Forcing

Relative Consistency of $ZFC + \neg CH$

S. M. Srivastava

INDIAN STATISTICAL INSTITUTE, Kolkata

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Calcutta Logic Circle.

Introduction.

Gödel (1939) gave an interpretation of $ZFC+GCH$ in an extension by definitions of ZF. This gives us the following theorem.

Theorem. *If ZF is consistent, so is $ZFC + GCH$.*

Moreover the corresponding class of this interpretation, denoted by $\mathbf{L}$ and called *the universe of constructible sets*, is proper (and not a set).

Even Gödel failed to prove that if ZF is consistent, so is $ZF + \neg AC$ or $ZFC + \neg CH$. What were the difficulties?

Gödel has showed that ZF can't prove its own consistency (the, so-called, second incompleteness theorem). In other words, we can't have a set model of ZF. In particular, we can't have a set model of ZF in which AC or CH, etc., fails. On the other hand, Gödel also showed that $\mathbf{L}$ is the minimal inner model of ZF, i.e.,

Theorem. *If $\mathbf{M}$ is a transitive proper class model of ZF containing all ordinals, then $\mathbf{L} \subset \mathbf{M}$.*

Definition. A class or a set $\mathbf{M}$ is called *transitive* if

$$\forall x (x \in \mathbf{M} \to x \subset \mathbf{M}).$$

Suppose there is a transitive proper class model $\mathbf{M}$ of ZF in which $\neg AC$ or $\neg CH$ holds. Since AC and GCH hold in $\mathbf{L}$, by the above theorem $\mathbf{L} \neq \mathbf{M}$. Since $\mathbf{V} = \mathbf{L}$ (the so-called *axiom of constructibility*, $\mathbf{V}$ the universe of sets) holds in $\mathbf{L}$, we have shown that $ZF \vdash \mathbf{V} \neq \mathbf{L}$. This is impossible since $\mathbf{V} = \mathbf{L}$ is consistent with ZF.

So, we can neither build a set model nor a proper class model of ZF in which $\neg AC$ or $\neg CH$ holds!!

The compactness theorem comes as a saviour. Recall that a theory T is consistent if and only if all its finite parts are consistent. So it is sufficient to prove that for any finite set P of axioms of ZFC, $P + \neg CH$ is consistent. The following theorem becomes quite useful now.

Theorem. (ZFC) *For every finite set of axioms P of ZFC, there is a countable, transitive set M that models P.*

In this note, we shall present a technique called *forcing*, invented by Paul Cohen (1963), to show that if ZF is consistent, so is $ZF + \neg CH$ or $ZF+$ some other hypopthesis, usually implying $\mathbf{V} \neq \mathbf{L}$. Since AC is relatively consistent, it is sufficient to prove that if ZFC is consistent, so is $ZFC + \neg CH$.

The proof of the relative consistency of $\neg AC$ is harder. We shall illustrate the technique by showing the relative consistency of mainly $\neg CH$.

One begins with a countable transitive set model M of a suitable finite part of ZFC and constructs a countable transitive set $N \supset M$ that models $P + \neg CH$, say. It is not necessary to specify in advance the finite part of ZFC which models M. For entire argument to go through, one will anyway use only finitely many axioms of ZFC. So, it will be assumed that M is a countable transitive model of a suitable finite part of ZFC to make entire argument go through.

In the rest of this note, the statement "M is a countable transitive model of ZFC" will mean that "M is a countable transitive model of a suitable finite part of ZFC to make entire argument go through".

Forcing: A curtain raiser.

Collapsing cardinals.

We begin with an example—probably a shocking example. Let X be any set. We now attempt to show that "X *is countable.*" We want to show that there is a surjection $f : \omega \to X$, where $\omega = \{0, 1, 2, \ldots\}$. Set

$$\mathbb{P} = X^{<\omega},$$

the set of all functions whose domain is a finite subset of ω (including the empty subset) and whose range is contained in X. We define a partial order $\leq$ on $\mathbb{P}$ by

$$p \leq q \Leftrightarrow p \text{ extends } q,$$

$p, q \in \mathbb{P}$. Let 1 denotes the empty function (the function with empty domain). Note that 1 is the largest element (hence, a maximal element) of $\mathbb{P}$. Elements of $\mathbb{P}$ are called *conditions.*

Call p and q *compatible* if

$$\exists r (r \leq p \wedge r \leq q).$$

If p and q are not compatible, we shall call them *incompatible* and write $p \perp q$.

