ELLIPTIC COMPLEXES AND INDEX THEORY

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1. Sobolev Theory on $\mathbb{R}^n$

As general references for this section, see the books [Nar], Ch.3, [Gil], Ch.1, [Hor], Ch I, II and [Rud], Ch. 6,7.

1.1. **Test functions and distributions.** We introduce some standard notation. For a multi-index $\alpha = (\alpha_1, .., \alpha_n)$ of length n, the symbol $\mid \alpha \mid := \sum_i \alpha_i$, and $\alpha! := \alpha_1!, .., \alpha_n!$.

For derivatives, we denote:

$$d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ...... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \partial_j = \left(\frac{\partial}{\partial x_j}\right), \quad D_j = \frac{1}{\sqrt{-1}}\partial_j$$

Finally $x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ will be denoted simply by x^α.

We can define some standard function spaces on $\mathbb{R}^n$. For us, functions will always be *complex valued*.

Definition 1.1.1 (Standard function spaces on $\mathbb{R}^n$)**.** . The function spaces defined below are all complex vector spaces, and to define a topology on them, it is enough to define convergence to zero.

(i):

$$C^\infty(\mathbb{R}^n) = \{\text{smooth functions on } \mathbb{R}^n\}$$

Define $f_n \to 0$ if $d_x^\alpha f_n \to 0$ uniformly on compact sets for all $\mid \alpha \mid \geq 0$. This space is also sometimes denoted $\mathcal{E}$ by analysts.

(ii):

$$C_0^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \lim_{x \to \infty} d_x^\alpha f = 0 \text{ for all } \mid \alpha \mid \geq 0\}$$

This is the space of all smooth functions whose derivatives of all orders vanish at infinity. The topology in this space is the subspace topology from $C^\infty(\mathbb{R}^n)$. It is often denoted $\mathcal{E}_0$.

(iii):

$$C_c^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{ support} f \text{ is compact}\}$$

Its topology is defined by $f_n \to 0$ if there exists a compact set K such that supp$f_n \subset K$ for all n and $d_x^\alpha f_n \to 0$ uniformly on K for all $\mid \alpha \mid \geq 0$. Note that this is *not the subspace topology* from $C^\infty(\mathbb{R}^n)$, for if we define a take a non-zero function ψ on $\mathbb{R}$ with compact support $[-1, 1]$ say, and let $f_n(x) = \psi(x - n)$, (which has support $[n - 1, n + 1]$), then $f_n \to 0$ in $C^\infty(\mathbb{R})$, but not in $C_c^\infty(\mathbb{R})$. It is, in fact easily seen to be strictly finer than the subspace topology. This space is denoted $\mathcal{D}$ by analysts, and also called the space of test functions.

(iv):

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \mid x^\alpha d_x^\beta f(x) \mid \leq C_{\alpha\beta} \text{ for all} \mid \alpha \mid, \mid \beta \mid \geq 0\}$$

This is called the *Schwartz space* of *rapidly decreasing functions.* Define the topology by declaring $f_n \to 0$ if

$$\sup_{x \in \mathbb{R}^n} \mid x^\alpha d_x^\beta f_n(x) \mid \to 0$$

for each $\mid \alpha \mid, \mid \beta \mid \geq 0$.

It is an easy exercise to see that there are natural inclusions:

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C_0^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

all of which are continuous, and all of which are strict. The reader may also check that the inclusion $C_c^\infty \subset C^\infty$ is dense, (by using multiplication with cutoff functions ϕ_n which are identically 1 on a ball of radius n and identically zero outside a ball of radius say $2n$), and hence all the inclusions above are dense.

On $\mathcal{S}$ or C_c^∞, we may introduce the L_p-norm defined by:

$$\|f\|_{L_p} := \left(\int_{\mathbb{R}^n} \mid f \mid^p dx\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Upon completing either of these two spaces with respect to this norm, one gets the Banach space $L_p(\mathbb{R}^n)$. For $p = \infty$, this is false, as can be seen by looking at the non-zero constant functions. $L_\infty(\mathbb{R}^n)$ is got by taking all measurable functions on $\mathbb{R}^n$ which are essentially bounded.

1.2. **The Fourier Transform and Plancherel Theorem.** In the sequel we will simply write C^∞ for $C^\infty(\mathbb{R}^n)$, and so on, if no confusion is likely. Also, to eliminate annoying powers of 2π, we introduce the measure (volume element) dx on $\mathbb{R}^n$ by the formula:

$$dx := (2\pi)^{-n/2} dx_1 ... dx_n$$

Definition 1.2.1. For $f \in \mathcal{S}$, define the *Fourier Transform of* f by the formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-i\xi . x} dx$$

which makes sense for any $f \in L_1$, and in particular for $f \in \mathcal{S}$. Here $\xi . x = \sum_i \xi_i x_i$ is the usual Euclidean inner product of vectors in $\mathbb{R}^n$. Similarly, for $f \in \mathcal{S}$, define the Inverse Fourier Transform of f by the formula:

$$f^\vee(\xi) := \widehat{f}(-\xi) = \int_{\mathbb{R}^n} f(x) e^{i\xi . x} dx$$

Finally, for $f, g \in \mathcal{S}$, define the *convolution product*

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y) dy = \int_{\mathbb{R}^n} f(-z) g(z+x) dz$$

It is easy to verify (taking limits inside the integral sign after appealing to Lebesgue's Dominated Convergence Theorem) that $f * g$ is also in $\mathcal{S}$, and that $f * g = g * f$.

