triangulation, and is done as follows.

Definition 4.1.3 (Geometric chains and cochains on a PL-space)**.** By definition, the admissible triangulations on a PL-space X are a directed set. Let us write $T' \geq T$ (or $T \leq T'$) if T' is a rectilinear subdivision of T. This defines a directed system via the maps:

$$\alpha_{TT'} : C_i^{T,BM}(X,k) \to C_i^{T',BM}(X,k)$$

which take an oriented i-simplex σ of T to the sum of the finitely many compatibly oriented i-simplices whose union is σ. It is easily verified that this is a chain map, and hence the direct limit of this system, denoted:

$$C_.^{BM}(X,k) := \lim_{\to T} C_.^{T,BM}(X,k)$$

is a chain complex. An element of the k-module $C_i^{BM}(X,k)$ is called a *geometric i-chain* or *geometric Borel-Moore i-chain* with coefficients in k.

The chain maps $\alpha_{TT'}$ above restrict to chain maps of the subcomplexes of finite chains, i.e.

$$\alpha_{TT'} : C_.^{T}(X,k) \to C_.^{T'}(X,k)$$

and their direct limit is the complex:

$$C_.(X,k) := \lim_{\to T} C_.^{T}(X,k)$$

which is a k-subcomplex of $C_.^{BM}(X,k)$. An element of $C_i(X,k)$ is called a *finite geometric i-chain.* Clearly X is compact iff $C_i(X,k) = C_i^{BM}(X,k)$.

We can now easily carry over the above considerations to cochains. Namely:

$$\begin{aligned} C^i(X,k) &:= \hom_k(C_i(X,k),k) = \hom_k(\lim_{\to} C_i^T(X,k),k) \\ C_c^i(X,k) &:= \{f \in C^i(X,k) : \operatorname{supp} f \text{ is compact}\} \end{aligned}$$

Remark 4.1.4. If $\xi = \sum_\sigma \xi(\sigma)\sigma \in C_i^{T,BM}(X,k)$ is a locally-finite chain, then we define $\operatorname{supp}\xi := \cup_{\xi(\sigma)\neq 0} \mid \sigma \mid$. Since being a closed set is a local property at each $x \in X$, and ξ is locally finite at each point of X, it follows that $\mid \xi \mid$ is a closed subset of X. Since each element ξ of $C_i^{BM}(X,k)$ is actually a chain in some $C_i^{T,BM}(X,k)$, and supports don't change under subdivision, it follows that $\mid \xi \mid$ is closed for each geometric Borel-Moore i-chain ξ.

A geometric Borel-Moore i-chain ξ must be locally finite at *each point of* X, not just each point of $\mid \xi \mid$. For example, in the space $X = [0,1]$ say, if we define σ_i to be the 1-simplex $[2^{-i-1}, 2^{-i}]$, then $\sum_{i=0}^{\infty} \sigma_i$ is *not* a geometric Borel-Moore 1-chain in X. Each term of this chain is from a finer triangulation, and indeed the sum 'chain' fails to be locally finite at 0.

However, if we let $U \subset X$ be the open subset $(0,1]$, and give it the (induced) PL-structure coming from the triangulation consisting of vertices $\{2^{-i}\}_{i\geq 0}$ and 1-simplices $\{\sigma_i\}_{i\geq 0}$, then the chain ξ above is a valid Borel-Moore 1-chain in U. This shows that there isn't a natural inclusion map for Borel-Moore simplicial chains of an open PL-subspace into the whole space. (For a forward homomorphism f_* in Borel-Moore homology, one needs f to be *proper*, as is easily seen from the definition above).

Remark 4.1.5. Even though i-cochains of arbitrary support $C^i(X,k)$ come from dualising $C_i(X,k)$, viz. the i-chains of finite support, the i-cochains of compact support *do not come from dualising Borel-Moore i-chains.* For example, let $X = \mathbb{N}$ with the discrete topology, and $k = \mathbb{Q}$. Then all triangulations of X are the same, and the $\mathbb{Q}$-vector space of geometric 0-chains $C_0^{BM}(X,\mathbb{Q}) = \prod_{i\in\mathbb{N}} \mathbb{Q}$. The $\mathbb{Q}$-dual of the last group surjects onto $\hom_{\mathbb{Q}}(\oplus_{i\in\mathbb{N}}\mathbb{Q},\mathbb{Q}) = \prod_{i\in\mathbb{N}} \mathbb{Q}$, so is a vector space of uncountable cardinality, and certainly not equal to the countable set $C_c^0(X,\mathbb{Q}) = \oplus_{i\in\mathbb{N}}\mathbb{Q}$.

4.2. **Sheaves of geometric chains and cochains.** Let X be a PL-space as above. For an open subset $U \subset X$, there exists a natural *induced PL-space structure* on U. In fact, if we fix an admissible triangulation T of X, then there is a triangulation T_U of U so that every i-simplex of U is an i-simplex of some subdivision T' of T, and is therefore contained in a unique i-simplex of T. (Exercise: Construct T_U from T).

Clearly any finite i-chain $\xi \in C_i^{T_U}(U,k)$ is a finite sum $\xi = \sum_{n=1}^r \xi_n \sigma_n$, and hence becomes a finite chain in $C_i^{T'}(X,k)$ for some some fixed admissible triangulation T' of X which is a subdivision of T. Composing with the natural map $C_i^{T'}(X,k) \to C_i(X,k)$, we have a natural map:

$$C_i^{T_U}(U,k) \to C_i(X,k)$$

for every induced triangulation T_U from an admissible triangulation T of X. Taking a limit of the left hand side over the directed set of admissible triangulations T of X (whose corresponding T_U's form a cofinal subfamily in the family of triangulations of U admissible with respect to its induced PL-structure), one has a natural map of *finite* geometric i-chains:

$$j_{UX} : C_i(U,k) \to C_i(X,k)$$

which is a chain map, and on dualising leads to the restriction maps of geometric cochains

$$\rho_{XU} : C^i(X,k) \to C^i(U,k)$$

More generally, if $V \supset U$, the same considerations as above apply to X replaced by V, and we have natural restrictions ρ_{VU}. So it is natural to make the following:

Definition 4.2.1. Define the *k-presheaf of geometric i-cochains* by the sections:

$$\mathcal{C}^i = \Gamma(U, \mathcal{C}^i) := C^i(U,k)$$

with the sheaf maps ρ_{VU} as in the last para. It is clearly a k-presheaf.

Proposition 4.2.2. Let X be a PL-space. Then:

(i): The k-presheaf $\mathcal{C}^i$ of geometric i-cochains defined above is a sheaf.

(ii): $\mathcal{C}^i$ is a soft sheaf.

Proof: There is an obvious Mayer-Vietoris short exact sequence (exercise, remembering that direct limits preserve exactness) of chain complexes:

$$0 \to C_i(U \cap V, k) \stackrel{(j_{U\cap V,U}, j_{U\cap V,V})}{\longrightarrow} C_i(U,k) \oplus C_i(V,k) \stackrel{j_{U,U\cup V} - j_{V,U\cup V}}{\longrightarrow} C_i(U \cup V, k) \to 0$$

which leads to exactness of:

$$0 \to C^i(U \cup V, k) \to C^i(U,k) \oplus C^i(V,k) \to C^i(U \cap V, k)$$

since $\hom(-,k)$ is a left-exact functor. This gives the sheaf condition for a pair of opens U, V, and the usual appeal to Zorn etc, gives it for arbitrary collections. This proves (i).

To see (ii) first note that for a fixed triangulation T of X, and $Y \subset X$ a subcomplex of X with respect to T, by definition $\Sigma_T^i(Y) \subset \Sigma_T^i(X)$, so that there is a splitting map:

$$\pi : C_i^T(X,k) \to C_i^T(Y,k)$$

which is a left inverse for the inclusion $j : C_i^T(Y,k) \to C_i^T(X,k)$. π sends all i-simplices not contained in Y to 0, and all those in Y to themselves. This map is compatible with subdivisions, so on passing to the limit over subdivisions of T, we get a splitting:

$$\pi : C_i(X,k) \to C_i(Y,k)$$

Hence, for a subcomplex, the restriction map of simplicial i-cochains with respect to the triangulation T, namely:

$$\rho = j^* : C^i(X,k) = \hom_k(C_i(X,k),k) \to C^i(Y,k) = \hom_k(C_i(Y,k),k)$$

is a split surjection.

Now we need to show that $\Gamma(X, \mathcal{C}^i) \to \Gamma(F, \mathcal{C}^i)$ is surjective for each compact subset F of X. Let $s \in \Gamma(F, \mathcal{C}^i)$. By the definition of the right hand side, there is a cochain $f \in C^i(U,k)$ extending s, for some open set U of

X. For each admissible triangulation T, we may regard $f \in C^i_{T_U}(U,k) = \hom(C_i^{T_U}(U,k),k)$ in the notation introduced above.

Since F is compact, there exists a finite subcomplex Y (with respect to any fixed T_U) of U such that $F \subset Y$. (For example, take Y to be the union of all simplices of the triangulation T_U that intersect F, and note that this is a finite set by the compactness of F). Note that $f_{|Y} \in C^i(Y,k)$ also restricts to s.

Since Y is a finite union of simplices of T_U, and each simplex of T_U is a simplex of some subdivision of X, it follows that there exists a subdivision $T' \geq T$ triangulating X such that each simplex of Y is a simplex of T'. In other words, Y is a subcomplex of X with respect to T'.

By the foregoing, $C^i(X,k) \to C^i(Y,k)$ is surjective. Thus there is a $g \in C^i(X,k)$ which lifts $f_{|Y}$. Its image in $C^i(X,k)$ is a lift of $s \in \Gamma(F, \mathcal{C}^i)$. This proves (ii).

□

4.3. **The Dualising sheaf for a PL-space.** By (ii) of the Proposition 4.2.2 above, for a PL-space X (which is automatically locally contractible, since it is a polyhedron) the map:

$$\underline{k}_X \to \mathcal{C}^\cdot$$

gives a soft resolution of k. In the case of $k = \mathbb{Z}$, or a field say, it is also flat.

Say we take k a field. Then the global sections of the dualising sheaf $D^\cdot(\mathcal{C}^\cdot, k)$ will be given by definition as:

$$\Gamma(X, D^{-i}(\mathcal{C}^\cdot, k)) = \hom_k(\Gamma_c(X, \mathcal{C}^i), k) = \hom_k(C^i_c(X,k), k)$$

It is a fact that the map $C^i_c(X,k) \to C^i_{c,T}(X,k)$ is a quasi-isomorphism, and hence the natural map

$$\hom_k(C^i_{c,T}(X,k),k) \to \Gamma(X, D^{-i}(\mathcal{C}^\cdot, k))$$

is a quasi-isomorphism. The same statement holds for U an open set in place of X (and T_U in place of T).

On the other hand, since $C^i_{c,T}(X,k) = \oplus_{\sigma \in \Sigma^i_T(X)} k$, it follows that $\hom_k(C^i_{c,T}(X,k),k) \simeq \prod_{\sigma \in \Sigma^i_T(X)} k = C_i^{T,BM}(X,k)$. Thus:

Proposition 4.3.1 (Dualising sheaves and geometric chains)**.** Let k be a field, and X a PL-space. Then there is a quasi-isomorphism between the global sections of the dualising sheaf, viz. $\Gamma(X, D^{-i}(\mathcal{C}^\cdot, k))$ and the locally finite (Borel-Moore) simplicial i-chains with respect to T given by $C_i^{T,BM}(X,k)$. Under this correspondence, compactly supported sections of both sides correspond to each other, i.e. finite simplicial i-chains $C_i^T(X,k)$ correspond to compactly supported sections $\Gamma_c(X, D^{-i}(\mathcal{C}^\cdot, k)$.

5. Geometric Intersection Homology

5.1. **Stratified pseudomanifolds.** We now introduce the class of topological spaces for which geometric (=PL) intersection homology will be developed.

Definition 5.1.1 (n-pseudomanifold)**.** A PL-space (in the sense of §4 above) is called an *n-dimensional pseudomanifold* if in some admissible triangulation T of X,

(i): For each i, each i-simplex of T is a face of some n-simplex of T.

(ii): Each $(n-1)$ simplex of T is a face of exactly two n-simplices.

One readily checks that these notions are independent of the admissible triangulation chosen, and depend only on the PL-structure. If, in addition, each n-simplex of T can be simultaneously and compatibly oriented (i.e. in such a way that each $(n-1)$-simplex acquires opposite orientations induced from the two faces adjacent to it), then say X is *orientable*. If every point of X has a neighbourhood (PL)-homeomorphic to a ball, then call it an n-dimensional PL manifold.

Definition 5.1.2 (Stratified n-pseudomanifold)**.** A *stratified n-pseudomanifold* is a pseudomanifold X with a filtration (called a *PL-stratification*):

$$X = X^n \supset X^{n-1} = X^{n-2} \supset X^{n-3}..... \supset X^0 \supset X^{-1} = \phi$$

by closed PL-subspaces X^{n-k}. X^{n-k} is called the *closed stratum of X of codimension k*. These strata are required to satisfy:

(i): The set $S^{n-k} := X^{n-k} \setminus X^{n-k-1}$ is either empty or a PL-manifold of dimension $n-k$.

(ii): [Local triviality condition] Each point $x \in S^{n-k}$ has a neighbourhood U which is PL-homeomorphic to $B^{n-k} \times cL$ where B^{n-k} is the open ball of dimension $(n-k)$, cL is the open cone cL on the *link* L where L is required to be a stratified $(k-1)$-pseudomanifold by a stratum preserving PL-homemorphism.

Some remarks about (ii) are in order. Note the definition is not circular, it is inductive on the dimension n, since $(k-1) < n$. An open set $U \subset X$ inherits the stratification of X by intersecting the strata of X with U. The open cone cL on L is the topological space $cL := [0,1) \times L/\sim$ with strata $(cL)^{k-j} = cL^{k-1-j}$ (cone on the codimension-j stratum of L is the codimension j stratum of cL) and $(cL)^0 =$ cone point. Finally $B^{n-k} \times cL$ has the strata $B^{n-k} \times (cL)^i$. X^{n-2} is often denoted Σ and called *singular locus* for historic reasons.

Proposition 5.1.3. An n-pseudomanifold always admits a PL-stratification.