A subset D of $\mathbb{P}$ is called *dense* in $\mathbb{P}$ if

$$\forall p \exists q (q \in D \wedge q \leq p).$$

For any $n \in \omega$, the set

$$D_n = \{p \in \mathbb{P} : n \in \text{ domain}(p)\}$$

is dense in $\mathbb{P}$ (provided, of course, X is non-empty). Similarly, for each $x \in X$,

$$D_x = \{p \in \mathbb{P} : x \in \text{ range}(p)\}$$

is dense.

Definition. A non-empty subset G of $\mathbb{P}$ is called a *filter* if

(1) $\forall p \forall q((p \in G \wedge p \leq q) \rightarrow q \in G)$, i.e., G is *closed upwards.*

(2) Any two elements of G are compatible.

Definition. A filter G on $\mathbb{P}$ is called $\mathbb{P}$-*generic* if for every dense set D, $G \cap D \neq \emptyset$.

If possible, suppose $\mathbb{P}$ contains a generic filter G. Then $\cup G$ is a map from from ω onto X. So, in such a case, X is countable.

If X is uncountable, a generic filter does not exist. We are about to present the first idea for forcing. We make an observation first.

Lemma. *Suppose the set of all dense sets is countable. Then a* $\mathbb{P}$*-generic filter* G *exists.*

Proof. Let $\{D_n\}$ be an enumeration of all the dense sets in $\mathbb{P}$. We define a sequence $\{p_n\}$ in $\mathbb{P}$ by induction as follows. Choose any $p_0 \in D_0$. Having chosen $p_n \in D_n$, choose a $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Since D_{n+1} is dense, such a p_{n+1} exists. Note that p_n's are pairwise compatible. Now set

$$G = \{p \in \mathbb{P} : \exists n(p_n \leq p)\}.$$

Let M be a countable transitive model of ZFC and $X \in M$. Then $(\mathbb{P}, \leq, 1) \in M$. Suppose X is not countable in M. This means that there is no (partial) function f from ω onto X that belongs to M. But there is a possibility that such a function exists in the universe (of sets). Cohen gave an ingenious method to build a countable transitive model $N \supset M$ of ZFC that contains a generic G. (Since N is a model of ZFC, this implies that $\cup G \in N$, and so,

X is countable in N.) More precisely, Cohen showed the following.

Let M be a countable transitive model of a suitable finite part of ZFC and $X \in M$ uncountable $\underline{\text{in } M}$. By choosing the finite set of axioms suitably, we can ensure that $(\mathbb{P}, \leq, 1) \in M$. Since M is countable, the set of all dense sets that belong to M is countable (not in M but in the universe). Hence, there is a $\mathbb{P}$-generic filter G (over M) in the universe. Cohen's method of forcing gives us a countable transitive set $M[G] \supset M$ that is a model of a desired finite part of ZFC containing G. Further, $M[G]$ and M will have the same ordinals. So, $\cup G \in M[G]$ and witnesses that X is countable in $M[G]$. This method of forcing is known as *collapsing cardinals*. In the above argument, replacing ω by any infinite $Y \in M$, we can get a model $M[G] \supset M$ in which $|X| \leq |Y|$.

Ordinals and Cardinals

Continum Hypothesis.

Definition. A set α is called an *ordinal number* if it is transitive and if $\in |\alpha$ is linear. We shall use **ON** to denote the class of all ordinal numbers.

The class of all ordinal numbers are "well-ordered" by $\in$. For ordinals α and β, we shall write $\alpha < \beta$ if $\alpha \in \beta$. We have the following facts:

(1) Exactly one of the following hold:

$$\alpha < \beta \text{ or } \alpha = \beta \text{ or } \beta < \alpha.$$

(2) If A is a non-empty class of ordinals, then there is a unique $\alpha \in A$ such that for every $\beta \in A$, $\alpha \leq \beta$.

(3) $\alpha < \beta$ if and only there is a one-to-one order-preserving map from α onto an initial segment of β.

Definition. A *cardinal number* is an ordinal number κ such that for every ordinal $\alpha < \kappa$, $|\alpha| < |\kappa|$.