Before proving the main proposition of this section, we need a couple of useful lemmas. Note that the Gaussian function

$$\psi(x) = e^{-\frac{|x|^2}{2}}$$

is in $\mathcal{S}$. Also its integral $\int_{\mathbb{R}^n} \psi(x) dx = 1$.

Lemma 1.2.2. For the Gaussian ψ above, we have $\widehat{\psi} = \psi$.

Proof: We have:

$$\begin{aligned} \widehat{\psi}(\xi) &= \int e^{-i\xi . x} e^{-\frac{|x|^2}{2}} dx \\ &= e^{-\frac{|\xi|^2}{2}} \int e^{\frac{(x+i\xi).(x+i\xi)}{2}} dx \end{aligned}$$

Let $x = (x_1, .., x_n)$ and $\xi = (\xi_1, .., \xi_n)$. By choosing a rectangular contour in $\mathbb{C}$ with vertices $-a, a, -a + i\xi_1, a + i\xi_1$, noting that the integral of the holomorphic function $e^{-z_1^2/2}$ around this contour is zero, and also that the contributions along the vertical edges $(-a + it)$ and $(a + it)$ for $0 \leq t \leq \xi_1$ (if $\xi_1 \geq 0$) (resp. $\xi_1 \leq t \leq 0$ if $\xi \leq 0$) converge to zero as $a_1 \to \infty$, we see that:

$$\int_{\mathbb{R}} e^{-(x_1 + i\xi_1)^2/2} dx_1 = \int_{\mathbb{R}} e^{-x_1^2/2} dx_1 = \sqrt{2\pi}$$

apply the argument variable by variable to conclude that:

$$\int_{\mathbb{R}^n} e^{-(x+i\xi).(x+i\xi)/2} dx = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} ... \int_{\mathbb{R}} e^{-\sum_j (x_j+i\xi_j)^2/2} dx_1 ... dx_n = 1$$

which proves our assertion. □

Lemma 1.2.3 (Approximate identities)**.** Let $\phi \in \mathcal{S}$, such that $\phi(x) \geq 0$ for all x and $\int \phi(x)dx = 1$. For $\epsilon > 0$, define the *approximate identity* or *mollifier*:

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$$

Then for any $f \in \mathcal{S}$, we have $f * \phi_\epsilon$ converges uniformly to f as $\epsilon \to 0$.

Proof: Since $\int_{\mathbb{R}^n} \phi_\epsilon dx = 1$ for each $\epsilon > 0$, we have:

$$\begin{aligned} \mid (f * \phi_\epsilon)(x) - f(x) \mid &= \left| \int \phi_\epsilon(y)\,(f(x-y) - f(x))\,dy \right| \\ &\leq \int \mid \phi_\epsilon(y)\,(f(x-y) - f(x)) \mid dy \end{aligned}$$

Let $C > 0$ be such that $\int \mid f(x) \mid dx \leq C$. Now let $\eta > 0$ be any positive number. Choose a $\delta > 0$ (by uniform continuity of f) such that $\mid f(x-y) - f(x) \mid \leq \eta$ for all $\mid y \mid \leq \delta$, and all x. Then

$$\int_{|y|\leq\delta} \mid \phi_\epsilon(y)\,(f(x-y) - f(x)) \mid dy \leq \eta \int_{|y|\leq\delta} \phi_\epsilon(y)dy \leq \eta \int_{\mathbb{R}^n} \phi_\epsilon(y)dy = \eta$$

for all $\epsilon > 0$.

Now choose an $\epsilon_0 > 0$ small enough so that $\int_{|y|>\delta} \phi_\epsilon(y)dy < \eta/2C$ for $\epsilon \leq \epsilon_0$. Then, we have:

$$\int_{|y|>\delta} \mid \phi_\epsilon(y)\,(f(x-y) - f(x)) \mid dy \leq \int_{|y|>\delta} \phi_\epsilon(y)(2C)dy \leq \eta$$

for $\epsilon < \epsilon_0$. Combining the integrals for $|y| \leq \delta$ and $|y| > \delta$, we get:

$$\mid f * \phi_\epsilon(x) - f(x) \mid \leq 2\eta \quad \text{for all} \quad \epsilon < \epsilon_0$$

independent of x. That is, $f * \phi_\epsilon \to f$ uniformly as $\epsilon \to 0$. □

Remark 1.2.4. It follows from the above that one can take any non-negative compactly supported function ϕ and define the approximate identities ϕ_ϵ. Similarly, by starting with the Gaussian ψ defined above, we get that the functions:

$$\psi_\epsilon(x) = \epsilon^{-n} e^{-|x|^2/2\epsilon^2}$$

are approximate identities.

Proposition 1.2.5. We have the following facts about the Fourier transform on the Schwartz class $\mathcal{S}$.

(i): The map $f \mapsto \widehat{f}$ is an isomorphism of $\mathcal{S}$ with itself, of order 4. In fact,

$$\left(\widehat{f}\right)^{\wedge}(x) = f(-x), \quad \left(\widehat{f}\right)^{\vee}(x) = f(x) \quad \text{for all} \quad f \in \mathcal{S}$$

(The second formula is called the *Fourier Inversion Formula*.)