Proof: Take $X^{n-1} = X^{n-2}$ to be the singular stratum $\Sigma \subset X$, which we can assume to be a subcomplex in some triangulation T of X. (By a regular point we mean a point which has a neighbourhood PL- homeomorphic to an n-ball, and a singular point is a point which is not regular. It is easy to see (using (ii) in the Definition 5.1.1 above) that Σ is contained in the $(n-2)$-skeleton of X. Now define X^{n-j} to be the $(n-j)$-skeleton of Σ. The details of verifying that this is a stratification are left as an exercise. □

Theorem 5.1.4 (Lojasiewicz)**.** If X is a complex analytic space of pure complex dimension n, then it has the structure of a stratified $2n$-pseudomanifold, with only even dimensional strata. i.e. $S^{2j+1} = X^{2j+1} \setminus X^{2j} = \phi$ for all $j \geq 0$.

Remark 5.1.5. If X is a complex analytic space of pure $\mathbb{C}$-dimension n, it might be tempting to stratify it by taking $X^{2n-2} := \Sigma(X)$ (the singular locus of X), $X^{2n-4} = \Sigma(\Sigma(X))$ and so on by induction. Unfortunately, even though this filtration trivially satisfies (i) in the Definition 5.1.2, it does not satisfy the local-triviality condition (ii) in general. A classic example is the affine cubic surface $V(y^2z - x^2) \subset \mathbb{C}^3$. It is easy to check that the entire z-axis is the singular locus $\Sigma(X)$, but the origin has no neighbourhood of the form $B^2 \times cL$ required. One needs to put in a zero stratum $X^0 = (0,0,0)$ for (ii) to be restored.

In general, the filtration by inductive singular loci defined here can be refined to give a genuine locally trivial stratification. In general, the less strata there are, the better.

5.2. **Intersection homology.** For X a PL-space, and Y a closed PL-subspace, the dimension of Y makes unambiguous sense (as the maximum dimension of all simplices occurring in Y, after provisionally fixing some admissible triangulation on X with respect to which Y is a subcomplex).

Definition 5.2.1 (Perversity and allowability)**.** Let X be a stratified n-pseudomanifold (see Definition 5.1.2). A *perversity* associated to X is a sequence $\overline{p} = (p_2, ..., p_n)$ of non-negative integers satisfying:

(i): $p_2 = 0$.

(ii): $p_k \leq p_{k+1} \leq p_k + 1$ for all $2 \leq k \leq n$

The *bottom perversity* $\overline{0} := (0,0,...,0)$ and the *top perversity* $\overline{t} := (0,1,2,..,n-2)$. The *lower middle perversity* (resp. *upper middle perversity*) are defined as:

$$\overline{m} := (0,0,1,1,2,2,...,m_k = [(k-2)/2]....) \quad (\text{resp. } \overline{n} := (0,1,1,2,2,...,[(k-1)/2],...))$$

Finally say that the perversities $\overline{p}$ and $\overline{q}$ are said to be *complementary* if $\overline{p}+\overline{q} = \overline{t}$. Note that the lower and upper middle perversities above are complementary.

Definition 5.2.2 (Allowable chains, intersection homology)**.** Let X be a stratified n-pseudomanifold, and $\overline{p}$ a perversity. We say a geometric i-chain $\xi \in C_i^{BM}(X,k)$ is $(\overline{p},i)$-*allowable* if:

$$\dim(|\xi| \cap X^{n-k}) \leq i-k+p_k, \quad 2 \leq k \leq n$$

where $|\xi|$ is the (closed) support of ξ viz. $\xi := \cup_{\xi(\sigma)\neq 0} \ |\sigma|$.

A finite chain $\xi \in C_i(X,k)$ is $(\overline{p},i)$-allowable if it is $(\overline{p},i)$-allowable considered as an element of $C_i^{BM}(X,k)$. Notice that by definition, $|\ \xi\ | \cap X^{n-k}$ is actually a subcomplex of X in some fine enough admissible triangulation of X (viz. a closed PL-subspace), so the dimension in the definition above refers to the dimension as a subcomplex.

The *intersection chain complex of perversity* $\overline{p}$ of geometric chains is the k-subcomplex of $C_{\cdot}^{BM}$ defined as:

$$IC_i^{\overline{p},BM}(X,k) = \{\xi \in C_i^{BM}(X,k) : \ \xi \ \text{ is } (\overline{p},i)\text{-allowable and } \partial\xi \ \text{ is } (\overline{p},i-1)\text{-allowable}\}$$

Finally, the intersection chain complex of perversity $\overline{p}$ of finite geometric i-chains is defined as:

$$IC_i^{\overline{p}}(X,k) = IC_i^{\overline{p},BM}(X,k) \cap C_i(X,k)$$

For X compact, of course, $IC_i^{\overline{p}}(X,k) = IC_i^{\overline{p},BM}(X,k)$.

The *intersection homologies of* X *with perversity* $\overline{p}$ (and coefficients k) are defined as:

$$IH_i^{\overline{p},BM}(X,k) := H_i(IC_{\cdot}^{\overline{p},BM}(X,k); \quad IH_i^{\overline{p}}(X,x) := H_i(IC_{\cdot}^{\overline{p}}(X,k)$$

Remark 5.2.3. From the definitions above, it is clear that if a stratified n-pseudomanifold X happens to be a n-manifold, and we equip it with the best possible stratification (with empty strata in dimensions $< n$), i.e. $X^{n-2} = X^{n-3} = ... = X^0 = X^{-1} = \phi$, then all geometric i-chains, finite or otherwise, are vacuously allowable, and Borel-Moore and finitely supported intersection homology (with any perversity $\overline{p}$) coincide with usual simplicial Borel-Moore and finitely supported simplicial homology respectively. What isn't clear at this point is that if we artificially stratify X with some lower dimensional strata, *the answer doesn't change.* The usual limitations of doing things with triangulations and geometric chains ! There is a formulation of intersection homology modelled after singular chains (due to Henry King) which shows that intersection homology is topologically invariant (see [K], ??). The technical complication with it is the notion of dimension, which has to be topologically formulated.

Remark 5.2.4. Note that if a simplicial chain ξ had its support $|\ \xi\ |$ meeting each closed stratum X^{n-k} *transversally*, then we would have dim $(|\ \xi\ | \cap X^{n-k}) = i-k$ for $k = 2,3,..,n$. Hence the perversity p_k measures departure from transversality in codimension k. ("perverse"="not transverse"). In particular, if a 0 or 1- chain ξ is to be allowable, it must be transverse to the "singular locus" $\Sigma = X^{n-2}$ of codimension 2 (since $p_2 = 0$), so must be disjoint from it. That is, $|\ \xi\ | \subset X \setminus X^{n-2}$ for an allowable 0 or 1-chain ξ.

It is instructive to work out the case of a stratified pseudomanifold with *isolated singularities*, the simplest departure from the manifold situation. Before we do so, we recall the notion of a family of supports. (See [BM], [God]).

Definition 5.2.5 (Homology with supports)**.** A collection Φ of closed subsets of a topological space X is called a *family of supports* if:

(i): $A, B \in \Phi$ implies $A \cup B \in \Phi$.

(ii): $A \in \Phi$, and $C \subset A$ a closed subset, impies $C \in \Phi$.

The key example, of course, is $\Phi = \{K \subset X : K \quad \text{is compact}\}$. Another example is the family of closed sets in X which is disjoint from some fixed subset A. It is quite easy to show (using both (i) and (ii) in the definition above) that the k-submodules of *geometric chains with supports in* Φ defined by:

$$C_i^{\Phi}(X,k) = \{\xi \in C_i^{BM}(X,k) : \mid \xi \mid \in \Phi; \text{ and } \mid \partial\xi \mid \in \Phi\}$$

form a subcomplex of $C_{\cdot}^{BM}$. The i-th homology $H_i(C_{\cdot}^{\Phi})$ is denoted $H_i^{\Phi}(X,k)$ and called the i-th *homology with supports in* Φ (and coefficients in k).

For example, if we let $\Phi = \{K : \ K \ \text{ compact}\}$, then $C_{\cdot}^{\Phi} = C_{\cdot}$, the finitely supported chain complex, and $H_i^{\Phi} = H_i$, the finitely supported homology.

Exercise 5.2.6. Let X be a topological space, and $A \subset X$ a proper closed subset. Let Φ denote the family of supports consisting of closed subsets of X disjoint from A. Then show that:

(i): If X *is compact*, we have:

$$H_i^{\Phi}(X,k) = H_i(X \setminus A, k)$$

(ii): When X is not assumed compact, assume it is paracompact and locally compact, hausdorff ($\Rightarrow$ normal). Then, prove that:

$$C_{\cdot}^{\Phi}(X,k) = \varinjlim_{U \in \mathcal{U}} C_{\cdot}^{BM}(X \setminus U, k)$$

where $\mathcal{U}$ is the directed set of all neighbourhoods of A ordered by reverse inclusion. Hence it follows that:

$$H_i^{\Phi}(X,k) = \varinjlim_{U \in \mathcal{U}} H_i^{BM}(X \setminus U, k) \quad \text{for all } \ i$$

[**Note:** Though $\lim_{U \in \mathcal{U}}(X \setminus U) = (X \setminus A)$ as a topological space, the direct limit on the right side above is *not* $H_i^{BM}(X \setminus A, k)$, since Borel-Moore homology does *not* commute with direct limits!]

Proposition 5.2.7 (Stratified Pseudomanifolds with Isolated Singularities)**.** Let X be a stratified n-pseudomanifold with isolated singularities. That is $X^{n-2} = X^{n-3} = ... = X^0$, with X^0 being a discrete set of points. Let Φ be the family of closed subsets of $X \setminus X^0$ disjoint from X^0. Then for a perversity $\overline{p} = (p_2, .., p_n)$ we have:

$$\begin{aligned} IH_i^{\overline{p},BM}(X,k) &= H_i^{\Phi}(X,k) \quad \text{for } \ i \leq n - p_n - 2 \\ &= \text{Im}\left[H_i^{\Phi}(X,k) \longrightarrow H_i^{BM}(X,k)\right] \quad \text{for } \ i = n - p_n - 1 \\ &= H_i^{BM}(X,k) \quad \text{for } \ i \geq n - p_n \end{aligned} \tag{14}$$

Proof: First note that:

$$p_{k+1} - (k+1) \leq p_k + 1 - (k+1) = p_k - k$$

by the definition of a perversity, so that $p_n - n = \min_{2 \leq k \leq n}(p_k - k)$. It follows that the allowability condition for $k = n$ (i.e. with respect to X^0), namely:

$$\dim(\mid \xi \mid \cap X^0) \leq i - n + p_n, \ \ \dim(\mid \partial\xi \mid \cap X^0) = i - n + p_n - 1$$

is the most stringent condition, and forces all the other allowability conditions for $k = 2, .., n-1$, since $X^{n-2} = X^{n-3} = ... = X^0$. It follows that:

$$\begin{aligned} IC_i^{\overline{p}}(X,k) &= C_i^{\Phi}(X,k) \quad \text{for } \ i \leq n - p_n - 1 \\ &= C_i^{BM}(X,k) \cap \partial^{-1} C_{i-1}^{\Phi}(X,k) \quad \text{for } \ i = n - p_n \\ &= C_i^{BM}(X,k) \quad \text{for } \ i \geq n - p_n + 1 \end{aligned} \tag{15}$$

From this it immediately follows that the intersection homologies are as stated in (5.2.7) abvoe. □

Corollary 5.2.8 (Compact case)**.** Let X be a *compact* stratified n-pseudomanifold with isolated singularities, viz., $X^{n-2} = X^{n-3} = ... = X^0$. Then for any perversity $\overline{p}$, the intersections homologies of X are given by:

$$\begin{aligned} IH_i^{\overline{p}}(X,k) &= H_i(X \setminus X^0, k) \ \text{ for } \ i \leq n - p_n - 2 \\ &= \operatorname{Im}\left[H_i(X \setminus X^0, k) \longrightarrow H_i(X,k)\right] \ \text{ for } \ i = n - p_n - 1 \\ &= H_i(X,k) \ \text{ for } \ i \geq n - p_n \end{aligned} \tag{16}$$

Proof: Follows immediately from the Exercise 5.2.6 and Proposition 5.2.7 above. □

Exercise 5.2.9. Verify that the intersection homologies $IH_{\cdot}^{\overline{p}}(X,k) = H_{\cdot}(X,k)$ (for all $\overline{p}$) when X is smooth and compact, and X^0 is taken to be any discrete subset of X.

5.3. **Poincare Duality Again.** We saw in Proposition 3.2.3 and its Corollary 3.2.4 that for a $\mathbb{Z}$-orientable (i.e. orientable) n-manifold M, we have Poincare Duality isomorphism:

$$D : H_c^i(M,\mathbb{Z}) \to H_{n-i}(M,\mathbb{Z})$$

This allows us to define the following *intersection pairing*:

$$\begin{aligned} . : H_i^{BM}(M,\mathbb{Z}) \otimes H_{n-i}(M,\mathbb{Z}) &\to \mathbb{Z} \\ \alpha \otimes \beta &\mapsto \alpha.\beta := \left\langle \alpha, D^{-1}\beta \right\rangle \end{aligned}$$

where we are using the natural map $H_i^{BM}(M,\mathbb{Z}) \to \hom_{\mathbb{Z}}(H_c^i(M,\mathbb{Z}),\mathbb{Z})$ which is the right hand arrow in the short exact sequence of Proposition 2.2.7. If one assumes that X is compact, $H_*(M,\mathbb{Z}) = H_*^{BM}(M,\mathbb{Z})$ and $H^i(M,\mathbb{Z}) = H_c^i(M,\mathbb{Z})$ are all finitely generated, and the fact that D is an isomorphism combined with Proposition 2.2.7 shows that:

Proposition 5.3.1 (Intersection pairing on manifolds)**.** Let M be a compact orientable n-manifold. Then, denoting the free part of a finitely generated abelian group G by $F(G)$, the restriction of the intersection pairing:

$$. : F(H_i(M,\mathbb{Z})) \otimes F(H_{n-i}(M,\mathbb{Z})) \to \mathbb{Z}$$

is non-degenerate over $\mathbb{Z}$ (viz. $F(H_i(M,\mathbb{Z})) \simeq \hom_{\mathbb{Z}}(F(H_{n-i}(M,\mathbb{Z})),\mathbb{Z})$ via the map $\alpha \mapsto \alpha.(\,)$).