Note that the class of all cardinal numbers are themselves well-ordered by $<$. We enumerate the class of all infinite cardinals by $\aleph_0 < \aleph_1 < \aleph_2 < \ldots,$.

We are in a position to explain the collapse of cardinals now. Let M be any transitive model. $o(M)$ will denote the set of all ordinal numbers in M and $\aleph_0^M < \aleph_1^M < \aleph_2^M < \ldots$, the enumeration of all infinite cardinals $\underline{\text{in } M}$. Given any cardinal $\aleph_\alpha^M$ in M, using the forcing described above we can get a model $M[G] \supset M$ in which $\aleph_\alpha^M$ is countable.

In order to give a model in which CH fails, naturally we should know all the cardinals in that model.

Definition. If $\mathbb{P} \in M$, we say that $\mathbb{P}$ *preserves cardinalities* if for every $\mathbb{P}$-generic filter G over M,

$$\forall \beta \in o(M)((\beta \text{ is a cardinal})^M) \Leftrightarrow (\beta \text{ is a cardinal})^{M[G]}),$$

where a filter G on $\mathbb{P}$ is called generic over M, if $G \cap D \neq \emptyset$ for every dense set D that belongs to M.

Definition. Let $\mathbb{P} \in M$. Then $(\mathbb{P} \text{ has c.c.c.})^M$ if every set $A \subset \mathbb{P}$ belonging to M consisting of pairwise incompatible elements is countable $\underline{\text{in } M}$.

Theorem. *Let $\mathbb{P} \in M$ and $(\mathbb{P}$ has c.c.c.$)^M$. Then $\mathbb{P}$ preserves cardinalities.*

Note that if $\mathbb{P}$ preserves cardinalities, then for every generic extension $M[G]$ of M and for every $\alpha \in o(M)$,

$$\aleph_\alpha^M = \aleph_\alpha^{M[G]}.$$

Continuum Hypothesis (CH). $2^{\aleph_0} = \aleph_1$.

Generalised Continuum Hypothesis (GCH). For every ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

What is required to give a model of $\neg CH$?

First note that $\aleph_0$ is the first infinite cardinal, $\aleph_1$ is the first uncountable cardinal, $\aleph_2$ is the second uncountable cardinal, and so on. But we have already seen that a set may be uncountable in a model M but countable in a larger model N. Similarly, an ordinal may be $\aleph_1$ in a model M but may be countable in a larger model N, or an ordinal may be $\aleph_2$ in a model M but may be $\leq \aleph_1$ in a larger model N, and so on. So, a model N satisfies $\neg CH$ if "<u>in N</u>, there is a surjection

$$f : \mathcal{P}^N(\aleph_0) \to (\aleph_2)^N,$$

where $\mathcal{P}^N(\aleph_0)$ stands for all subsets of $\aleph_0$ that are in N and $(\aleph_2)^N$ is the second uncountable cardinal in N."

So, to show the relative consistency of $\neg CH$, one begin with a suitable countable transitive model M, a suitable $\mathbb{P} \in M$ such that $(\mathbb{P}$ has c.c.c.$)^M$. Then M and $M[G]$, G $\mathbb{P}$-generic in M, have the same cardinal numbers. Now $\mathbb{P}$ is so chosen that in every $M[G]$, $|\mathcal{P}(\aleph_0)| > \aleph_2$.

Absoluteness.

We have seen that a cardinal in a model may not be so in a larger model; If $N \supset M$ and $x \in M$, $\mathcal{P}(x)^M$ may be a proper subset of $\mathcal{P}(x)^N$ because the larger model N may contain a subset of x that does not belong to M. On the other hand, if M and N are transitive and if $x \in M$ is an ordinal in M, it remains so in N. This brings us to the very important notion of absoluteness.

Definition. Let $\mathbf{M}$ be a class and $\varphi[x_1, \cdots, x_n]$ a formula of ZF. The *relativization* of φ to $\mathbf{M}$, denoted by $\varphi^{\mathbf{M}}$, is defined by induction on the length of φ and by replacing every subformula of the form $\exists x \psi$ by $\exists x(x \in \mathbf{M} \wedge \psi^{\mathbf{M}})$.