(ii): For all multi-indices α,

$$\left(D_x^\alpha f\right)^\wedge(\xi) = \xi^\alpha \widehat{f}; \quad D_\xi^\alpha \widehat{f}(\xi) = (-1)^{|\alpha|}\left(x^\alpha f\right)^\wedge$$

In particular, by the first formula, if P is an n-variable polynomial with complex coefficients, then for the constant coefficient differential operator $P(D)$ we have:

$$(P(D)f)^\wedge = P(\xi)\widehat{f}$$

(iii):

$$\widehat{f}\,\widehat{g} = (f*g)^{\wedge}\,, \quad \widehat{f}*\widehat{g} = (fg)^{\wedge}$$

(iv): (*Plancherel Theorem*) The map $f \mapsto \widehat{f}$ is a unitary isomorphism of $\mathcal{S}$ to itself with respect to L_2-norm. Thus (in view of (i) above), it extends to a unitary isomorphism of $L_2(\mathbb{R}^n)$ to itself.

(v): (*Riemann-Lebesgue Lemma*) There is an inclusion:

$$(L_1(\mathbb{R}^n))^{\wedge} \subset C_0(\mathbb{R}^n)$$

where the space on the right is the space of all continuous functions vanishing at ∞, with the topology of uniform convergence on compact sets.

Proof: We first prove (ii). Since all derivatives of $f \in \mathcal{S}$ are also in $\mathcal{S}$, and hence Lebesgue integrable, one can use the Lebesgue Dominated Convergence Theorem, and differentiate under the integral sign to get:

$$\begin{aligned} D_\xi^\alpha \widehat{f}(\xi) &= \int_{\mathbb{R}^n} D_\xi^\alpha \left(e^{-i\xi.x} f(x)\right) dx \\ &= \int_{\mathbb{R}^n} (-1)^{|\alpha|} x^\alpha e^{-i\xi.x} f(x) dx = (-1)^{|\alpha|} \left(x^\alpha f\right)^{\wedge} \end{aligned}$$

In particular, we have that $\widehat{f}$ is a smooth function. This proves the second part of (ii). To prove the first part, one uses repeated integration by parts and the fact that all derivatives of f vanish at ∞ to conclude that:

$$\begin{aligned} (D_x^\alpha f)^\wedge = \int_{\mathbb{R}^n} e^{-i\xi.x} D_x^\alpha f(x) dx &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \left(D_x^\alpha e^{-i\xi.x}\right) f(x) dx \\ &= \xi^\alpha \widehat{f}(\xi) \end{aligned}$$

This proves (ii). Thus we also have $(P(D)f)^\wedge = P(\xi)\widehat{f}$. From (ii) it also follows that

$$\xi^\alpha D_\xi^\beta \widehat{f} = \pm \xi^\alpha (x^\beta f)^\wedge = \pm (D_x^\alpha (x^\beta f))^\wedge$$

and since the function on the right is bounded by the L_1 norm of $D_x^\alpha(x^\beta f)$ (a Schwartz class function), it follows that $\widehat{f} \in \mathcal{S}$ as well.

(v) is easy by noting that for an L^1 function f, we have: $\left\|\widehat{f}\right\|_\infty = \sup_{\xi\in\mathbb{R}^n} |\widehat{f}(\xi)| \leq \|f\|_1$, and that for any L_1 function f, there is a sequence of $f_n \in \mathcal{S}$ with $\|f - f_n\|_1 \to 0$. Which implies that $\widehat{f_n} \to \widehat{f}$ uniformly, so that since $\widehat{f_n} \in \mathcal{S}$, we have $\lim_{\xi\to\infty} \widehat{f}(\xi) = 0$. This proves (v).

To prove (i), define the operator $T : \mathcal{S} \to \mathcal{S}$ by $Tf(x) = (\widehat{f})^\wedge(-x)$. We need to show that $Tf(x) = f(x)$ for all x. First suppose $f \in \mathcal{S}$ with $f(0) = 0$. Then, by the first order Taylor formula we may write:

$$f(x) = \sum_{j=1}^n x_j g_j(x)$$

where g_j are some smooth functions. Let ϕ be a non-negative compactly supported function which is identically =1 in a neighbourhood of the origin. Then

$$f(x) = \phi(x)f(x) + (1-\phi(x))f(x) = \sum_j x_j \phi g_j + \sum_j x_j \left(\frac{x_j(1-\phi)f}{|x|^2}\right)$$

Since ϕ has compact support, the functions $\phi g_j \in \mathcal{S}$. On the other hand, since $\phi \equiv 1$ near the origin, the functions $\left(\frac{x_j(1-\phi)f}{|x|^2}\right) \in \mathcal{S}$ as well. Thus:

$$f = \sum_{j=1}^n x_j h_j$$

where $h_j \in \mathcal{S}$. However, by (ii) proved above, we have

$$\widehat{(x_j h_j)} = i\frac{\partial \widehat{h}_j}{\partial \xi_j}$$

so that $\widehat{f} = i\sum_j \frac{\partial \widehat{h}_j}{\partial \xi_j}$. Thus:

$$Tf(0) = (\widehat{f})^\wedge(0) = \int_{\mathbb{R}^n} \sum_j \frac{\partial \widehat{h}_j}{\partial \xi_j} d\xi$$

But by the divergence theorem, the last integral is the limit:

$$\lim_{R\to\infty} \int_{S(R)} \mathbf{h}.\nu d\mu$$

where $S(R)$ is the sphere of radius R, and ν is the unit normal to $S(R)$, and $\mathbf{h} = (\hat{h}_1, .., \hat{h}_n)$, and $d\mu$ is the suitably normalised measure on the sphere. Since $\|\mathbf{h}\|$ decreases faster than all powers of R, and the volume of $S(R)$ grows as R^{n-1}, the limit above is zero. This proves that $Tf(0) = f(0)$ for $f(0) = 0$.