Example 5.3.2 (Nodal cubic)**.** The nodal cubic curve is the complex curve defined as:

$$X := \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : xy^2 - z^2(z + x) = 0\}$$

It is pictured in Fig. 1, and maybe viewed as a "degeneration" of the elliptic curve $V(xy^2 - z(z+\lambda x)(z+x)) \subset \mathbb{P}^2(\mathbb{C})$ as the root λ and the root 0 merge (i.e. $\lambda \to 0$). The intersection pairing on $H_1(X,\mathbb{Z}) = \mathbb{Z}b$ (see the Figure 1) is easily seen to be $b.b = 0$ (since the cohomology class β dual to b is a 1-dimensional class, it satisfies $\beta \cup \beta = -\beta \cup \beta$ and $H^2(X,\mathbb{Z})$ is infinite cyclic implies $\beta \cup \beta = 0$!) Hence the free part $F(H_1(X,\mathbb{Z})) = H_1(X,\mathbb{Z})$ is *not* non-degenerately paired with itself, and Poincare Duality has broken down for X.

The key reason for introducing intersection homology is to restore the non-degenerate pairing of Poincare Duality. We first need a lemma on Borel-Moore homology.

Lemma 5.3.3 (One-point compactification and Borel-Moore homology)**.** Let X be any locally compact hausdorff topological space, and let $\widehat{X} := X \cup \infty$ denote its one point compactification. Assume $\widehat{X}$ is locally contractible. Then for k any Noetherian ring with 1, we have:

$$H_c^i(X,k) \simeq H^i(\widehat{X},\infty;\ k) \ \text{ for all } \ i$$

and similarly:

$$H_i^{BM}(X,k) \simeq H_i(\widehat{X},\infty;\ k) \ \text{ for all } \ i$$

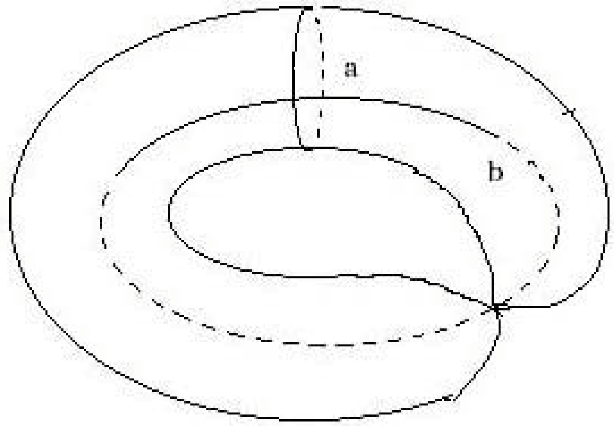

FIGURE 1. The Nodal Cubic

Proof: If $\alpha \in C^i(X,k)$ is an i-cochain of compact support K, then by definition the restriction of α to $C^i(X \setminus K, k)$ is zero, so that that $\alpha \in C^i(X, X \setminus K, k)$. It easily follows that the compactly supported cochains are given as the direct limit:

$$C_c^i(X,k) = \lim_{K \subset X \text{ compact}} C^i(X, X \setminus K; k)$$

and since homology and direct limit commute, it follows that $H_c^i(X,k) = \lim_{K \subset X \text{ compact}} C^i(X, X \setminus K; k)$.

Using excision the fact that ∞ has a fundamental system $\mathcal{U}$ of contractible neighbourhoods in the one point compactification $\widehat{X} = X \cup \infty$, we find that

$$\lim_K H^i(X, X \setminus K; k) = \lim_{U \in \mathcal{U}} H^i(\widehat{X} \setminus \infty, U \setminus \infty; k) = \lim_{U \in \mathcal{U}} H^i(\widehat{X}, U; k) = H^i(\widehat{X}, \infty; k)$$

which proves the first assertion. For the Borel-Moore statement, use the Universal coefficient theorem Proposition 2.2.7) □

Before we establish Poincare Duality for a stratified n-pseudomanifold with isolated singularities, we quote a result from [Dold], p 297.

Lemma 5.3.4 (Alexander-Lefschetz-Poincare)**.** Let $L \subset K \subset X$ be topological spaces such that (i) L is closed in K, (ii) $K \setminus L$ is closed in $X \setminus L$, and $X \setminus L$ is an n-manifold which is oriented along $K \setminus L$, then:

$$\check{H}_c^i(K, L) \simeq \check{H}_c^i(K \setminus L) \simeq H_{n-i}(X \setminus L, X \setminus K)$$

Proposition 5.3.5 (Poincare Duality for $IH_.$)**.** Let X be a stratified and oriented n-pseudomanifold with isolated singularities $X^{n-2} = ... = X^0$. Assume that X^0 is a *finite* set. Let $\overline{p}$ and $\overline{q}$ be complementary perversities. Then $IH_i^{\overline{p},BM}(X,\mathbb{Q})$ and $IH_{n-i}^{\overline{q}}(X,\mathbb{Q})$ are dual to each other.

Proof: We drop the coefficient ring from both homologies and intersection homologies, since it is assumed to be $\mathbb{Q}$ throughout the ensuing proof. By the Proposition 5.2.7, we have:

$$\begin{aligned} IH_i^{\overline{p},BM}(X) &= H_i^{\Phi}(X) \quad \text{for } i \le n - p_n - 2 \\ &= \operatorname{Im}\left[H_i^{\Phi}(X) \longrightarrow H_i^{BM}(X)\right] \quad \text{for } i = n - p_n - 1 \\ &= H_i^{BM}(X) \quad \text{for } i \ge n - p_n \end{aligned} \tag{17}$$

where Φ is the family of closed subsets of X disjoint from X^0.

By a minor variation of the same proof, the analogue of 5.2.7 for finitely supported homology (for the complementary verversity $\overline{q}$) is easily seen to be:

$$\begin{aligned} IH^{\overline{q}}_{n-i}(X) &= H_{n-i}(X \setminus X^0) \quad \text{for } n-i \leq n-q_n-2 \ (\Leftrightarrow i \geq n-p_n) \\ &= \operatorname{Im}\left[H_i(X \setminus X^0) \longrightarrow H_i(X)\right] \quad \text{for } n-i = n-q_n-1 \ (\Leftrightarrow i = n-p_n-1) \\ &= H_{n-i}(X) \quad \text{for } n-i \geq n-q_n \ (\Leftrightarrow i \leq n-p_n-2) \end{aligned} \tag{18}$$

Let $X^0 = \cup_{i=1}^m \{x_i\}$, and let $U_i = cL_i$ be a cone-like open neighbourhood of x_i (by the local triviality condition (ii) in Definition 5.1.2). Letting $U = \coprod_{i=1}^r U_i$ be an open neighbourhood of X^0, set $M_U := X \setminus U$. Now, let us substitute $X = X$, $L = X^0$ and $K = M_U \coprod X^0$ in the Lemma 5.3.4 above. Then $L = X^0$ is closed in K, and $K \setminus L = M_U$ is closed in $X \setminus L = X \setminus X^0$, and the last set is an n-manifold which is orientable. So by that Lemma we have:

$$H^i_c(M_U) \simeq H_{n-i}(X \setminus X^0, U \setminus X^0) = H_{n-i}(X, U) = H_{n-i}(X, X^0)$$

where Cech cohomology is replaced by usual cohomology, and the last equality comes from the strong deformation retraction of U to X^0. By (ii) of Exercise 5.2.6, we find that for a fundamental system $\mathcal{U}$ of such conical neighbourhoods of X^0, we have:

$$H^\Phi_i(X) = \lim_{U \in \mathcal{U}} H^{BM}_i(X \setminus U) \simeq \lim_{U \in \mathcal{U}} H^{BM}_i(M_U) = \lim_{U \in \mathcal{U}} \hom(H^i_c(M_U), \mathbb{Q}) \simeq \hom(H_{n-i}(X, X^0); \mathbb{Q}) \quad \text{for all } i \tag{19}$$

Now getting back to the intersection homologies, substituting in (17) and (18) above, we note that when $i \leq n - p_n - 2$, we have $n - i \geq p_n + 2 = (n - 2 - q_n) + 2 = n - q_n$. Also $n - i \geq p_n + 2 \geq 2$, and $H_{n-i}(X, X^0) = H_{n-i}(X)$. Thus:

$$IH^{\overline{p},BM}_i(X) = H^\Phi_i(X) \simeq \hom(H_{n-i}(X, X^0), \mathbb{Q}) \simeq \hom(IH^{\overline{q}}_{n-i}(X), \mathbb{Q}) \quad \text{for all } i \leq n - p_n - 2$$

by using (19) above.

For $i \geq n - p_n$, we have $n - i \leq n - q_n - 2$, and by (17) and Lemma 5.3.3 we have that:

$$IH^{\overline{p},BM}_i(X, \mathbb{Q}) = H^{BM}_i(X) = H_i(\widehat{X}, \infty)$$

But $n - p_n \geq 2$, so $H_i(\widehat{X}, \infty) = H_i(\widehat{X}, X^0 \cup \infty)$. But again by the Lemma 5.3.4, we have (taking $\widehat{X}$ for $X = K$ and $L = X^0 \cup \infty$) that

$$H_i(\widehat{X}, X^0 \cup \infty) \simeq \hom(H_{n-i}(\widehat{X} \setminus (X^0 \cup \infty)), \mathbb{Q}) \simeq \hom(H_{n-i}(X \setminus X^0), \mathbb{Q}) \simeq \hom(IH^{\overline{q}}_{n-i}(X), \mathbb{Q})$$

where the last relation follows from (18). Thus $IH^{\overline{p},BM}_i(X, \mathbb{Q}) \simeq \hom(IH^{\overline{q}}_{n-i}(X), \mathbb{Q})$ for $i \geq n - p_n$ as well.

Finally, in the critical dimension, $i = n - p_n - 1$, $n - i = p_n + 1 = n - q_n - 1$ is the critical dimension for the complementary perversity $\overline{q}$, and both i and $n - i$ are ≥ 1. Thus the map induced by inclusion:

$$H_{n-i}(X) \to H_{n-i}(X, X^0)$$

is injective, and we may as well write

$$IH^{\overline{q}}_{n-i}(X, \mathbb{Q}) = \operatorname{Im}\left[H_{n-i}(X \setminus X^0) \to H_{n-i}(X, X^0)\right]$$

Analogously, since $i = n - p_n - 1 \geq 1$, the natural map:

$$H^{BM}_i(X) = H_i(\widehat{X}, \infty) \to H_i(\widehat{X}, X^0 \cup \infty)$$

is also injective. Also by the foregoing, $H^\Phi_i(X) \simeq \hom(H_{n-i}(X, X^0), \mathbb{Q})$. Thus

$$IH^{\overline{p},BM}_i(X, \mathbb{Q}) = \operatorname{Im}\left[H^\Phi_i(X) \to H_i(\widehat{X}, X^0 \cup \infty)\right]$$

By the preceding paragraphs $H^\Phi_i(X)$ is non-degenerately paired with $H_{n-i}(X, X^0)$. Similarly by the preceding paragraphs, $H_i(\widehat{X}, X^0 \cup \infty)$ is non-degenerately paired with $H_{n-i}(X \setminus X^0)$. Thus the two images defining $IH^{\overline{p},BM}_i$ and $IH^{\overline{q}}_{n-i}$ above are non-degenerately paired. The proposition follows.

□

Example 5.3.6 (Nodal cubic again). Let us go back to the nodal cubic curve of Example 5.3.2 and see what happens to the intersection homologies. In this case $X \setminus X^0$ is homotopy equivalent to S^1, and for any perversity $\overline{p}$, we have $p_2 = 0$, so the critical dimension is $2 - p_2 - 1 = 1$. Also the generator a of $H_1(X \setminus X^0)$ is nullhomologous in X (see Figure 1), so $IH_1^{\overline{p}}(X) = 0$. Also $IH_0^{\overline{p}}(X) = H_0(X \setminus X^0) = \mathbb{Q}$ and $IH_2^{\overline{p}}(X) = H_2(X) = \mathbb{Q}$. Thus, with any perversity $\overline{p}$, $IH_*^{\overline{p}}(X)$ is isomorphic to $H_*(S^2, \mathbb{Q})$, and Poincare duality is restored. (This is not an accident, since S^2 is the normalisation of X, and it will turn out later that intersection homology remains invariant under normalisation.)

It is natural to ask whether Poincare duality holds over $\mathbb{Z}$, i.e. whether the *free parts of* $IH_i^{\overline{p}}$ and $IH_{n-i}^{\overline{q}}$ are non-degenerately paired over $\mathbb{Z}$. That this is not so will follow from the next example.

Example 5.3.7 (Thom Spaces of vector bundles). Let E be a real rank k orientable bundle over a compact manifold M of dimension $n - k$. The *Thom space of* E is the quotient space $T(E) := D(E)/S(E)$, where $D(E)$ (resp, $S(E)$) is the disc bundle (resp. sphere bundle) of E, with respect to a suitable Riemannian metric on E. Then $X = T(E)$ is a space with one isolated singularity $X^0 = \infty$. Note that $X \setminus X^0 = E$, and $H_i(X, \mathbb{Z}) = H_i(T(E), \infty; \mathbb{Z}) = H_i(D(E), S(E); \mathbb{Z}) \simeq H_{i-k}(M, \mathbb{Z})$ via the *Thom isomorphism* for $i \geq 1$. Thus the image:

$$H_i(X \setminus X^0, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

is precisely the image:

$$H_i(M, \mathbb{Z}) \xrightarrow{\cap e} H_{i-k}(M, \mathbb{Z})$$

for all i, where $e \in H^k(M, \mathbb{Z})$ denotes the *Euler class of* E. Thus, by the above Corollary 5.2.8 we have:

$$\begin{aligned} IH_i^p(X, \mathbb{Z}) &= H_i(M, \mathbb{Z}), \quad 0 \leq i \leq n - p_n - 2 \\ &= \mathrm{Im}[(- \cap e) : H_i(M, \mathbb{Z}) \to H_{i-k}(M, \mathbb{Z})], \quad i = n - p_n - 1 \\ &= H_{i-k}(M, \mathbb{Z}), \quad i > n - p_n \end{aligned}$$

If, for example one takes $M = S^2$, and E the real rank 2 bundle given by the tangent bundle $E = \tau(S^2)$ whose Euler class is $e = 2\mu \in \mathbb{Z}\mu = H^2(S^2, \mathbb{Z})$. Then for the lower middle perversity $\overline{m}$ and upper middle perversity $\overline{n}$, we have $m_4 = n_4 = 1$, so that the critical dimension for both $\overline{m}$ and $\overline{n}$ is $i = 4 - m_4 - 1 = 4 - n_4 - 1 = 2$, and we have the intersection homology in the dimension 2, by the above, is

$$IH_2^{\overline{m}}(X, \mathbb{Z}) = 2\mathbb{Z} = IH_2^{\overline{n}}(X, \mathbb{Z})$$

which on pairing gives determinant 4 and is not non-degenerate over $\mathbb{Z}$.