Definition. A class $\mathbf{M}$ is called a *model* of ZF (or ZFC or $ZFC + GCH$) if $ZF \vdash \varphi^{\mathbf{M}}$ for every axiom φ of ZF (or ZFC or $ZFC + GCH$).

Definition. A formula $\varphi[x_1, \cdots, x_n]$ is called *absolute* for a class $\mathbf{M}$ if

$$ZF \vdash \forall x_1, \cdots, x_n \in \mathbf{M}(\varphi[x_1, \cdots, x_n] \leftrightarrow \varphi^{\mathbf{M}}[x_1, \cdots, x_n]).$$

A "function," e.g., $x \to \cup x$ or $(x, y) \to x \times y$ or $(x, y) \to \{x, y\}$ or a "relation," e.g., $x \subset y$ or x is a function etc., is called absolute with respect to $\mathbf{M}$ if the corresponding graph is so.

The next few results are quite useful in proving absoluteness.

Definition. The smallest set of formulas of ZF containing all atomic formulas and closed under $\vee$, $\neg$ and *bounded quantification* $\exists x \in y \cdots$, defined by $\exists x(x \in y \wedge \cdots)$, is denoted by Δ_0.

Note that if φ is a Δ_0-formula, so is $\forall x \in y\varphi$.

Theorem. Every Δ_0 formula is absolute for every transitive class $\mathbf{M}$.

Theorem *If* $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ *is a "function" such that its graph*

$$\{(x, y) \in \mathbf{V} \times \mathbf{V} : y = F(x)\}$$

is a class, then there is a unique "function" $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ *defined by*

$$\forall \alpha \in \mathbf{ON}(G(\alpha) = F(\mathbf{G}|\alpha))$$

such that

$$\{(\alpha, \mathbf{G}(\alpha)) : \alpha \in \mathbf{ON}\}$$

is a class. ($G|\alpha$ denotes the "restriction" of $\mathbf{G}$ *on* $\alpha = \{\beta \in \mathbf{ON} : \beta < \alpha\}$*.) Moreover, if* $\mathbf{F}$ *is absolute for a transitive model* $\mathbf{M}$ *of* ZF*, so is* $\mathbf{G}$*.*

The next result uses the foundation axiom.

Theorem. *If* $\mathbf{F} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ *is a "function" whose "graph" is a class, then there is a unique* $\mathbf{G} : \mathbf{A} \to \mathbf{V}$ *defined by*

$$\forall x \in \mathbf{A}[\mathbf{G}(x) = \mathbf{F}(x, \mathbf{G}|(\{y \in \mathbf{A} : y \in x\}))]$$

whose "graph" is a class. Moreover, if $\mathbf{F}$ *and* $\mathbf{A}$ *are absolute for a transitive model* $\mathbf{M}$ *of* ZF*, so is* $\mathbf{G}$*.*

Remark. Most of the commonly used functions, relations and constants (**with notable exceptions of power set** $x \to \mathcal{P}(x)$ **and cardinals**) are absolute with respect to every transitive model of ZF. This actually means that there is a finite set of axioms $\varphi_1, \cdots, \varphi_n$ of ZF such that for every transitive model $\mathbf{M}$ of $\varphi_1, \cdots, \varphi_n$ the concerned function (or relation or constant) is absolute with respect to $\mathbf{M}$. Readers are referred to Kunen's book for this.

A model in which CH fails.

In this section, we present (without proof) a partially ordered set $\mathbb{P}$ such that forcing with $\mathbb{P}$ yields a model in which CH fails.

Lemma. *Let I be an arbitrary set and J countable. Let*

$$\mathbb{P} = Pfn^{<\omega}(I, J)$$

denote the set of all partial functions with domain a finite subset of I and range contained in J. For $p, q \in \mathbb{P}$, set

$$p \leq q \leftrightarrow p \text{ extends } q.$$

Then $\mathbb{P}$ is c.c.c.

Since one uses only finitely many axioms of ZFC to prove the above lemma, we have the following result.

Lemma. *There is a finite set P of axioms of ZFC such that for every countable transitive model M of P we have the following: Let, in M, I be an arbitrary set and J countable. Let*

$$\mathbb{P} = Pfn^{<\omega}(I, J)$$

denote the set of all partial functions with domain a finite subset of I and range contained in J. For $p, q \in \mathbb{P}$, set

$$p \leq q \leftrightarrow p \text{ extends } q.$$

Then $(\mathbb{P}, \leq, 1) \in M$ and $(\mathbb{P}$ is c.c.c.$)^M$.