Now for $f \in \mathcal{S}$ arbitrary, we write:

$$f = f(0)\psi + (f - f(0)\psi) = f(0)\psi + g$$

where ψ is the Gaussian. Clearly, $g \in \mathcal{S}$, with $g(0) = 0$. Thus $Tf(0) = f(0)(T\psi)(0) + (Tg)(0)$. But since ψ is its own Fourier transform, have $T\psi(0) = \psi(0) = 1$, and $Tg(0) = 0$ by the case done in the last para, we have $Tf(0) = f(0)$ for all $f \in \mathcal{S}$. Finally, to deduce the result for all points, we just translate coordinates. That is, for $f \in \mathcal{S}$, and $a \in \mathbb{R}^n$, define $g(x) = f(x+a)$, so that $g(0) = f(a)$, and g is also in $\mathcal{S}$. Then

$$\begin{aligned} cccf(a) = g(0) = (Tg)(0) &= \int\int e^{-i\xi.x} f(x+a)dxd\xi \\ &= \int\int e^{-i\xi.x}e^{i\xi.a} f(x)dxd\xi \\ &= \int e^{i\xi.a}\widehat{f}(\xi)d\xi \\ &= (\widehat{f})^\wedge(-a) = Tf(a) \end{aligned}$$

This proves the first part of (i). The second part (about the inverse Fourier transform) follows immediately. Thus (i) is proved.

To see (iii), note that:

$$\begin{aligned} (\widehat{f}\widehat{g})(\xi) &= \int\int e^{-i\xi.x} f(x) e^{-i\xi.y} g(y)dxdy \\ &= \int\int e^{-i\xi.(x-y)} f(x-y) e^{-i\xi.y} g(y)dxdy \\ &= \int\int e^{-i\xi.x} f(x-y)g(y)dxdy = \widehat{f*g}(\xi) \end{aligned}$$

where we have used Fubini to get the last line, because the double integral is absolutely convergent (since $f, g \in \mathcal{S}$), and a change of variables in the second line. The second part of (iii) follows immediately from (i) by replacing f and g by $\widehat{f}$ and $\widehat{g}$ respectively.

It finally remains to prove (iv), the Plancherel Theorem. We denote the L_2 inner product of f and g by $(f, g) = \int f(x)\overline{g}(x)dx$, which is $\mathbb{C}$-linear in the first slot, and $\mathbb{C}$-antilinear in the second. We compute for $f, g \in \mathcal{S}$:

$$\begin{aligned} (f, \widehat{g}) = \int f(x)\overline{\widehat{g}}(x)dx &= \int\int f(x)e^{ix.y}\overline{g}(y)dydx \\ &= \int \left(\int f(x)e^{ix.y}dx\right)\overline{g}(y)dy \\ &= (\widehat{f}(-x), g) \end{aligned}$$

where Fubini is used to interchange the order integration in an absolutely convergent double integral. Replacing $\widehat{f}(-y)$ by $g(y)$, we have $\widehat{g} = f$ by (i) above, so that:

$$(\widehat{g}, \widehat{g}) = (g, g)$$

which shows (using (i)) that the Fourier transform is a unitary map on $\mathcal{S}$ with respect to L_2-norm. Thus it extends to a unitary isomorphism of $L^2(\mathbb{R}^n)$, since $\mathcal{S}$ is dense in it. This proves the proposition. □

1.3. **Distributions.**

Definition 1.3.1 (Distributions)**.** We define a *distribution* T on $\mathbb{R}^n$ to be an element of the topological vector space dual of $C_c^\infty = \mathcal{D}$. That is, T is a linear functional on C_c^∞ and continuous with respect to the topology on it. The space of all distributions on $\mathbb{R}^n$ is clearly a complex vector space, and denoted $\mathcal{D}'$. Distributions $T \in \mathcal{D}'$ which extend to a continuous linear functional on the larger space $\mathcal{S}$ are called *tempered distributions*, and the vector space of tempered distributions is denoted $\mathcal{S}'$. Finally, distributions which extend as a continuous linear functional on all of $C^\infty = \mathcal{E}$ are called *compactly supported distributions*, and their vector space is denoted by $\mathcal{E}'$. Clearly, we have the inclusions of vector spaces:

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$$

Here are some basic examples:

Example 1.3.2. Let f be a measurable and locally L_1 function on $\mathbb{R}^n$. Then f defines the distribution $T_f \in \mathcal{D}'$ by the formula:

$$T_f(g) = \int_{\mathbb{R}^n} f(x)g(x)dx \quad \text{for } \ g \in C_c^\infty$$

which makes sense since g is compactly supported. By the way the topology is defined on C_c^∞ and the dominated convergence theorem, it follows that $T_f(g_n) \to 0$ in $\mathbb{C}$ if $g_n \to 0$ in C_c^∞.