An interesting special case of the Example 5.3.7 above is the cone over a complex projective hypersurface.

Example 5.3.8 (Projective cone over a smooth hypersurface). Let M of $\dim_{\mathbb{C}} M = m - 1$ be a smooth hypersurface in $\mathbb{P}^m(\mathbb{C})$. Regarding $\mathbb{P}^m(\mathbb{C}) \subset \mathbb{P}^{m+1}(\mathbb{C})$ as the hyperplane $V(X_{m+1})$, we define the *projective cone over* M as:

$$X = \{[\alpha X_0 :, ..., : \alpha X_m : \beta] \in \mathbb{P}^{m+1}(\mathbb{C}) : [X_0 :, .., X_m] \in M, \ \ (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}\}$$

If one removes the point $\infty := [0 :, .., : 0 : 1]$ from X, the remainder $X \setminus \infty$ is the total space of the line bundle E on M given by restricting the hyperplane bundle λ on $\mathbb{P}^{m+1}(\mathbb{C})$ to M. Thus X is the Thom space $T(E)$ of E, and the discussion of Example 5.3.7 above applies. The Euler class e in this case is the Kahler class of the hypersurface M.

Again, the interesting dimension is the critical dimension, when $i = n - p_n - 1 = 2m - p_{2m} - 1$. Let us take coefficients in $\mathbb{C}$. In this case, $\cap e : H_i(M, \mathbb{C}) \to H_{i-2}(M, \mathbb{C})$ is the Lefschetz map, corresponding to the cohomology Lefschetz map:

$$L = (-) \cup e : H^{2m-2-i}(M, \mathbb{C}) \to H^{2m-i}(M, \mathbb{C})$$

via Poincare Duality on M. By the Lefschetz decomposition, the cokernel of this map is well-known to be the primitive cohomology $P^{2m-i} \subset H^{2m-i}$, which is defined as the kernel of $L^{i+1} : H^{2m-i} \to H^{2m+i+2}$. If we denote by P_{i-2} the subspace of $H_{i-2}(M,\mathbb{C})$ which is complementary to the inverse image of P^{2m-i} in H_{i-2}, then

$$IH_i^{\overline{p}}(M,\mathbb{C}) = \mathrm{Im}(-\cap e) = P_{i-2}, \quad i = 2m - p_{2m} - 1$$

5.4. **Cones and Products with $\mathbb{R}$.**

Definition 5.4.1. Let X be a stratified n-pseudomanifold. The *open cone* $cX := [0,1) \times X/\sim$ where $\sim$ is the equivalence relation of identifying $0 \times X$ to a single point p called the *cone point*. It is given the stratification described in Definition 5.1.2. That is:

$$(cX)^j = c(X^{j-1}); \quad (cX)^0 = \{p\}$$

Note that the cone on an i-simplex is an $(i+1)$-simplex, and hence the *closed cone* $\overline{c}X = [0,1] \times X/\sim$ has the structure of a PL-space coming naturally from that of X. As an open subset of $\overline{c}X$, the space cX also acquires the structure of a PL-space (see outset of §4.2). We omit the details of checking that the filtration above defines the structure of a stratified $(n+1)$-pseudomanifold on cX. The local triviality condition follows at points of all strata (from the corresponding condition for X), except at the cone point, where it holds by definition!

With X as above, it is possible to give the structure of a stratified $(n+1)$-pseudomanifold to $\mathbb{R}\times X$ as well. Its structure as a PL-space follows by the well-known triangulation of $[i,i+1]\times\sigma$ for a k-simplex $\sigma = \langle v_0,..,v_k\rangle$. That is, let the set

$$V^0 := \{(s,v_p) : \ s = i \ \text{ or } i+1; \ 0 \le p \le k, \ \}$$

be the vertex set for $[i,i+1]\times\sigma$. Define a partial order on V^0 by the prescription $(a,v_r) \le (b,v_s)$ iff $a \le b$ and $r \le s$, for $a,b \in \{i,i+1\}$ and $r,s \in \{0,..,k\}$. Then define a set $\langle S\rangle$ to be a simplex of $[i,i+1]\times\sigma$ iff S is a totally ordered subset of V^0. This gives a PL-space structure to $[i,i+1]\times X$ for each $i \in \mathbb{Z}$, and all of these match up to give one on all of $\mathbb{R}\times X$. The strata of $\mathbb{R}\times X$ are defined by:

$$(\mathbb{R}\times X)^j = \mathbb{R}\times X^{j-1}$$

for all $j = 0,..,n$. Note therefore that $\mathbb{R}\times X$ has no 0-dimensional stratum.

Lemma 5.4.2. Let X be a stratified n-pseudomanifold, and let $Y := \mathbb{R}\times X$ be the product stratified $(n+1)$-pseudomanifold as indicated above. Let ξ be a $(\overline{p},i)$-allowable cycle in Y which is supported in $[0,\infty)\times X$. Further assume that ξ is in "general position" so that all vertical lines $\mathbb{R}\times\{x\}$ meet $|\,\xi\,|$ transversely. (Equivalently, the projection $\pi_2 : \mathbb{R}\times X \to X$ is of full rank on all of $|\,\xi\,|$). Then there exists a $(\overline{p},i+1)$-allowable chain $K^+\xi$ in Y such that the homology class $\partial K^+\xi = \xi$. Further, $K^+\xi$ has support in $[0,\infty)\times X$.

Proof: For notational convenience, we denote $[0,\infty)$ by $\mathbb{R}_+$. We consider the map:

$$\begin{array}{rcl} \mu : \mathbb{R}_+ \times \mathbb{R}_+ & \to & \mathbb{R}_+ \\ (s,t) & \mapsto & s+t \end{array}$$

which is linear. Thus we have a PL-map:

$$\alpha := \mu \times \mathrm{id}_X : \mathbb{R}_+ \times \mathbb{R}_+ \times X \to \mathbb{R}_+ \times X$$

The general position assumption on ξ guarantees that $\alpha_*(\mathbb{R}_+\times\xi)$ is a geometric $(i+1)$-chain, where

$$\alpha_* : C^{BM}_{i+1}(\mathbb{R}_+\times\mathbb{R}_+\times X, k) \to C^{BM}_{i+1}(\mathbb{R}_+\times X, k)$$

is the chain map induced by α. Define $K^+\xi := (\alpha)_*(\mathbb{R}_+\times\xi))$ (see Fig. 2). By definition, $|\,K^+\xi\,| \subset \mathbb{R}_+\times X$.

Furthermore:

$$\partial K^+(\xi) = \partial(\alpha)_*(\mathbb{R}_+\times\xi) = \alpha_*(\partial(\mathbb{R}_+\times\xi)) = \alpha_*(0\times\xi) = \xi$$

where the second equality follows because α_* is a chain map. To check the $(\overline{p}, i+1)$-allowability of $K^+\xi$, note that $\mathbb{R}_+ \times X^{n-k} = \alpha(\mathbb{R}_+ \times \mathbb{R}_+ \times X^{n-k})$, so that:

$$\begin{aligned} \dim(|K^+\xi| \cap (\mathbb{R} \times X^{n-k})) &= \dim(|K^+\xi| \cap (\mathbb{R}_+ \times X^{n-k})) = \dim(\alpha(|\mathbb{R}_+ \times \xi|) \cap \alpha(\mathbb{R}_+ \times \mathbb{R}_+ \times X^{n-k})) \\ &= \dim(\alpha(\mathbb{R}_+ \times (|\xi| \cap (\mathbb{R}_+ \times X^{n-k})))) = \dim(\alpha(\mathbb{R}_+ \times (|\xi| \cap (\mathbb{R} \times X^{n-k})))) \\ &\leq 1 + \dim(|\xi| \cap (\mathbb{R} \times X^{n-k})) \leq 1 + (i - k + p_k) = (i+1) - k + p_k \end{aligned}$$

where the second inequality of the last line is from the $(\overline{p}, i)$-allowability of ξ. Hence $K^+\xi$ is $(\overline{p}, i+1)$-allowable. The lemma follows.

□

Definition 5.4.3 (Dimensional transversality)**.** Let A and B be closed PL-subspaces of a stratified n-pseudomanifold X. Let $S^{n-k} := X^{n-k} \setminus X^{n-k-1}$, which is a manifold of dimesnion $(n-k)$ by (i) of Definition 5.1.2. Say that $A \pitchfork_k B$, i.e. A is *dimensionally transverse to B in the codimension k stratum* if:

$$\dim(A \cap B \cap S^{n-k}) \leq \dim(A \cap S^{n-k}) + \dim(B \cap S^{n-k}) - (n-k)$$

This simply means that in the manifold S^{n-k}, the intersection $A \cap S^{n-k}$ and $B \cap S^{n-k}$ meet PL-transversely. Finally, say that $A \pitchfork B$ if $A \pitchfork_k B$ for all k.

Proposition 5.4.4. Let X be a stratified n-pseudomanifold. Give $\mathbb{R} \times X$ the structure of a stratified $(n+1)$-pseudomanifold as in Definition 5.4.1 above. Then:

$$IH_i^{\overline{p},BM}(X,k) \simeq IH_{i+1}^{\overline{p},BM}(\mathbb{R} \times X, k)$$

for all i and all perversities $\overline{p}$.

Proof: Let $\xi \in IC_i^{\overline{p},BM}(X,k)$ be a $(\overline{p}, i)$-allowable i-chain. Define the *suspension of* ξ by:

$$\Sigma(\xi) = \mathbb{R} \times \xi$$

This simply means that if $\xi = \sum_\sigma \xi(\sigma)\sigma$ in some admissible triangulation T of X, then triangulating $\mathbb{R} \times \sigma$ as described in Definition 5.4.1 above, we set $\Sigma(\sigma) := \mathbb{R} \times \sigma$, which can be viewed as an $(i+1)$-chain in $\mathbb{R} \times X$, for each i-simplex σ of T. Finally set $\Sigma\xi := \sum_\sigma \xi(\sigma)\Sigma(\sigma)$, which is an $(i+1)$-chain in $\mathbb{R} \times X$. Define the map:

$$\begin{aligned} \Sigma : C_i^{BM}(X,k) &\to C_{i+1}^{BM}(\mathbb{R} \times X, k) \\ \xi &\mapsto \Sigma(\xi) \end{aligned}$$

It is easily verified that the j-th face operator commutes with Σ, and hence that Σ is a chain map. Now we claim that if ξ is a $(\overline{p}, i)$-allowable chain in X, then $\Sigma(\xi)$ is a $(\overline{p}, i+1)$-allowable chain in $\mathbb{R} \times X$. For,

$$\begin{aligned} \dim(|\mathbb{R} \times \xi| \cap (\mathbb{R} \times X)^{n+1-k}) &= \dim(|\mathbb{R} \times \xi| \cap (\mathbb{R} \times X^{n-k})) = 1 + \dim(|\xi| \cap X^{n-k}) \\ &\leq 1 + (i - k + p_k) = (i+1) - k + p_k \end{aligned}$$

which shows that $\Sigma(\xi)$ is $(\overline{p}, i+1)$-allowable. Clearly if $\partial\xi$ is $(\overline{p}, i-1)$-allowable, then $\partial(\Sigma(\xi)) = \Sigma(\partial\xi)$ is $(\overline{p}, i)$-allowable, by the same reasoning, so that Σ restricts to a chain map:

$$\Sigma : IC_{\cdot}^{\overline{p},BM}(X,k) \to IC_{\cdot+1}^{\overline{p},BM}(\mathbb{R} \times X, k)$$

We now claim that this map is a quasi-isomorphism.

Given a $(\overline{p}, i+1)$-allowable cycle $\xi \in \mathbb{R} \times X$, define $\theta(\xi) = \xi \cap (\{t\} \times X)$ after making sure that t is chosen from a generic set such that the PL-subspace $|\xi|$ is dimensionally transverse to the PL-subspace $\{t\} \times X$, so that $\theta(\xi)$ is a geometric i-chain. (For example, let ξ be an $(i+1)$-chain in the product triangulation $\mathbb{R} \times T$ where T is some triangulation of X, and take t to be distinct from all the countably many vertices that occur in $|\xi|$.) Let us check that $\theta(\xi)$ is $(\overline{p}, i)$-allowable. By dimensional transversality:

$$\begin{aligned} \dim(|\theta(\xi)| \cap S_{\mathbb{R}\times X}^{n-k+1}) &= \dim(|\xi| \cap (\{t\} \times X) \cap (\mathbb{R} \times S^{n-k})) \\ &\leq \dim(|\xi| \cap (\mathbb{R} \times S^{n-k})) + \dim((\{t\} \times X) \cap (\mathbb{R} \times S^{n-k})) - (n+1-k) \\ &\leq ((i+1) - k + p_k) + (n-k) - (n+1-k) = i - k + p_k \end{aligned}$$

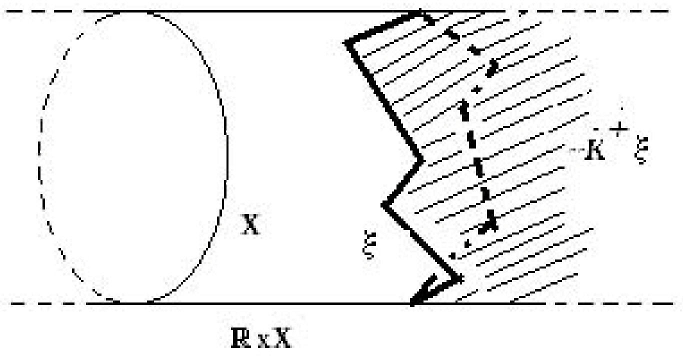

Figure 2. The operation K^+

where the last line follows from the $(\overline{p}, i+1)$-allowability of ξ. This shows that $\theta(\xi)$ is $(\overline{p}, i)$-allowable. Also ξ a cycle and above dimensional transversality implies that $\partial(\theta(\xi)) = \partial\xi \cap (\{t\} \times X) = 0$, so $\theta(\xi)$ is also a cycle.