Theorem. *Let M be a countable transitive model of ZFC. Let κ be an uncountable cardinal of M and*

$$\mathbb{P} = Pfn^{<\omega}(\kappa \times \aleph_0, 2).$$

Then for every $\mathbb{P}$-generic filter G over M,

$$(2^{\aleph_0} \geq \kappa)^{M[G]}.$$

Proof. Since $M[G]$ is a model of ZFC and $G \in M[G]$, the function

$$f = \cup G : \kappa \times \aleph_0 \longrightarrow 2$$

belongs to $M[G]$. Now define $F : \kappa \to \mathcal{P}(\aleph_0)$ as follows: for any $\alpha < \kappa$,

$$F(\alpha) = \{n < \aleph_0 : f(\alpha, n) = 1\}.$$

We show that F is one-to-one to complete the proof. Let $\alpha < \beta < \kappa$. Consider

$$D_{\alpha\beta} = \{p \in \mathbb{P} : \exists n(p(\alpha, n) = 0 \wedge p(\beta, n) = 1)\}.$$

Since M is a model, by absoluteness, $D_{\alpha\beta} \in M$. It is easily seen that $D_{\alpha\beta}$ is dense in $\mathbb{P}$. Since G is generic, $G \cap D_{\alpha\beta} \neq \emptyset$. So, $n \in F(\beta)$ but $n \notin F(\alpha)$.

Definition of $M[G]$

This is the stage at which we require some patience. We are going to present some very technical definitions. Once these concepts are understood and certain facts about them accepted, the use of forcing becomes relatively pleasant and easy.

At this stage, it would be helpful to keep in mind that based on the Zermelo-Fraenkel axioms, the universe of sets $\mathbf{V}$ has the following properties: First each set ($x \in \mathbf{V}$) is a "hereditary set" in the sense that members of any $x \in \mathbf{V}$ is again a member of $\mathbf{V}$. Secondly, *the foundation axiom* says

$$\forall x(x \neq \emptyset \to \exists y(y \in x \wedge \forall z(z \in x \to (z = y \vee z \notin y))))$$

i.e., $\in$ is **well-founded** on every $x \in \mathbf{V}$. We can now picture $\mathbf{V}$ as follows: By tranfinite induction, define

$$V(0) = \emptyset,$$

$$V(\alpha + 1) = \mathcal{P}(V(\alpha))$$

and for limit λ,

$$V(\lambda) = \cup_{\alpha<\lambda} V(\alpha).$$

One can show (in ZF) that

$$\mathbf{V} = \cup_{\alpha} V(\alpha),$$

i.e.,

$$ZF \vdash \forall x \exists \alpha (x \in V(\alpha)).$$

For any $x \in \mathbf{V}$, we define rank(x) to be the first α such that $x \in V(\alpha + 1)$. Thus rank$(\emptyset) = 0$ and rank$(y) <$ rank(x) if $y \in x$. The notion of rank enables us to do induction on the class of sets.

$\mathbb{P}$-names.

Let M be a countable transitive model of ZFC, and $(\mathbb{P}, \leq, 1) \in M$.

Definition. τ is a $\mathbb{P}$-*name* iff τ is a relation and

$$\forall \langle \sigma, p \rangle \in \tau [\sigma \text{ is a } \mathbb{P} - \text{name} \wedge p \in \mathbb{P}].$$

$\mathbf{V}^{\mathbb{P}}$ will denote the class of all $\mathbb{P}$-names. More precisely, $\mathbf{V}^{\mathbb{P}}[\tau]$ is the class defined by the formula

$$relation(\tau) \wedge \forall (\sigma, p) \in \tau [\mathbf{V}^{\mathbb{P}}[\sigma] \wedge p \in \mathbb{P}].$$

As an example note that $\emptyset$ is a $\mathbb{P}$-name. If σ, μ are $\mathbb{P}$-names, so is $\{\langle \sigma, 1 \rangle, \langle \mu, 1 \rangle\}$. Further, for any set $x \in \mathbf{V}$, we define a $\mathbb{P}$-name $\check{x}$, called *the canonical name* of x, by induction on rank(x) as follows:

$$\check{x} = \{\langle \check{y}, 1 \rangle : y \in x\}.$$

Since M is a model, $\check{x} \in M$ whenever $x \in M$. In fact, the restriction of $x \to \check{x}$ to any set in M is in M. We can easily check that $\check{\emptyset} = \emptyset$ and $\{\emptyset\}\check{} = \{\langle \check{\emptyset}, 1 \rangle\}$. Another important example of a $\mathbb{P}$-name is

$$\Gamma = \{\langle \check{p}, p \rangle : p \in \mathbb{P}\}.$$

By the remark made above $\Gamma \in M$.