Example 1.3.3. Let f be a measurable function on $\mathbb{R}^n$ such that $(1+ \mid x \mid)^{-N} f(x)$ is in $L_1(\mathbb{R}^n)$, for some $N \in \mathbb{N}$. Such functions are called *tempered functions*. Then defining T_f by the same formula as in the example above, and letting $g \in \mathcal{S}$, we get a *tempered distribution*. The formula makes sense because, for the N as above:

$$\int_{\mathbb{R}^n} f(x)g(x)dx = \int_{\mathbb{R}^n} (1+ \mid x \mid)^{-N} f(x)(1+ \mid x \mid)^N g(x)dx$$

and we have that the function $(1 + x|)^N g(x)$ is bounded since $g \in \mathcal{S}$, and $(1+ \mid x \mid)^{-N} f(x)$ is L_1. Again, the proof of its continuity is a consequence of the Dominated Convergence Theorem, and the topology defined earlier on $\mathcal{S}$. In particular, since $(1+ \mid x \mid)^{-n-1}$ is integrable on $\mathbb{R}^n$, all polynomials, bounded continuous functions, or continuous functions with at most polynomial growth define tempered distributions.

Note that if we take a function like $f(x) = e^x$, it can be checked that this is a distribution which is not tempered, so the inclusion $\mathcal{S}' \subset \mathcal{D}'$ is strict.

Example 1.3.4. Let f be a compactly supported function, and define an element T_f of $\mathcal{E}'$ via the same formula as in the above two examples, but $g \in \mathcal{E}$. It is checked easily that this is a compactly supported distribution.

If one wants to see a distribution which is *not* defined by a function, it is the very celebrated Dirac distribution of the next example.

Example 1.3.5 (Dirac distribution)**.** Define the distribution δ_a by the formula:

$$\delta_a(g) = g(a) \quad \text{for } \ g \in C^\infty$$

Again, it is trivial to check continuity, so that $\delta_a \in \mathcal{E}'$.

Exercise 1.3.6. Show that the inclusion $\mathcal{E}' \subset \mathcal{S}'$ is also strict.

Remark 1.3.7. A locally L^1-function f which defines a tempered distribution (via integration against $g \in \mathcal{S}$) *need not be a tempered function* in the sense of Example 1.3.3 above. For example, the locally L_1 function $e^x \sin e^x$ defines a tempered distribution on $\mathbb{R}$, (because it is the derivative of the bounded continuous function $-\cos e^x$, which is therefore a tempered distribution by 1.3.3 above, and the fact proved in the next subsection that all derivatives of tempered distributions are tempered distributions). However, it is not a tempered function, as we check below. For each N, we have a $C_N > 0$ such that:

$$(1+ \mid x \mid)^{-N} \mid e^x \mid \geq C_N e^{x/2}$$

for all $x \in [0, \infty)$, and thus we have an inequality of the integrals:

$$\int_{\mathbb{R}} (1+ \mid x \mid)^{-N} \mid e^x \sin e^x \mid dx \geq C_N \int_1^\infty e^{x/2} \mid \sin e^x \mid dx \geq C_N \int_0^\infty \left| \frac{\sin y}{\sqrt{y}} \right| dy$$

by a change of variables $y = e^x$. The right hand integral is infinite by comparing with the infinite series $\sum_n n^{-1/2}$. Some authors (e.g. Folland) define a tempered function to be a locally L_1 function which is a tempered distribution, to avoid this inconsistency.

We will see later after defining convolutions that if f is a *real-valued non-negative* locally integrable function on $\mathbb{R}^n$, then it is a tempered function in our sense if it is a tempered distribution. The rapid oscillation of say $e^x \sin e^x$ which causes the problem above is thereby eliminated.

As a final example of a distribution which is not a function, we have:

Example 1.3.8. Fix a multi-index α, and a point $a \in \mathbb{R}^n$. Then the higher derivative $D^\alpha_{x|a}$ at a clearly maps $\mathcal{E} \to \mathbb{C}$ in a continuous fashion with respect to the given topology, and defines a compactly supported distribution. For $\alpha = (0, 0, ..0)$, we recover the Dirac distribution. When we later define derivatives of distributions, we will see that this distribution is nothing but $\pm D^\alpha_x \delta_a$.

Definition 1.3.9 (Support of a distribution)**.** For an open subset $U \subset \mathbb{R}^n$, we say that the distribution $T \in \mathcal{D}'$ *vanishes on* U if $T(f) = 0$ for all f with compact support in U. For example, the Dirac distribution δ_a vanishes on $\mathbb{R}^n \setminus \{a\}$. Similarly the distribution $D^\alpha_{x|a}$ vanishes on $\mathbb{R}^n \setminus \{a\}$. By using a partition of unity subordinate to an open covering $\{U_i\}_{i \in \Lambda}$, one easily sees that if a distribution T vanishes on U_i for each $i \in \Lambda$, then it vanishes on the union $U = \cup_{i \in \Lambda} U_i$. Hence there is a largest open set U (possibly empty) on which a distribution T vanishes. The complement of this open set is called *support of* T, and denoted $\operatorname{supp} T$.

Lemma 1.3.10 ($\mathcal{E}'$ and distributions of compact support)**.** A distribution $T \in \mathcal{D}'$ is in $\mathcal{E}'$ iff $\operatorname{supp} T$ is compact. (Hence the terminology "compactly supported distribution" for elements of $\mathcal{E}'$.)