Verify that $\xi \cap ([t, s] \times X)$ gives an (allowable) homology between $\xi \cap (\{t\} \times X)$ and $\xi \cap (\{s\} \times X)$, so that $\theta(\xi)$ is well-defined upto (intersection) homology class, and clearly $\theta(\Sigma\xi) = \xi$. Also if $\eta \in IC_{i+2}^{\overline{p},BM}(\mathbb{R} \times X, k)$ such that $\partial\eta = \xi$, then after choosing t so that η and ξ are both dimensionally transverse to $\{t\} \times X$, we get $\eta \cap (\{t\} \times X) \in IC_{i+1}^{\overline{p},BM}(X, k)$ and also $\partial(\eta \cap (\{t\} \times X)) = \theta(\xi)$. Thus we have that

$$\Sigma. : IH_i^{\overline{p},BM}(X, k) \to IH_{i+1}^{\overline{p},BM}(\mathbb{R} \times X, k)$$

is a split injection. We need to show it is surjective. Let ξ be $(\overline{p}, i+1)$-cycle, and let $\theta(\xi)$ be defined as in the last para. We need to show that these two cycles ξ and $\Sigma\theta(\xi)$ are (intersection) homologous in $\mathbb{R} \times X$. Translate and take t of the preceding para to be $t = 0$, so that $\theta(\xi) = \xi \cap (\{0\} \times X)$. Let $\xi^+ := \xi \cap (\mathbb{R}_+ \times X)$ and $\xi^- := \xi \cap (\mathbb{R}_- \times X)$. One chooses the orientations of $\xi^{\pm}$ compatibly with those of $\mathbb{R}_{\pm}$ so that $\xi = \xi^+ - \xi^-$ and $\partial\xi^+ = \theta(\xi) = \partial\xi^-$. Now consider the $(i+1)$-chain $\eta := \xi^+ - (\mathbb{R}_+ \times \theta(\xi))$. Then $\partial\eta = \theta(\xi) - \theta(\xi) = 0$. η is supported in $\mathbb{R}_+ \times X$ by definition. Again, by putting ξ in general position, we can assume that η is also in the same general position as required by Lemma 5.4.2. Thus, there exists a $(i+2)$-chain $K^+\eta$ so that $\partial(K^+\eta) = \eta = \xi^+ - (\mathbb{R}_+ \times \theta(\xi))$. Similarly, one finds an $(i+2)$-chain $K^-\zeta$ such that $\partial K^-\zeta = \zeta := \xi^- - (\mathbb{R}_- \times \theta(\xi))$. Then it follows that:

$$\partial(K^+\eta - K^-\zeta) = \xi^+ - (\mathbb{R}_+ \times \theta(\xi)) - \xi^- + (\mathbb{R}_- \times \theta(\xi)) = \xi - (\mathbb{R} \times \theta(\xi)) = \xi - \Sigma(\theta(\xi))$$

Also, by the proof of Lemma 5.4.2, both $K^+\eta$ and $K^-\zeta$ are $(\overline{p}, i+2)$-allowable, and hence so is $K^+\eta - K^-\zeta$. Thus $[\xi] = [\Sigma\theta(\xi)]$ in $IH_{i+1}^{\overline{p},BM}(\mathbb{R} \times X, k)$ and the proposition is proved. □

Remark 5.4.5. We note that for $\mathbb{R} \times X$, which has no codimension $(n+1)$-stratum, we only need an "n"-perversity $\overline{p} = (p_2, .., p_n)$ on the left hand side of the above proposition.

We now do the finitely supported version of the above result.

Proposition 5.4.6 (Finitely supported intersection homology of $\mathbb{R} \times X$)**.** Let X be a stratified n-pseudomanifold, and let $Y := \mathbb{R} \times X$ be given the product stratification. Then:

$$IH_i^{\overline{p}}(Y, k) \simeq IH_i^{\overline{p}}(X, k) \quad \text{for all} \quad i \text{ and all } \overline{p}$$

Proof: Again, in view of Remark 5.4.5 above, only n-perversities $\overline{p} = (p_2, .., p_n)$ are relevant to this proposition. Consider the inclusion map $j : X \hookrightarrow Y$ defined by $j(x) = (0, x)$, and the projection map $\pi : Y \to X$ defined by $\pi(t, x) = x$. Since $j(S) \cap Y^{n-k} = S \cap X^{n-k}$ for all subsets $S \subset X$, it is clear that j induces a chain map

$$j_* : IC_i^{\overline{p}}(X, k) \to IC_i^{\overline{p}}(Y, k)$$

for all i and $\overline{p}$. We note that if σ is an i-simplex of Y which is in general position with respect to vertical rays $\mathbb{R} \times \{x\}$, then π homeomorphically maps σ to its image $\pi(\sigma)$. Thus if ξ is a $(\overline{p}, i)$-allowable chain in Y which is in general position, we have:

$$\begin{aligned} \dim (\pi \mid \xi \mid \cap X^{n-k}) &= \dim (\pi(\mid \xi \mid \cap \pi^{-1} X^{n-k})) = \dim (\mid \xi \mid \cap \pi^{-1} X^{n-k}) \\ &= \dim (\mid \xi \mid \cap Y^{n-k}) \le i - k + p_k \ \text{ for } \ k = 2, .., n \end{aligned}$$

and hence $\pi(\xi) = \pi_*(\xi)$ is $(\overline{p}, i)$-allowable for ξ a $(\overline{p}, i)$-allowable chain. For a chain ξ in general position, its boundary chain is also in general position, and the same reasoning shows that $\partial \pi_*(\xi)$ is $(\overline{p}, i-1)$-allowable if $\partial \xi$ is $(\overline{p}, i-1)$ allowable. It follows (after checking that it is well defined) that we have a map:

$$\pi_* : IH_{\cdot}^{\overline{p}}(Y, k) \to IH_{\cdot}^{\overline{p}}(X, k)$$

satisfying $\pi_* j_* = \text{id}$. We claim that $j_* \pi_*(\xi)$ is (intersection) homologous to ξ for each cycle $\xi \in IC_i(Y, k)$. This is quite easy. For two points $x = (s, x_1)$ and $y = (t, x_1)$ in the same vertical ray $\mathbb{R} \times \{x_1\}$, we denote $(\lambda s + \mu t, x_1)$ by $\lambda x + \mu y$. With this notation, for an i-simplex σ in Y which is in general position, define the geometric $(i+1)$-chain

$$K\sigma := \{t.x + (1-t)\pi(x) \in \mathbb{R} \times X : t \in [0, 1], \ \ x \in \sigma\}$$

which may be thought of as a prism with one end being σ and another end being $\pi(\sigma)$. This defines $K\xi$ as a finitely supported $(i+1)$-chain for each finitely supported i-chain ξ. It follows quite easily that if ξ is a cycle, then $\partial K\xi = \xi - \pi(\xi)$. This also shows that $\partial K\xi$ is $(\overline{p}, i)$-allowable whenever ξ is $(\overline{p}, i)$-allowable. It is clear that for a i-simplex σ in general position, $\dim (\mid K\sigma \mid \cap (\mathbb{R} \times X^{n-k})) = 1 + \dim (\mid \sigma \mid \cap (\mathbb{R} \times X^{n-k}))$, which proves that $K\xi$ is the $(\overline{p}, i+1)$-allowable if ξ is $(\overline{p}, i)$-allowable. The proposition follows. □

Remark 5.4.7. Note that if we take just a locally finite chain ξ in $\mathbb{R} \times X$ which is not finite, then $\pi(\xi)$ will not in general be locally finite. For example take $\omega = e^{2\pi i\alpha} \in S^1$ with α irrational, and set $x_n = (n, \omega^n), \ \ n \in \mathbb{Z}_+$. Then $\xi = \sum_n x_n$ is a locally finite (Borel Moore) 0-chain in $\mathbb{R} \times S^1$, but $\mid \pi(\xi) \mid$ is a countable dense set in S^1. This is precisely why the calculations for Borel-Moore homology and finitely supported homology are radically different.

We first prove an analogue of Lemma 5.4.2 for the case of cones.

We also assume henceforth, whenever we take a cone, that X is compact so that cX is locally finite and locally compact.

Lemma 5.4.8. Let X be a compact stratified $(n-1)$-pseudomanifold, and let $Y := cX = [0, \infty) \times X/\sim$ be the cone on X, which is given its natural structure of a stratified n-pseudomanifold as in Definitions 5.1.2, 5.4.1. Let $\overline{p} = (p_2, .., p_n)$ be any perversity. Then:

- (i): Let $i \geq n - p_n - 1$ and let ξ be a $(\overline{p}, i)$-allowable cycle supported in the closed a-cone $c_{\leq a}X := [0, a] \times X/\sim$ contained inside cX. Assume ξ is 'general position' so that each ray $c\{x\}$ intersects $\mid \xi \mid$ transversely. Then there exists a $(\overline{p}, i+1)$-allowable chain $K^+\xi$, also supported in $c_{\leq a}X$, satisfying $\partial K^+\xi = \xi$.
- (ii): Let ξ be a $(\overline{p}, i)$-allowable cycle supported in $c_{\geq a}X := [a, \infty) \times X$ contained inside cX, in general position as above. Then there exists a $(\overline{p}, i+1)$-allowable chain $K^-\xi$, also supported in $c_{\geq a}X$, satisfying $\partial K^-\xi = \xi$

Proof: For the proof of (i), set $K^+\xi := \overline{c}\xi = \sum_\sigma \xi(\sigma) c(\sigma)$, where $\overline{c}(\sigma)$ is the $(i+1)$-simplex obtained by (closed)-coning the i-simplex σ from the cone point p of cX (i.e. $\overline{c}(\sigma)$ is the *simplicial join* $p * \sigma$, and the general position hypothesis of ξ guarantees that this is an $(i+1)$-simplex). For codimensions $0 \leq k \leq n-1$, the allowability condition $\dim (\mid K^+\xi \mid \cap Y^{n-k}) \leq (i + 1 - k + p_k)$ follows from the $(\overline{p}, i)$-allowability of ξ, exactly as in Lemma 5.4.2. On the other hand, in codimension $k = n$, we have that $\mid K^+\xi \mid \cap Y^0 = \{p\}$ is 0-dimensional, and $0 \leq (i+1) - n + p_n$ by the condition imposed on i. Thus $K^+\xi$ is $(\overline{p}, i+1)$-allowable and (i) is established.

The proof of (ii) is completely analogous to Lemma 5.4.2, using the translation map:

$$\begin{array}{rcl}\beta : \mathbb{R}_+ \times (cX \setminus \{p\}) & \to & cX \setminus \{p\} \\ (s, [(t,x)]) & \mapsto & [(t+s, x)]\end{array}$$

instead of the map α used in that lemma. We skip the details. □

Now one can calculate the intersection homologies of a cone.

Proposition 5.4.9 (Borel-Moore intersection homology of a cone)**.** Let X be a compact stratified $(n-1)$-pseudomanifold, and let $Y := cX$ be the open cone, a stratified n-pseudomanifold with stratification as in Definitions 5.1.2, 5.4.1 above. Then, for all perversities $\overline{p}$, we have:

$$\begin{array}{rcl}IH_i^{\overline{p},BM}(Y,k) & = & 0 \quad \text{for} \;\; i \leq n - p_n - 1 \\ & = & IH_{i-1}^{\overline{p}}(X,k) \quad \text{for} \;\; i \geq n - p_n\end{array}$$

Proof: Let $i \leq n - p_n - 1$. Then if $\xi \in C_i^{BM}(Y,k)$ is a $(\overline{p}, i)$-allowable cycle, it follows that $\dim(|\,\xi\,| \cap Y^0) \leq i - n + p_n \leq -1$, so ξ is a cycle supported in $Y \setminus \{p\}$. Since $|\,\xi\,|$ is closed, it is actually contained in $c_{\geq a}X$ for some $a > 0$. Now (ii) of the foregoing Lemma 5.4.8 applies (after putting ξ is general position if needed), and $\xi = \partial K^- \xi$ shows that ξ is nullhomologous in $IH_i^{\overline{p},BM}(Y,k)$. This proves the first assertion of the proposition.

Next, define the coning operation:

$$\begin{array}{rcl}c : C_i(X,k) & \to & C_{i+1}(Y,k) \\ \xi & \mapsto & c\xi\end{array}$$

where $c\xi = \sum_{j=1}^r \xi_j c(\sigma_j)$ for $\xi = \sum_{j=1}^r \xi_j \sigma_j$. It is easy to check that if ξ is $(\overline{p}, i)$-allowable in X (where the entry p_n of $\overline{p}$ is ignored since $\dim X = n-1$) and $i \geq n - p_n - 1$, then $c(\xi)$ is $(\overline{p}, i+1)$-allowable in Y. The key point is that $\dim(|\,c\xi\,| \cap Y^0) = \dim\{p\} = 0 \leq (i+1) - n + p_n$. The intersections with other strata fall in line from the $(\overline{p}, i)$-allowability of ξ. We claim this map c induces isomorphisms in intersection homology in the dimensions $i + 1 \geq n - p_n$.

If $i+1 \geq n - p_n$, a generic a-slice $\{a\} \times X \subset cX = Y$ meets $|\,\xi\,|$ (dimensionally) transversely, as in the proof of Proposition 5.4.4. Then set $\theta(\xi) = \xi \cap (\{a\} \times X)$, which is a $(\overline{p}, i)$-allowable cycle by analogous reasoning. Also set $\xi^+ := \xi \cap c_{\leq a}X$ and $\xi^- := \xi \cap c_{\geq a}X$. Then, again choosing orientations compatibly, we have $\xi = \xi^+ - \xi^-$. The cycle $\eta := \xi^+ - c_{\leq a}(\theta(\xi))$ is $(\overline{p}, i)$-allowable, supported in $c_{\leq a}X$ and (if we move it into general position by initially perturbing ξ) satisfies the conditions in (i) of Lemma 5.4.8 above. Thus $\eta = \partial K^+ \eta$. Similarly, by (ii) of the Lemma 5.4.8 above, we have $\zeta := \xi^- - c_{\geq a}(\theta(\xi)) = \partial K^- \zeta$. Thus we finally have:

$$\xi - c(\theta(\xi)) = (\xi^+ - c_{\leq a}(\theta(\xi)) - (\xi^- - c_{\geq a}(\theta(\xi)) = \eta - \zeta = \partial K^+ \eta - \partial K^- \zeta$$

shows that ξ is (intersection) homologous to $c(\theta(\xi))$. Thus the maps $\xi \mapsto \theta(\xi)$ with inverse map $\rho \mapsto c\rho$ (after checking that the map θ is well-defined at the level of homology) produces the desired isomorphism $IH_{i+1}^{\overline{p},BM}(cX,k) \simeq IH_i^{\overline{p}}(X,k)$ for $i + 1 \geq n - p_n$, and the proposition follows. □

It is quite natural to ask what happens to the above results in the context of finitely supported intersection homology instead of Borel-Moore intersection homology. We have the following proposition.