Value of $\mathbb{P}$-names in a generic extension.

Definition. Let G be a $\mathbb{P}$-generic filter over M. For each $\mathbb{P}$-name τ, we define *the value of* τ, and denote it by τ_G, by induction as follows:

$$\tau_G = \{\sigma_G : \exists p \in G(\langle \sigma, p\rangle \in \tau)\}.$$

It is easy to check that $\{\langle \sigma, 1\rangle, \langle \mu, 1\rangle\}_G = \{\sigma_G, \mu_G\}$. Further, for all sets x, $\check{x}_G = x$ and $\Gamma_G = G$.

We set $M^{\mathbb{P}} = \mathbf{V}^{\mathbb{P}} \cap M$. So, $M^{\mathbb{P}}$ is the set of all $\mathbb{P}$-names that belong to M. Finally, we define

$$M[G] = \{\sigma_G : \sigma \in M^{\mathbb{P}}\}.$$

Any model obtained by forcing is of the form $M[G]$ and is called a *generic extension* of M (obtained by forcing with $\mathbb{P}$). We first record some properties of $M[G]$ that follow directly from the definition. For instance $M[G]$ is countable since M is. $M[G]$ is transitive. Since $\check{x} \in M$ whenever $x \in M$ and since $\check{x}_G = x$, $M \subset M[G]$. Since $\Gamma \in M$ and $\Gamma_G = G$, $G \in M[G]$. Further, $o(M) = o(M[G])$.

The difficulty arises when we try to determine whether a particular statement is true in $M[G]$ or not. The only information that we have is that of the truth in M. Note that $M[G]$ is described in terms of G which, most often, does not belong to M. Thus, the people residing in M do not know precisely which sets belong to $M[G]$. But they have a name for each element in $M[G]$. We can draw a parallel. We prove results on polynomial rings $k[X_1, \cdots, X_n]$ by looking at just the coefficients of polynomials, which belong to the field k.

Suppose we want to see when does $M[G]$ satisfy the pairing axiom. This amounts to showing that whenever σ_G, μ_G belong to $M[G]$, there is a set in $M[G]$ containing both of these. Now suppose M satisfies the pairing axiom and some more axioms including comprehension. Then clearly $\{\langle \sigma, 1\rangle, \langle \mu, 1\rangle\} \in M^{\mathbb{P}}$. But $\{\langle \sigma, 1\rangle, \langle \mu, 1\rangle\}_G = \{\sigma_G, \mu_G\}$. So, by definition, $\{\sigma_G, \mu_G\} \in M[G]$.

Unfortunately, most often, it is fairly hard to show that a certain statement is true in $M[G]$. Now we have arrived at a stage to describe the deepest observation of Cohen giving a very powerful and a very successful technique to connect the truth in $M[G]$ with that in M.

The Forcing Notion

Theoughout this section we fix a countable transitive model M of ZFC, a partially ordered set $(\mathbb{P}, \leq, 1)$ in M, a $p \in \mathbb{P}$, a formula $\varphi[x_1, \cdots, x_n]$ of ZFC, and $\mathbb{P}$-names σ_1, $\cdots$, σ_n in M. The following is the single most important definition in our topic.

Definition We say that $\mathbb{P}$ *forces* $\varphi[\sigma_1, \cdots, \sigma_n]$, and write $p \Vdash \varphi[\sigma_1, \cdots, \sigma_n]$, if $\varphi[(\sigma_1)_G, \cdots, (\sigma_n)_G]$ is true in all those generic extensions $M[G]$ for which $p \in G$.

We make some very simple observations first. Since 1 belongs to every filter G, if 1 forces $\varphi[\sigma_1, \cdots, \sigma_n]$, then $\varphi[(\sigma_1)_G, \cdots, (\sigma_n)_G]$ is true in all generic extensions $M[G]$ of M.