Proof: Suppose $\operatorname{supp} T = K$ a compact set. Let $\psi \in \mathcal{D}$ be a compactly supported smooth real-valued function with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on K. For a function $\phi \in \mathcal{E}$, define:

$$T(\phi) = T(\psi\phi)$$

Note that this definition is independent of the cut-off function ψ chosen above, for if ψ_1 is another cut-off function satisfying the same properties as ψ above, then $\psi\phi - \psi_1\phi$ will be a smooth compactly supported function whose support lies in K^c, so that $\operatorname{supp} T = K$ will imply that $T(\psi\phi) = T(\psi_1\phi)$. Now if $\phi_n \to 0$ in $\mathcal{E}$, we have (on applying Leibniz formula for derivatives of products) that $\psi\phi_n$ are compactly supported with support contained in the fixed compact set $L := \operatorname{supp} \psi$ for all n, and that $D^\alpha(\psi\phi_n) \to 0$ uniformly on L. Thus $\psi\phi_n \to 0$ in $\mathcal{D}$, and hence $T(\phi_n) = T(\psi\phi_n) \to 0$ since $T \in \mathcal{D}'$. Thus $T \in \mathcal{E}'$.

Conversely, suppose $\operatorname{supp} T$ is not compact, so T does not vanish on $\mathbb{R}^n \setminus \overline{B(0,n)}$ for each ball $B(0,n)$ of radius $n = 1, 2, , , ,$. Thus there exists a function ϕ_n with compact support $K_n \subset \mathbb{R}^n \setminus \overline{B(0,n)}$ with

$$T(\phi_n) = \lambda_n \neq 0 \quad \text{for} \qquad n = 1, 2, ...$$

Then it is trivial to verify that the functions $f_n := \lambda_n^{-1}\phi_n$ converge to 0 in $\mathcal{E}$, since on each compact set $L \subset \mathbb{R}^n$, we have $f_n \equiv 0$ on L for n large enough. On the other hand $T(f_n) = \lambda_n^{-1} T(\phi_n) = 1$ for all n, so that T is not continuous on $\mathcal{E}$, and hence $T \notin \mathcal{E}'$. The lemma follows. □

More examples of distributions will emerge as soon as we define some basic operations on distributions. Since tempered distributions are the ones of interest to us, we will concentrate mainly on them.

1.4. **New distributions out of old.** The most important operation on distributions is that of differentiation. Historically, distributions were invented by Dirac, to differentiate functions which had singularities, i.e. points of non-differentiability. Dirac realised that these are not going to be functions, but it was possible to do some self-consistent manipulations with them, so he called them "generalised functions". It took another thirty years for Laurent Schwartz to rigorise these ideas mathematically, and thanks to him, *every* distribution can be differentiated to get another distribution.

The starting point is to note that if $f \in \mathcal{E}$ and $g \in \mathcal{D}$, then we have

$$\int_{\mathbb{R}^n} D_x^\alpha f(x) g(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) D_x^\alpha g(x) dx$$

by using integration by parts, and noting that $\lim_{|x|\to\infty} fg = 0$ because of compact support of g. The same identity holds if $f \in \mathcal{E}$ and of slow (at most polynomial, for all derivatives) growth in x, and $g \in \mathcal{S}$. Thus it makes sense to make the:

Definition 1.4.1 (Derivative of a distribution)**.** For $T \in \mathcal{D}'$, define the distribution $D_x^\alpha T$ by:

$$D_x^\alpha T(g) = (-1)^{|\alpha|} T(D_x^\alpha g) \quad g \in \mathcal{D}$$

If $g_n \to 0$ in $\mathcal{D}$, then by definition, $D_x^\alpha g_n \to 0$ in $\mathcal{D}$ as well, and hence $D_x^\alpha T$ defined as above is a continuous linear functional on $\mathcal{D}$. Hence it is also in $\mathcal{D}'$. The factor $(-1)^{|\alpha|}$ has been chosen for consistency with derivatives of smooth functions, i.e. if $f \in \mathcal{E} = C^\infty$, the distribution T_f defined by f will satisfy $D_x^\alpha T_f = T_{D_x^\alpha f}$, viz. it is the distribution defined by $D_x^\alpha f$ in view of the last paragraph. The derivative of a distribution is often called a *distributional derivative.*

Exercise 1.4.2. For a fixed $a \in \mathbb{R}$, consider the distribution defined by the locally L_1 *Heaviside* function:

$$\begin{array}{rcl} f : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & 0 \quad \text{for } x < a \\ x & \mapsto & 1 \quad \text{for } x \geq a \end{array}$$

(This is just the *indicator* (or *characteristic*) function $\chi_{[a,\infty)}$ of $[a, \infty)$.) Show that the distributional derivative $\frac{df}{dx}$ is the Dirac distribution δ_a.

Exercise 1.4.3. For $T \in \mathcal{S}'$ a tempered distribution, $D_x^\alpha T$ is also tempered. If $T \in \mathcal{E}'$ is any compactly supported distribution, then so is $D_x^\alpha T$. For any distribution $T \in \mathcal{D}'$, the support of the derivative obeys

$$\operatorname{supp} D_x^\alpha T \subset \operatorname{supp} T$$

Definition 1.4.4 (Multiplication by a smooth function)**.** If $f \in \mathcal{E}$, then the linear multiplication mapping $\mathcal{D} \to \mathcal{D}$ defined by $g \mapsto fg$ is clearly continuous. Thus we may define for a distribution $T \in \mathcal{D}'$ the product fT by the formula:

$$fT(g) = T(fg) \quad \text{for } g \in \mathcal{D}$$

By the remark above, fT is also a distribution. Likewise for $\mathcal{E}$, the mapping $\mathcal{E} \to \mathcal{E}$ defined by $g \mapsto fg$ is continuous, and we can again define fT as a compactly supported distribution for $T \in \mathcal{E}'$ a compactly supported distribution by the same procedure as above.