Proposition 5.4.10 (Finite support intersection homology for cones)**.** Let X and $Y = cX^{n-1}$ be as in the last Proposition 5.4.9. Then, for any perversity $\overline{p} = (p_2, .., p_n)$ the finitely supported intersection homology of Y is given by:

$$\begin{array}{rcl}IH_i^{\overline{p}}(Y,k) & = & IH_i^{\overline{p}}(X,k) \quad \text{for} \;\; 0 \leq i \leq n - p_n - 2 \\ & = & 0 \quad \text{for} \;\; i \geq n - p_n - 1\end{array}$$

Proof: Let $\xi \in IC_i^{\overline{p}}(Y,k)$ be a $(\overline{p},i)$-allowable chain of finite support in Y for $i \leq n-p_n-1$. Then $i-n+p_n \leq -1$, from which it follows that $|\,\xi\,|$ does not meet the cone point $p = Y^0$. Hence it is a finitely supported chain in $Y \setminus \{p\} = c_{>0}X$. Since ξ is a finite chain, it actually lies in $c_{>a}X$ for some $a > 0$. Thus we have an identification:

$$IC_i^{\overline{p}}(Y,k) = \lim_{a\to 0} IC_i^{\overline{p}}(c_{>a}X,k) \ \text{ for } \ 0 \leq i \leq n-p_n-1$$

It follows that $IH_i^{\overline{p}}(Y,k) = \lim_{a\to 0} IH_i^{\overline{p}}(c_{>a}X,k)$ for $0 \leq i \leq n-p_n-2$, and from the Proposition 5.4.6 (since $c_{>a}X \simeq (a,\infty)\times X$), this last limit is $IH_i^{\overline{p}}(X,k)$.

For $i \geq n-p_n-1$, we note that for ξ a $(\overline{p},i)$-allowable chain, the cone $K^+\xi$ is an allowable $(\overline{p},i+1)$-chain, where K^+ is the operator defined in the proof of Lemma 5.4.8. As noted there, for ξ a cycle, we have $\partial K^+\xi = \xi$. This shows that ξ is nullhomologous. The proposition follows. □

Remark 5.4.11. Again, in analogy with the Remark 5.4.7, we note that the operation K^+ of coning off a cycle from the cone point p can only be applied to finitely supported chains. Coning off an infinite chain will produce a chain that isn't locally-finite at p. Hence the difference between the calculations of Propositions 5.4.9 and 5.4.10.

Corollary 5.4.12 (Poincare Duality for cones)**.** Let X be a compact stratified $(n-1)$-pseudomanifold. Assume that X satisfies k-Poincare duality, viz. for each pair of complementary perversities $\overline{r} = (r_2,..,r_{n-1})$ and $\overline{s} = (s_2,..,s_{n-1})$ we have

$$IH_i^{\overline{r}}(X,k) \simeq IH_{n-1-i}^{\overline{s}}(X,k)$$

for all $0 \leq i \leq (n-1)$. Then the open cone $Y := cX$ also satisfies k-Poincare Duality, viz.,

$$IH_i^{\overline{p},BM}(Y,k) \simeq IH_{n-i}^{\overline{q}}(Y,k)$$

for all complementary perversities $\overline{p}$, $\overline{q}$ and all $0 \leq i \leq n$.

Proof: Let $\overline{p}+\overline{q} = \overline{t} = (0,1,2,..,n-2)$, so that $p_k + q_k = k-2$ for all k. Then for $i \leq n-p_n-1$, we have $n-i \geq p_n+1 = n-2-q_n+1 = n-q_n-1$, and we have:

$$IH_i^{\overline{p},BM}(Y,k) = 0 = IH_{n-i}^{\overline{q}}(Y,k)$$

where the first equality follows from Proposition 5.4.9 and the second from Proposition 5.4.10. On the other hand, for $i \geq n-p_n$, we have $(n-i) \leq p_n = n-q_n-2$, so that:

$$IH_i^{\overline{p},BM}(Y,k) \simeq IH_{i-1}^{\overline{p}}(X,k);\ \ IH_{n-i}^{\overline{q}}(Y,k) \simeq IH_{n-i}^{\overline{q}}(X,k)$$

where the first equality follows from Proposition 5.4.9 and the second from Proposition 5.4.10. But then, X compact and Poincare duality hypothesis on X implies that for all i we have $IH_{i-1}^{\overline{p}}(X,k) \simeq IH_{n-i}^{\overline{q}}(X,k)$, since $\dim X = (n-1)$. Thus $IH_i^{\overline{p},BM}(Y,k) \simeq IH_{n-i}^{\overline{p}}(Y,k)$ for $i \geq n-p_n$ as well, and the corollary follows. (Note we haven't specified the Poincare Duality map, which will be done later) □

5.5. **Comparison of intersection homology and usual homology.** We note the following differences between intersection homology and ordinary homology.

Remark 5.5.1 (Intersection homology with finite support is not homotopy invariant)**.** Note that if we take $Y = cX$, where X is a compact stratified $(n-1)$-pseudomanifold, then it is always contractible (to the cone point $p = Y^0$). However, if we ensure that X has non- trivial homology, then by the computation of Proposition 5.4.10 its intersection homology with finite support is non-zero in general. Hence homotopy invariance is no longer true for finitely supported intersection homology, in contrast with finitely supported homology.

Remark 5.5.2 (Subdivision and intersection homology)**.** We recall that simplicial homology can be calculated with respect to any admissible triangulation. Unfortunately, this is *false* for intersection homology. One must take geometric chains, i.e. consider all admissible trinagulations simultaneously. As an example, take the triangulation T of S^3 illustrated in the Fig. 3, with seven vertices (The two solid 3-balls depicted are to be glued along the common boundary S^2 by the vertex identifications $1 \leftrightarrow 1$, $2 \leftrightarrow 2$, $3 \leftrightarrow 3$, $4 \leftrightarrow 4$ and $5 \leftrightarrow 5$ as shown). Set $\overline{p} = (p_2,p_3) = (0,0)$ (the bottom perversity) and set $X^0 = \{1,5,6,7\}$. If one writes down the chain

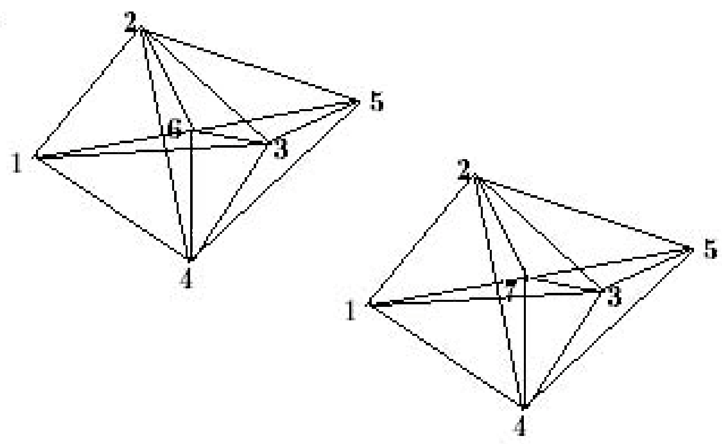

Figure 3. A triangulation of S^3

complex of allowable simplicial chains (whose boundaries are allowable) with respect to this triangulation T, the homologies one gets are *not* the intersection homologies of S^3. Indeed, for a 2-chain ξ the $(\overline{p}, 2)$-allowability condition means that the dimension $\dim(|\, \xi \,| \cap X^0) \leq 2 - 3 + p_3 = -1$, so $|\, \xi \,|$ must be disjoint from X^0. But notice that every 2-simplex of T intersects X^0, and hence it follows that the trivial chain 0 is the only allowable 2-chain in this triangulation. Hence the allowable 1-cycle $\langle 23 \rangle + \langle 34 \rangle + \langle 42 \rangle$ is *not* the boundary of an allowable 2-chain, and generates 1-homology. However, when we pass to a subdivision, it is easy to see that it will bound an allowable 2-cycle which avoids X^0, and lead to $IH_1^{\overline{p}}(S^3, k) = 0$, as it should (since it should be $H_1(S^3, k) = 0$, by Exercise 5.2.9).

Remark 5.5.3. Intersection homology with coefficients in $k = \mathbb{Z}$ does not generally obey duality of free parts in complementary dimensions and complementary perversities for oriented n-pseudomanifolds as ordinary $\mathbb{Z}$-coefficient homology does for n-manifolds (see Example 5.3.7). It will only obey it after tensoring with $\mathbb{Q}$.

6. Sheaf Theoretic Intersection Homology

6.1. **Softness of the Intersection Sheaves.** We recall the dualising (homology) sheaf from §4.3. The geometric Borel-Moore i-chains give rise to a sheaf $\mathcal{C}_i^{BM} = \mathcal{D}^{-i}(\mathcal{C}^{\cdot}, k)$, whose sections on an open set $U \subset X$ are given as:

$$\Gamma(U, \mathcal{C}_i^{BM}) = \hom(\Gamma_c(U, \mathcal{C}^i), k) \simeq C_i^{BM}(U, k) = \lim_{T} C_i^{BM, T_U}(U, k)$$

where the limit on the right is a direct limit over all induced triangulations T_U on U as T ranges through the directed set of all admissible triangulations T on X. We note that if $V \subset U$ are open sets, then there is a geometric description of the sheaf restriction maps. Let T be an admissible triangulation on X. There are compatible triangulations T_U and T_V such that for each i-simplex σ of T_V, there is a unique i-simplex $\tau(\sigma)$ of T_U which is its carrier, viz. $\sigma \subset \tau(\sigma)$, which leads to the restriction maps:

$$\begin{aligned} \rho_{UV} : C_i^{BM, T_U}(U, k) \quad &\to \quad C_i^{BM, T_V}(V, k) \\ \xi = \sum_{\alpha \in T_U} \xi(\alpha)\alpha \quad &\mapsto \quad \sum_{\alpha,\ \sigma \in T_V : \tau(\sigma) = \alpha} \xi(\alpha)\sigma \end{aligned}$$

Passing to limits over admissible triangulations leads to the restriction map

$$\rho_{UV} : \Gamma(U, \mathcal{C}_i^{BM}) \to \Gamma(V, \mathcal{C}_i^{BM})$$

The Borel-Moore intersection homology sheaf is defined as a suitable subsheaf of $\mathcal{C}_i^{BM}$. First note that for any closed PL-subspace $Y \subset X$, the germ Y_x of Y at any point $x \in X$ makes sense. Just take a neighbourhood U of x, and look at the equivalence class of the intersection $U \cap Y$, the equivalence relation of course being that of germs (=equality on restriction to a smaller neighbourhood). The dimension of a germ (since it is locally a closed subcomplex) also makes sense, and hence we may make the:

Definition 6.1.1 (The Intersection chain complex of sheaves)**.** Let X be a stratified n-pseudomanifold, and let $\overline{p} = (p_2, .., p_n)$ be a perversity. Define:

$$\begin{aligned} \Gamma(U, \mathcal{IC}_i^{\overline{p}}) := \{\xi \in \Gamma(U, \mathcal{C}_i^{BM}) : \dim(|\,\xi\,| \cap X^{n-k})_x &\leq i - k + p_k, \\ \dim(|\,\partial\xi\,| \cap X^{n-k})_x &\leq i - 1 - k + p_k \quad \text{for all } \; x \in U, \quad \text{and all } \; 2 \leq k \leq n\} \end{aligned}$$

By definition, it follows that (i) For $V \subset U$ open, ρ_{UV} maps $\Gamma(U, \mathcal{IC}_i^{\overline{p}})$ into $\Gamma(V, \mathcal{IC}_i^{\overline{p}})$, and (ii) $\mathcal{IC}_{\cdot}^{\overline{p}}$ is a subcomplex of sub-presheaves of the complex of sheaves $\mathcal{C}_{\cdot}^{BM}$, and hence a chain-complex of sheaves on X. One can turn it into a cochain complex $\mathcal{IC}_{\overline{p}}^{\cdot}$ as usual, by changing the sign of the grading and making it a superscript.

Notation : 6.1.2. *In the rest of this section, the perversity $\overline{p}$ will be held fixed throughout and suppressed from the notation.*

Lemma 6.1.3. The sheaves $\mathcal{IC}_{\cdot}$ are soft.

Proof: Let X be a stratified n-pseudomanifold, and $K \subset X$ be a compact set. Let $\xi \in \Gamma(K, \mathcal{IC}_i)$ be an i-chain. We need to find an i-chain in $\Gamma(X, \mathcal{IC}_i)$ whose restriction to K is ξ. We first do the simpler case:

Case 1: K is a compact PL-subspace of X.

By the definition of $\Gamma(K, \mathcal{IC}_i)$ and the local compactness, paracompactness etc. of X, there is an open neighbourhood U of K, and a chain, also denoted ξ, in $C_i^{BM,T_U}(U, k)$ satisfying:

$$\dim(|\,\xi\,| \cap U^{n-k}) \leq i - k + p_k; \quad \dim(|\,\partial\xi\,| \cap U^{n-k}) \leq i - 1 - k + p_k$$

where $U^{n-k} := U \cap X^{n-k}$ are the strata of the induced stratification on U. We assume (without loss) that the triangulation T_U of U is such that all of the U^{n-k} are subcomplexes of U (for example, by passing to an admissible triangulation T of X under which all X^{n-k} are subcomplexes, and then taking T_U compatible with T). Similarly, since K is compact, we can assume that K is a subcomplex of T and of T_U. Write $\xi = \sum_{\sigma \in T_U} \xi(\sigma)\sigma$.

At this point it is natural to take the chain τ defined by:

$$\tau := \sum_{\sigma \cap K \neq \phi} \xi(\sigma)\sigma$$

This will be a finite chain by the compactness of K, and hence may be regarded as a chain in X. Further, since ξ is $(\overline{p}, i)$-allowable and $|\,\tau\,| \subset |\,\xi\,|$, it will follow that τ is also $(\overline{p}, i)$-allowable. Clearly the restriction of τ to K is the same as that of ξ to K. However, the trouble is that $|\,\partial\tau\,|$ has nothing to do with $|\,\partial\xi\,|$. Indeed ξ may be a cycle, and the set $|\,\partial\xi\,|$ may consequently be empty, but the set $|\,\partial\tau\,|$ may not be $(\overline{p}, i-1)$-allowable. The trouble can be remedied by taking a subdivision.