We are making some obvious notational simplification in the following easy observations.

1. If $q \Vdash \varphi$ and $p \leq q$, then $p \Vdash \varphi$
2. $p \Vdash \varphi$ & $p \Vdash \psi$ iff $p \Vdash \varphi \wedge \psi$.

The observation 1. says that if a fact φ is forced by some condidtion or information, it is also forced by a condidtion $\mathbb{P}$ which has more information than q. We refer the reader to the book of Kunen ([K]) for further details. There is precisely one result, but a very deep result, that we need to know now. In the result stated below we are following the notation that we have been using in this section.

The first observation below is the formalization of "*truth in $M[G]$ can be decided in M*." The main contention of the second statement is that any true statement in $M[G]$ must be forced by some condition $p \in G$.

Forcing Lemma

(a) *To each $p \in \mathbb{P}$, $\varphi[x_1, \cdots, x_n]$ and $\mathbb{P}$-names $\sigma_1, \cdots, \sigma_n$ we can associate a formula $p \Vdash^* \varphi[\sigma_1, \cdots, \sigma_n]$ such that*

$$p \Vdash \varphi[\sigma_1, \cdots, \sigma_n] \Leftrightarrow p \Vdash^* \varphi[\sigma_1, \cdots, \sigma_n] \text{ is true in } M.$$

(b) For all $\mathbb{P}$-generic G over M

$$(\varphi[(\sigma_1)_G, \cdots, (\sigma_n)_G])^{M[G]} \Leftrightarrow \exists p \in G[p \Vdash \varphi[\sigma_1, \cdots, \sigma_n]].$$

Interesting part of the forcing lemma is that we never need to know what the formula $p\Vdash^* \varphi[\sigma_1, \cdots, \sigma_n]$ is. It is sufficient to know that $p \Vdash \varphi[\sigma_1, \cdots, \sigma_n]$ can be decided in M in the sense of part (a) of the Forcing Lemma. To verify $\varphi[(\sigma_1)_G, \cdots, (\sigma_n)_G]$ is true in $M[G]$ it is more convenient to use part (b) of the forcing lemma. Thus almost entire proof uses very little metamathematics and is almost like proving any other mathematical theorem.

Using forcing lemma, we can prove, rather easily, that for every axiom φ of ZFC, there exists finitely many axioms $\psi_1, \cdots, \psi_n$ of ZFC such that if we begin with a countable transitive model M of $\psi_1, \cdots, \psi_n$, then $M[G]$ is a model of φ. This has already been shown by Cohen and so we can as well assume it.

Difficulties arise only to show that $M[G]$ also satisfies some other hypothesis such as $\neg CH$. If we accept the forcing lemma and the broad description of the concepts described above, the arguments involved are like usual mathematical arguments involving not too much of metamathematics. The real challenge lies in cooking up the right partially ordered set $\mathbb{P}$.

To sum up, **the forcing prescription** for showing the relative consistency of a statement φ is the following.

1. Take any countable transitive model M (called the *ground model*) of ZFC. (Since we shall only use the fact that M is a model of some finitely many axioms ZFC, there is no harm in pretending that M is a model of the entire ZFC.)
2. Depending on φ, cook up a suitable poset $\langle \mathbb{P}, \leq, 1 \rangle$ in M. (This is where the real challenge lies.)
3. Consider the countable set $M^{\mathbb{P}}$ of all $\mathbb{P}$-names in M.
4. Take any $\mathbb{P}$-generic filter G over M. Now consider the countable transitive set $M[G]$, called *the generic extension* of M. Pretend that $M[G]$ is a countable transitive model of ZFC satisfying
   (a) $G \in M[G]$.
   (b) $o(M) = o(M[G])$.
   (c) Further, if $(\mathbb{P}$ is c.c.c$)^M$, M and $M[G]$ have the same cardinals.

   Now using part (b) of the forcing lemma, check that φ is true in $M[G]$. To do this, we shall only use the fact that $M[G]$ is a countable transitive model of some finitely many axioms of ZFC which we can assume by beginning with a suitable M. Thus there is no harm in pretending that $M[G]$ is a transitive model of ZFC.

Reference

K. Kunnen, Set Theory, North-Holland Publishing Company.