The story for tempered distributions is different. *Multiplication by an arbitrary smooth function f does not send the Schwartz space $\mathcal{S}$ to itself.* The best we can do is to observe that if f is a smooth function of *slow growth* (i.e. $| D_x^\beta f |< C_\beta(1+ | x |)^{N_\beta}$ for each β), then $g \mapsto fg$ is a continuous linear operator $\mathcal{S} \to \mathcal{S}$. Hence, by the procedure above, we can define fT for $T \in \mathcal{S}'$ and f a smooth function *of slow growth.*

Finally, we come to convolution of functions and distributions. For a function g, we define:

$$g^x(y) := g(x-y)$$

so that for smooth functions f, g, their convolution (whenever it is defined) maybe expressed as

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)g^x(y)dy$$

By a change of variables, $f*g = g*f$. It is clear that the linear mapping $g \mapsto g^x$ is a continuous map which takes $\mathcal{E}$ to $\mathcal{E}$, $\mathcal{D}$ to $\mathcal{D}$ and $\mathcal{S}$ to $\mathcal{S}$. Taking our cue from this, it is natural to make the following definition:

Definition 1.4.5 (Convolution of a distribution with a function)**.** Let $T \in \mathcal{D}$ be a distribution, and f a smooth function of *compact support*. Then define the *function* $f*T$ by the formula:

$$(f*T)(x) = T(f^x) \quad \text{for} \quad f \in \mathcal{E}$$

Similarly, if $T \in \mathcal{S}'$ is a tempered distribution and $f \in \mathcal{S}$, or if $T \in \mathcal{E}'$ is a compactly supported distribution and $f \in \mathcal{E}$ is any smooth function. These restrictions are natural, in view of the fact that even functions f, g need to obey some decay conditions in order to be convolved.

Example 1.4.6 (Convolution with the Dirac distribution)**.** Let $g \in \mathcal{E}$ be any smooth function, and δ_0 be the Dirac distribution. Then the convolution $g*\delta_0$ is the function g. (This shows that the identity element for the convolution product is a distribution). For, by definition,

$$(g*\delta_0)(x) = \delta_0(g^x) = g^x(0) = g(x)$$

Lemma 1.4.7. Whenever it makes sense by the definition above, the convolution $f*T$ is a smooth function. Furthermore, we have the identities:

$$D^\alpha(f*T) = f*D^\alpha T = D^\alpha f * T$$

Proof: We just prove it for the first partial derivative with respect to x_1, viz. $\partial_1 = iD_1$. Let e_1 denote the unit vector $(1,0,..,0)$, and the case of $f \in \mathcal{D}$ and $T \in \mathcal{D}'$. For a smooth function $f \in \mathcal{E}$, we have the Taylor formula:

$$f^{x+he_1}(y) - f^x(y) = f(x+he_1-y) - f(x-y) = hg(x,h,y) + h^2 r(x,h,y)$$

where g and r are smooth in all the variables, and $g(x,0,y) = (\partial_1 f)(x-y) = (\partial_1 f)^x(y)$. Because the supremum norm

$$\sup_{y\in K,\,|h|\le\epsilon} \mid r(x,h,y) \mid \le C(K)$$

for any compact set $K \subset \mathbb{R}^n$, it follows that, for a fixed x, and as a function of y:

$$\lim_{h\to 0}\left(\frac{f^{x+he_1} - f^x}{h}\right) \to g(x,0,-) = (\partial_1 f)^x$$

uniformly on compact sets. Similarly all y-derivatives of the functions $g_h := \frac{f^{x+he_1}-f^x}{h}$ converge uniformly to the corresponding derivatives of $(\partial_1 f)^x$ on all compact sets as $h \to 0$. If the function f is in $\mathcal{D}$, and compactly supported in K say, then it is easy to check that for all $\mid h \mid < 1$, the functions g_h are all supported in the fixed compact set $K' = x - K + \overline{B(0,1)}$, and $g_h \to (\partial_1 f)^x$ as $h \to 0$ in $\mathcal{D}$ as well. Thus by the continuity and linearity of $T \in \mathcal{D}'$ we have:

$$\partial_1(f*T) = \lim_{h\to 0}\left(\frac{T(f^{x+he_1}) - T(f^x)}{h}\right) = T\left[\lim_{h\to 0}\left(\frac{f^{x+he_1} - f^x}{h}\right)\right] = T((\partial_1 f)^x) = \partial_1 f * T$$

Also note that we have:

$$(\partial_1 f)^x(y) = (\partial_{y_1} f)^x(y) = (\partial_{y_1} f)(x-y) = -\partial_{y_1}(f^x)(y)$$

so that:

$$\partial_1 f * T = T\left((\partial_1 f)^x\right) = -T(\partial_1 f^x) = (\partial_1 T)(f^x) = f * \partial_1 T$$

by the definition of derivative of a distribution. This proves the lemma. □

Exercise 1.4.8.