Let T'_U be the first barycentric subdivision of T_U. For an oriented i-simplex $\sigma' \in T'_U$, define the *carrier of* σ', denoted $\mathrm{carr}(\sigma')$, to be the unique i-simplex $\sigma \in T_U$ such that $|\,\sigma'\,| \subset |\,\sigma\,|$. Now set:

$$F := \{\sigma' \in T'_U \;\text{ an oriented } i\text{-simplex} : \; \xi(\mathrm{carr}(\sigma')) \neq 0 \text{ and } \; \sigma' \text{ has a vertex in } K\}$$

Since K is compact, the set F is finite. Now consider the i-chain $\tau \in C_i^{T'_U}(U, k)$ defined by:

$$\tau := \sum_{\sigma' \in F} \tau(\sigma')\sigma'$$

where

$$\begin{aligned}\tau(\sigma') &= \xi(\sigma) \text{ if } \operatorname{carr}(\sigma') = \sigma \text{ and } \sigma', \sigma \text{ are compatibly oriented} \\ &= -\xi(\sigma) \text{ if } \operatorname{carr}(\sigma') = \sigma \text{ and } \sigma', \sigma \text{ are oppositely oriented} \\ &= 0 \text{ otherwise}\end{aligned}$$

Since F is a finite set, we can find a *single* admissible triangulation T_1 of X such that $\tau \in C_i^{T_1}(X,k)$. It is easily verified that τ restricts to ξ in $\Gamma(K, \mathcal{C}_i)$, and since $|\tau| \subset |\xi|$, it follows that τ is $(\overline{p}, i)$-allowable.

Now $\partial\tau$ is a sum of oriented $(i-1)$-simplices which are faces $\partial_j \sigma'$ of oriented i-simplices $\sigma' \in F$, with certain coefficients. Each oriented i-simplex σ' in T'_U is of the form:

$$\sigma' = \pm\langle b(\sigma_0), b(\sigma_1), ..., b(\sigma_i = b(\sigma)\rangle$$

where $\sigma_0 \prec \sigma_1 \prec .... \prec \sigma_i = \sigma$ is a full flag of faces of $\sigma \in T_U$ and $b(\sigma_j)$ the barycentre of σ_j. Thus σ' has a vertex in K iff $b(\sigma_0) = \sigma_0 \in K$.

Let $\sigma' \in F$ be as above, and let $\operatorname{carr}\sigma' = \sigma$, σ a $(\overline{p}, i)$-allowable i-simplex as above. Write the support of face $\partial\sigma'$ as the disjoint union:

$$|\partial\sigma'| = A \coprod B; \text{ where } A = |\partial\sigma'| \cap \sigma^\circ; \; B = |\partial\sigma'| \cap |\partial\sigma|$$

where σ° is the interior of σ.

Claim: $A \cap X^{n-k} = \phi$ for all $k \geq 2$. This is because σ being $(\overline{p}, i)$-allowable, satisfies:

$$\dim(|\sigma| \cap X^{n-2}) \leq i - 2 + p_2 \leq i - 2$$

Thus σ cannot be contained in X^{n-2}. Since X^{n-2} is a subcomplex of T_U, and σ is a simplex of T_U, it follows that the intersection $|\sigma| \cap X^{n-2}$ is a proper subcomplex of $|\sigma|$, and hence contained in $|\partial\sigma|$. Thus $\sigma^\circ \cap X^{n-2} = \phi$, and a fortiori $A \cap X^{n-k} = \phi$ for all $k \geq 2$. The Claim follows.

Thus, for $k \geq 2$, the intersection $|\partial\sigma'| \cap X^{n-k}$ is a union of faces:

$$|\partial\sigma'| \cap X^{n-k} = \bigcup_{j \in M} (|\partial_j\sigma'| \cap X^{n-k}) \subset B \cap X^{n-k}$$

where

$$M := \{j : \operatorname{carr}(\partial_j\sigma') = \partial_l\sigma \text{ for some } l\}$$

Now suppose $j \in M$, with $\operatorname{carr}(\partial_j\sigma') = \partial_l\sigma$ for some l. Then two cases arise:

Subcase 1: $\partial_l\sigma$ occurs with non-zero coefficient in $\partial\xi$. In this case by the $(\overline{p}, i-1)$-allowability of $\partial\xi$, we have

$$\dim(|\partial_l\sigma| \cap X^{n-k}) \leq i - 1 - k + p_k$$

for all $k \geq 2$, which implies that:

$$\dim(|\partial_j\sigma'| \cap X^{n-k}) \leq \dim(|\partial_l\sigma| \cap X^{n-k}) \leq i - 1 - k + p_k$$

for all $k \geq 2$.

Or

Subcase 2: $\partial_l\sigma$ *does not occur* in $\partial\xi$. In this event we have another simplex $\widetilde{\sigma}$ of T_U occurring in ξ such that $\partial_l\sigma = \partial_k\widetilde{\sigma}$ and $\xi(\sigma) = -\xi(\widetilde{\sigma})$. Since σ' has a vertex in K ($\sigma' \in F$ we recall), so $b(\sigma_r) \in K$ for some $r \geq 0$, which implies (since K is a subcomplex with resp. to T_U) that $\sigma_k \subset K$, which implies $b(\sigma_0), .., b(\sigma_r) \in K$ as well. Since $\partial_j\sigma'$ is contained in $\partial_l\sigma$, the barycenter $b(\sigma)$ does not occur in $\partial_j\sigma'$, so in fact :

$$\partial_j\sigma' = \langle b(\sigma_0), ..., b(\sigma_{i-1})\rangle$$

Now consider the simplex:

$$\widetilde{\sigma}' := \langle b(\sigma_0), ..., b(\sigma_{i-1}), b(\widetilde{\sigma})\rangle$$

This simplex $\widetilde{\sigma}'$ is a simplex of T'_U. Furthermore $\widetilde{\sigma}'$ qualifies for membership of F, since its first r-vertices are the same as the first r vertices of σ', and hence lie in K. Thus $\widetilde{\sigma}'$ occurs in τ, with coefficient $\xi(\widetilde{\sigma}) = -\xi(\sigma)$.

The boundary terms $\partial_t\widetilde{\sigma}'$ of $\partial\widetilde{\sigma}'$ therefore occur in $\partial\tau$ with coefficient $\xi(\widetilde{\sigma})$. In particular the term $\xi(\widetilde{\sigma})\partial_m\widetilde{\sigma}'$, where

$$\partial_m\widetilde{\sigma}' = \langle b(\sigma_0), ..., b(\sigma_{i-1})\rangle = \partial_l\sigma'$$

occurs in the boundary $\partial\tau$, and cancels out the term $\xi(\sigma)\partial_j\sigma'$ in $\partial\tau$. Thus in Subcase 2, the term $|\partial_j\sigma'|$ *does not occur* in $\partial\tau$, so is not a subset of $|\partial\tau|$, and hence plays no role in the $(\overline{p}, i-1)$-allowability of $\partial\tau$. Thus $\partial\tau$ is $(\overline{p}, i-1)$-allowable. □.

Case 2: Now let C be an arbitrary compact set, let $\xi \in \Gamma(C, \mathcal{IC}_i)$. As in Case 1, take a representative $\xi \in \Gamma(U, \mathcal{IC}_i)$ where U is an open neighbourhood of K. Look at all the finitely many simplices $\sigma \in T_U$ that meet C (since K is compact), and let K be the union of all these finitely many simplices. Clearly $C \subset K$, and K is a finite subcomplex. Apply case 1 above to $\xi_{|K}$. The proposition follows. □

Corollary 6.1.4 (IH_* as hyperhomology)**.** The Borel-Moore (resp. finitely supported) intersection homology of a stratified n-pseudomanifold X is the hypercohomology of X (resp. hypercohomology of X with compact supports) in the complex of sheaves $\mathcal{IC}.$. That is:

$$IH_i^{BM}(X,k) = \mathbb{H}_i(X, \mathcal{IC}.) = R^i\Gamma(X, \mathcal{IC}.); \;\; IH_i(X,k) = \mathbb{H}_{i,c}(X, \mathcal{IC}.) = R^i\Gamma_c(\mathcal{IC}.)$$

Proof: We recall the hyperhomology spectral sequence with:

$$E^1_{pq} = H_q(X, \mathcal{IC}_p)$$

which abuts to $\mathbb{H}_{p+q}(X, \mathcal{IC}.)$. Since $\mathcal{IC}_p$ is soft for all $p \geq 0$, and X is *paracompact, hausdorff*, it follows that $\mathcal{IC}_p$ is acyclic for the global section functor Γ. Thus the spectral sequence collapses, and $E^1_{p,0} = H_0(X, \mathcal{IC}_p) = \Gamma(X, \mathcal{IC}_p) = IC_p(X,k)$. As a consequence:

$$E^2_{p,0} = E^\infty_{p,0} = H_p(IC.(X,k)) = IH_p(X,k) = \mathbb{H}_p(X, \mathcal{IC}.)$$

The statement for finitely supported intersection homology is proved similarly, noting that $\mathcal{IC}.$ being soft, is acyclic for finitely supported homology. □

6.2. **The Local Computation.** The next step is to compute the derived sheaf $\mathcal{H}.(\mathcal{IC}.)$, in the spirit of what we did in the Proposition 3.2.3 for the complex of geometric chains. In that Proposition, the link of every point on an n-manifold is an $(n-1)$-sphere, and the derived sheaf turns out to be concentrated in dimension n. Unlike that scenario, the derived sheaf of $\mathcal{IC}.$ will no longer just be concentrated in dimension n. Indeed, for a point $x \in S^{n-k}$, the codimension k-stratum, it will contain local topological data about the link L_x, where $U \simeq B^{n-k} \times cL_x$ is a locally trivial neighbourhood of x.

Definition 6.2.1. We define the *truncation operator* $\tau_{\geq r}$ on a chain complex $C.$ by:

$$\begin{aligned}(\tau_{\geq r}C)_m &= C_m \quad \text{for } m > r \\ &= \ker\left[\partial_r : C_r \to C_{r-1}\right] \quad \text{for } m = r \\ &= 0 \quad \text{for } m < r\end{aligned}$$

Note that there is thus a natural chain map $\tau_{\geq r}C. \to C.$ which induces isomorphisms of homology H_i for $i \geq r$.

Lemma 6.2.2. Let X be a *compact* stratified $(k-1)$-pseudomanifold. Let $\overline{p} = (p_2, .., p_n)$ be any n-perversity, which is to be held fixed, and will be henceforth suppressed from the notation. Similarly, all intersection homology is with respect to some fixed coefficient ring k, which is also suppressed from the notation. Then there is a commutative diagram:

$$\begin{array}{ccc} \tau_{\geq n-p_k} IC_{*-(n-k+1)}(X) & \xrightarrow{\rho} & IC_{*-(n-k+1)}(X) \\ \downarrow \alpha_* & & \downarrow \beta_* \\ IC_*^{BM}(\mathbb{R}^{n-k} \times cX) & \xrightarrow{\gamma} & IC_*^{BM}(\mathbb{R}^{n-k} \times (cX \setminus \{p\})) \end{array}$$

where the vertical arrows are quasiisomorphisms. Here, the top horizontal map is from Definition 6.2.1 and the bottom horizontal map γ is the restriction map on sections of the Borel-Moore intersection homology sheaf defined in the last subsection.

Proof: For $i \geq n - p_k$, the left vertical arrow α_i is the composite:

$$IC_{i-(n-k+1)}(X) \xrightarrow{c} IC^{BM}_{i-n+k}(cX) \xrightarrow{\Sigma^{n-k}} IC^{BM}_i(\mathbb{R}^{n-k} \times cX)$$

where c is the coning map of Proposition 5.4.9 and Σ is the suspension map of Proposition 5.4.4.

The second map Σ^{n-k} is a quasiisomorphism when $i \geq n - p_k$, by Proposition 5.4.4. In this case, $j := i - n + k \geq k - p_k$. Thus the first map is a quasiisomorphism of complexes $\tau_{\geq n-p_k} IC_{*-(n-k+1)}(X) \to IC_{*-n+k}(cX)$, by Proposition 5.4.9. Hence the quasiisomorphism α_*.

That the second vertical map β_* is a quasiisomorphism follows from noting that $\mathbb{R}^{n-k} \times (cX \setminus \{p\}) \simeq \mathbb{R}^{n-k+1} \times X$, and the Proposition 5.4.4.

The commutativity of the diagram follows by noting that for a chain ξ in X, the restriction of $c\xi$ to $cX \setminus \{p\} \simeq \mathbb{R} \times X$ is the same as the suspension $\mathbb{R} \times \xi = \Sigma\xi$. This proves the commutativity for $n = k$. The commutativity for general k then follows by composing everything with Σ^{n-k}. $\square$

We now revert to cochain complexes instead of chain complexes. So we define:

Definition 6.2.3. The sheaf of *intersection cochains*, denoted $\mathcal{IC}^{\cdot}$ (with perversity $\overline{p}$, which is usually going to be suppressed from the notation) is defined as:

$$\mathcal{IC}^j = \mathcal{IC}_{-j}$$

(This is the convention in [GM2]. Some authors (e.g. [Bor]) use $\mathcal{IC}^j = \mathcal{IC}_{n-j}$, which has the advantage of keeping $\mathcal{IC}^{\cdot}$ a chain complex in non-negative degrees.)

Proposition 6.2.4. Let X be a compact $(k-1)$-pseudomanifold, and let $i : \mathbb{R}^{n-k} \times (cX \setminus \{p\}) \hookrightarrow \mathbb{R}^{n-k} \times cX$ denote the inclusion. Let $x \in \mathbb{R}^{n-k} \times \{p\}$ be any point. Then:

(i): The natural (sheaf restriction) maps:

$$\begin{aligned} IC^{BM}_{\cdot}(\mathbb{R}^{n-k} \times cX) &\to (\mathcal{IC}^{-\cdot})_x \\ IC^{BM}_{\cdot}(\mathbb{R}^{n-k} \times (cX \setminus \{p\})) &\to (i_*\mathcal{IC}^{-\cdot})_x \end{aligned}$$

are quasiisomorphisms.

(ii): The derived sheaf of $\mathcal{IC}^{\cdot}$ is given by

$$\begin{aligned} \mathcal{H}^j(\mathcal{IC}^{\cdot})_x &= \mathcal{H}^j(i_*\mathcal{IC}^{\cdot})_x \quad \text{for } j \leq p_k - n \\ &= 0 \quad \text{for } j > p_k - n \\ &= 0 \quad \text{for } j < -n \end{aligned}$$

for $x \in \mathbb{R}^{n-k} \times \{p\}$. Thus, at a point $x \in \mathbb{R}^{n-k} \times cX$, the stalk cohomology $\mathcal{H}^j(\mathcal{IC}^{\cdot})_x$ is concentrated in the range $-n \leq j \leq p_k - n$.

(iii): The two sheaves $\mathcal{IC}^{\cdot}$ and $i_*\mathcal{IC}^{\cdot}$ are *cohomologically constant* (that is $\mathcal{H}^j(\mathcal{IC}^{\cdot}_x)$ and $\mathcal{H}^j(i_*\mathcal{IC}^{\cdot}_x)$ are constant) for $x \in \mathbb{R}^{n-k} \times \{p\}$

Proof: For $x \in \mathbb{R}^{n-k} \times \{p\}$, one may choose a fundamental system of neighbourhoods V_ϵ such that $V_\epsilon \simeq B^{n-k}_\epsilon \times cX$, where $B^{n-k}_\epsilon \subset \mathbb{R}^{n-k}$ is an open ball of dimension $n-k$. Furthermore V_ϵ is stratified compatibly with the stratification on $\mathbb{R}^{n-k} \times cX$. For any such V_ϵ, the sheaf theoretic restriction map:

$$IC^{BM}_{\cdot}(\mathbb{R}^{n-k} \times cX) \to IC^{BM}_{\cdot}(V_\epsilon) = \Gamma(V_\epsilon, \mathcal{IC}.)$$

produces isomorphisms on homology by appeal to Proposition 5.4.9. Taking limits over V_ϵ, and noting that $\mathcal{IC}^{\cdot} = \mathcal{IC}_{-\cdot}$, we have the first assertion of (i). For the second statement of (i), note that by definition

$$\Gamma(V_\epsilon, i_*\mathcal{IC}.) = \Gamma(i^{-1}(V_\epsilon), \mathcal{IC}.) = IC^{BM}_{\cdot}(B^{n-k}_\epsilon \times (cX \setminus \{p\}))$$

Since $cX \setminus \{p\} \simeq \mathbb{R} \times X$, appeal to the Proposition 5.4.4 implies the second assertion of (i).

For the next assertion (ii), since the top horizontal arrow ρ of Lemma 6.2.2 is an isomorphism on H_i for $i \geq n - p_k$, it follows from that lemma that:

$$IC_*^{BM}(\mathbb{R}^{n-k} \times cX) \xrightarrow{\gamma} IC_*^{BM}(\mathbb{R}^{n-k} \times (cX \setminus \{p\}))$$

is a quasiisomorphism for $i \geq n - p_k$. That is, for $j := -i \leq p_k - n$. Part (i) then implies that $H^j(\mathcal{IC}^\cdot_x)$ is isomorphic to $H^j(i_*\mathcal{IC}^\cdot_x)$ for $j \leq p_k - n$. This is precisely the first statement of (ii). The second statement follows immediately from the fact that α_* of Lemma 6.2.2 is a quasiisomorphism. The third statement follows from the fact that $\mathcal{IC}_i \equiv 0$ for $i > n$, since $\mathbb{R}^{n-k} \times cX$ is a PL-space of dimension n. This proves (ii).

The constancy of the cohomology sheaves follows from the fact that the link X of the cone cX is constant for $x \in \mathbb{R}^{n-k} \times \{p\}$, and depending on j, the homologies of these sheaves are either zero or corresponding intersection homology of X. □

6.3. **Constructibility of the complex $\mathcal{IC}_\cdot$ and Deligne's Construction.** The local calculations of the last section allow the definition of the complex $\mathcal{IC}^\cdot$ by an inductive procedure proceeding from the top-dimensional stratum S^{n-2} and moving down to S^{n-k}.

Proposition 6.3.1 (Properties of the complex $\mathcal{IC}^\cdot$)**.** , Let k be a coefficient ring, and let X be a stratified n-pseudomanifold. Let $S^{n-k} := X^{n-k} \setminus X^{n-k-1}$ (which is either empty or an $(n-k)$-dimensional manifold) as before, and let $U_k := X \setminus X^{n-k}$. Let $i_k : U_k \to U_{k+1}$ denote the inclusion. Then for the intersection cochain complex $\mathcal{IC}^\cdot$ we have the following facts:

(i): $\mathcal{IC}^\cdot$ is a bounded complex, and zero in degrees $< -n$.

(ii): $\mathcal{IC}^\cdot_{|U_2}$ is quasisomorphic to $Or_{U_2}[n]$, where Or_{U_2} is the orientation sheaf of U_2.

(iii): The derived sheaf $\mathcal{H}^j(\mathcal{IC}_{|U_{k+1}}) = 0$ for $j > p_k - n$.

(iv): The natural map of complexes of sheaves (called the attaching map):

$$\mathcal{IC}^\cdot_{|U_{k+1}} \to \tau_{\leq p_k - n} Ri_{k*}\mathcal{IC}^\cdot_{|U_k}$$

is a quasisomorphism of complexes of sheaves on U_{k+1}.

(v): $\mathcal{IC}^\cdot$ is cohomologically locally constant on S^{n-k} for each k (i.e. $\mathcal{H}^j(\mathcal{IC}^\cdot)$ is a locally constant sheaf) on each S^{n-k}.)

Proof: (i) follows by Definition 6.1.1, since $\mathcal{IC}^{-j} = \mathcal{IC}_j$ is a sheaf of j-chains for $0 \leq j \leq n$.

For $k = 2$, we have $S^{n-2} = X \setminus X^{n-2} = U_2$. Then

$$\mathcal{H}_i(\mathcal{IC}_{n-\cdot|U_2})_x = \lim_{V}(H_i(\Gamma(V, \mathcal{IC}_{n-\cdot})) = \lim_{V} IH^{BM}_{n-i}(V)$$

where V ranges over a fundamental system of neighbourhoods of x. By the local triviality of (ii) of Definition 5.1.2, V is homeomorphic to $\mathbb{R}^n$, and $IH^{BM}_{n-i}(V) \simeq H^{BM}_{n-i}(V)$, and this last homology $= 0$ for $i \neq 0$, and $= Or_{U_2}$ for $i = n$. Thus $\mathcal{H}^j(\mathcal{IC}^\cdot_{|U_2})_x = 0$ for $j \neq -n$ and $= Or_{U_2}$ for $j = -n$. Thus $\mathcal{IC}^\cdot_{|U_2}$ is quasiisomorphic to $Or_{U_2}[n]$, and (ii) is proved.

To see (iii), first note that for $x \in U_k$, $H^j(\mathcal{IC}^\cdot_{|U_{k+1},x}) = H^j(\mathcal{IC}^\cdot_{|U_k,x}) = 0$ for $j > p_{k-1} - n$, by induction on k. Hence it is 0 a fortiori for $j > p_k - n$ since $p_{k-1} \leq p_k$. (The induction begins by (ii) proved above).

For $x \in S^{n-k} = U_{k+1} \setminus U_k$, there is a neighbourhood U of x which is PL-stratified homeomorphic to $\mathbb{R}^{n-k} \times cX$, where X is a stratified $(k-1)$-pseudomanifold. In this model, the stratum $S^{n-k} \cap U = \mathbb{R}^{n-k} \times \{p\}$, where p is the cone point of cX. Thus the stalk of the intersection complex has the same cohomology as the intersection complex of $U = \mathbb{R}^{n-k} \times cX$. For $x \in S^{n-k} \subset U_{k+1}$, it follows from the assertion (ii) of Proposition 6.2.4 that $H^j(\mathcal{IC}^\cdot_{|U_{k+1},x}) = 0$ for $j > p_k - n$. This proves the assertion (iii).

To see (iv), note that $\mathcal{IC}^\cdot = \mathcal{IC}_{-\cdot}$ is a complex of *soft* sheaves by Lemma 6.1.3. Since U_k is open for all k, $\mathcal{IC}_{\cdot|U_k}$ is also a soft complex of sheaves, by Corollary 1.1.19. Thus $i_{k*}(\mathcal{IC}^\cdot_{|U_k})$ is quasiisomorphic to $Ri_{k*}(\mathcal{IC}^\cdot_{|U_k})$, as complexes of sheaves on U_{k+1}. For $x \in U_k$, the stalks $i_*(\mathcal{IC}^\cdot_{|U_{k+1},x})$ and $\mathcal{IC}^\cdot_{|U_{k+1},x}$ are the same, and because of (iii), so are the cohomologies of the stalks $\tau_{\leq p_k - n} i_*(\mathcal{IC}^{\cdot}_{|U_{k+1},x})$ and $\mathcal{IC}^{\cdot}_{|U_{k+1},x})$. For $x \in U_{k+1} \setminus U_k = S^{n-k}$, the assertion follows by using a model neighbourhood $U = \mathbb{R}^{n-k} \times cX$ as in the last para, and appealing to the first statement of (ii) of Proposition 6.2.4

(v) follows straightaway from considerations as above and (iii) of Proposition 6.2.4. The key point again is that for $x \in S^{n-k}$, the link X_x of x is locally constant. □

Remark 6.3.2. The Proposition above shows that at points $x \in U_2$, the dense stratum, the stalk of the derived sheaf $\mathcal{H}^j(\mathcal{IC}^{\cdot}_{|U_2})_x$ is situated only at $j = -n$ (generalising the statement of Proposition 3.2.3). At points $x \in S^{n-k}$, the stalk of the derived sheaf $\mathcal{H}^j(\mathcal{IC}^{\cdot})_x$ is situated in the degrees $-n \leq j \leq n - p_k$. So as one moves to points in strata of higher codimension, the derived sheaf has more spread, and lives in more degrees.

Definition 6.3.3. Let X be a stratified n-pseudomanifold, with closed strata $X^j \supset X^{j-1}$.... Say a complex of sheaves $\mathcal{A}^{\cdot}$ is *cohomologically locally constructible* (denoted CLC, and sometimes just called *constructible* by abuse of language) with respect to this stratification if the derived sheaf $\mathcal{H}^m(\mathcal{A}^{\cdot})$ is a locally constant sheaf on each $S^j = X^j \setminus X^{j-1}$ for each m.

For example, by (v) of Proposition 6.3.1, it follows that $\mathcal{IC}.$ is a constructible complex of sheaves on a stratified N-pseudomanifold X. It is quite natural to ask what properties a constructible complex $\mathcal{P}^{\cdot}$ of sheaves on X must satisfy in order to be quasiisomorphic to $\mathcal{IC}^{\cdot}$. In this connection, we have the following:

Definition 6.3.4 (Deligne's Construction)**.** Let X be a stratified n-pseudomanifold, and let $S_{n-k} := X^{n-k} \setminus X^{n-k-1}$, $U_k = X \setminus X^{n-k}$, and $i_k : U_k \hookrightarrow U_{k+1}$ denote the inclusion. Let $\overline{p}$ be a perversity (which will be fixed, suppressed from the notation in the sequel), and let $\mathbb{R}$ be the fixed coefficient ring, which will also be suppressed. Let $D^b(Y)$ denote the derived category of bounded $\mathbb{R}$-sheaves on a topological space Y. Inductively define a complex $\mathcal{P}^{\cdot}_k \in D^b(U_k)$ by:

(i): $\mathcal{P}^{\cdot}_2 = Or_{U_2}[n]$, where Or_{U_2} denotes the $\mathbb{R}$-orientation sheaf of $U_2 = X \setminus X^{n-2} = X \setminus \Sigma$.

(ii): Assume $\mathcal{P}_k \in D^b(U_k)$ has been defined for $k \geq 2$. Define:

$$\mathcal{P}^{\cdot}_{k+1} := \tau_{\leq p_k - n} \, Ri_{k*} \mathcal{P}^{\cdot}_k$$

which makes sense as an element of $D^b(U_{k+1})$.

(iii): Set $\mathcal{P}^{\cdot} \in D^b(X)$ to be $\mathcal{P}^{\cdot} = P^{\cdot}_{n+1}$. (noting that $U_{n+1} = X$!)

A complex of sheaves $\mathcal{P}^{\cdot}$ on $D^b(X)$ obtained by Deligne's construction above is called a *perverse sheaf* with perversity $\overline{p}$ and coefficients $\mathbb{R}$. (More generally, one can define a perverse sheaf with perversity $\overline{p}$ and coefficients k in analogous fashion.)

Clearly, the Proposition 6.3.1 implies that the sheaf of intersection cochains $\mathcal{IC}^{\cdot}$ is a perverse sheaf. One may ask whether the properties proved in Proposition 6.3.1 somehow characterise perverse sheaves in $D^b(X)$ (i.e. upto quasiisomorphism). There is a set of axioms denoted (AX1) which codify the Proposition 6.3.1

Definition 6.3.5 (The Axioms (AX1))**.** Let X be as above, and $\mathcal{S}^{\cdot}$ be a complex of sheaves on X. Denote $\mathcal{S}_k := S_{|U_k}$, and let $\overline{p}$ be a perversity. Say that $\mathcal{S}^{\cdot}$ satisfies axioms $(AX1)$ if:

(i): (Normalisation): $\mathcal{S}^{\cdot}_2$ is quasiisomorphic to $\mathcal{F}[n]$, where $\mathcal{F}$ is a locally constant sheaf on U_2.

(ii): (Lower bound): $\mathcal{H}^j(\mathcal{S}^{\cdot}) = 0$ for $j < -n$.

(iii): (Vanishing Condition): $\mathcal{H}^j(\mathcal{S}^{\cdot}_{k+1}) = 0$ for $j > p_k - n$.

(iv): (Attaching condition): Letting $j_k : S^{n-k} \hookrightarrow U_{k+1}$ and $i_k : U_k \hookrightarrow U_{k+1}$ denote the natural inclusions, the natural "attaching" map:

$$\mathcal{H}^i(j_k^* \mathcal{S}^{\cdot}_{k+1}) \to \mathcal{H}^i(j_k^* R i_{k*} i_k^* \mathcal{S}^{\cdot}_{k+1})$$

is an isomorphism for all $i \leq p_k - n$

Theorem 6.3.6 (Deligne)**.** Let $\mathcal{S}^{\cdot}$ be a complex of sheaves on X as above, which satisfies $(AX1)$, with respect to the perversity $\overline{p}$ and with $\mathcal{F} = Or_{U_2}$ (the $\mathbb{R}$-orientation sheaf of U_2). Then $\mathcal{S}^{\cdot}$ is CLC, and quasisomorphic to a perverse sheaf (with perversity $\overline{p}$) with $\mathbb{R}$-coefficients. In particular it is quasiisomorphic to the complex of $\mathbb{R}$-intersection cochains $\mathcal{IC}^{\cdot}$ defined earlier.

Proof: See Theorem 3.5, [GM2]. □

References

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