(i): Let $T \in \mathcal{D}'$ be a distribution, and let

$$\rho(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$$

be a smooth function of compact support in $\mathbb{R}^{2n}$, say $\operatorname{supp}\rho \subset K \times K$ for some compact $K \subset \mathbb{R}^n$. Thus, the function $\int_{x\in\mathbb{R}^n} \rho(x,y)dx$ is a smooth function (of y) in $C_c^\infty(\mathbb{R}^n)$, which is supported in K. Then show that

$$T\left(\int_{x\in\mathbb{R}^n} \rho(x,y)dx\right) = \int_{\mathbb{R}^n} T(\rho(x,y))dx$$

where on the right hand side, T is operating on the function $\rho(x,y)$ considered as a function of y, and thus $T(\rho(x,y))$ is a function of x. (*Hint*: Find a sequence of Riemann sums, which are functions of y, say $S_n(y) := \sum_j \rho(x_j,y)\Delta_j$ where the Δ_j's are the volumes of cubes of side $\frac{1}{n}$ covering the compact set K, x_j the centre of Δ_j, and show that $S_n \to \int \rho(x,-)dx$ inside $\mathcal{D}(\mathbb{R}^n)$, and use the continuity and linearity of T.)

(ii): If f and g are compactly supported functions in $\mathcal{D}$, show that $\rho(x,y) := f(x)g(y-x)$ is in $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and conclude that $f * g$ is a smooth compactly supported function, and by (i) above, we further have:

$$T(f*g) = \int_x f(x)T(\widetilde{g}^x)dx = (\widetilde{g} * T)(f)$$

where $\widetilde{g}(z) := g(-z)$ and $\widetilde{g} * T$ is a function being regarded as a distribution.

Here is an important application of convolutions of functions with distributions. We can use compactly supported approximate identities (see the Lemma 1.2.3) to approximate any distribution $T \in \mathcal{D}'$ by smooth functions of compact support. First we need a topology on $\mathcal{D}'$ to make sense of the notion of approximation.

Definition 1.4.9 (Weak-star topology on $\mathcal{D}'$)**.** Say that a sequence of distributions $T_n \to 0$ in $\mathcal{D}'$ if for each $\phi \in \mathcal{D}$, the sequence $T_n(\phi) \to 0$. This is the topology of *pointwise convergence* in any dual vector space, and is usually called the *weak-star topology.* On the subspaces $\mathcal{S}'$ (resp. $\mathcal{E}'$) of tempered (resp. compactly supported) distributions, we induce the subspace topology from this weak star topology on $\mathcal{D}'$.

Proposition 1.4.10 (Approximation of distributions by compactly supported functions)**.** The space of smooth compactly supported functions $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$ is dense in the topological vector space $\mathcal{D}'$ of distributions on $\mathbb{R}^n$.

Proof: We first make a remark about convolution with approximate identities (see the Lemma 1.2.3). Take a real-valued $\phi \in C_c^\infty$, with $\phi(x) \geq 0$ all $x \in \mathbb{R}^n$ and $\int \phi(x)dx = 1$, $\operatorname{supp}\phi \subset \overline{B(0,1)}$ and $\phi(x) = \phi(-x)$ (even function). Then $\operatorname{supp}\phi_\epsilon \subset \overline{B(0,\epsilon)}$, and for a compactly supported function $g \in C_c^\infty$ with $\operatorname{supp} g = K$ a compact set, we have:

$$\operatorname{supp}(g * \phi_\epsilon) \subset K^\epsilon := \{x : d(x,K) \leq \epsilon\}$$

This is because if x is outside the set K^ϵ on the right, $(x-y)$ will lie outside K for all $y \in \overline{B(0,\epsilon)}$, and hence $g(x-y)$ will be zero. For $y \notin \overline{B(0,\epsilon)}$, $\phi_\epsilon(y)$ will be zero. Thus the product $g(x-y)\phi_\epsilon(y)$ will be identically zero, and hence

$$g * \phi_\epsilon(x) = \int_{\mathbb{R}^n} g(x-y)\phi_\epsilon(y) = 0$$

for $x \notin K^\epsilon$.

Now let $T \in \mathcal{D}'$ be a distribution. Let $\psi_j \geq 0$ be compactly supported cutoff functions which are identically 1 on $\overline{B(0,j)}$ and identically zero outside $V_j := \overline{B(0,j+1)}$, say. We claim that the distributions $\psi_j T$ converge to T in $\mathcal{D}'$. Indeed, for a fixed $g \in \mathcal{D}$

$$\psi_j T(g) = T(\psi_j g) \to T(g)$$

because $\psi_j g \equiv g$ for j large enough, g being compactly supported. So by the definition of the weak star topology, we have $\psi_j T \to T$.

Now we claim that the function $\phi_\epsilon * (\psi_j T)$ is a smooth function compactly supported in V_j^ϵ. We already know by the Lemma 1.4.7 above that the convolution $\phi_\epsilon * (\psi_j T)$ is a smooth function. Clearly, it is a compactly supported function iff it is a compactly supported distribution. To show that $\phi_\epsilon * \psi_j T$ vanishes on the complement of V_j^ϵ, let $g \in \mathcal{D}$ be a smooth function with $\operatorname{supp} g = L$ a compact set, and satisfying $L \cap V_j^\epsilon = \phi$. Then, by (ii) of Exercise 1.4.8 above:

$$(\phi_\epsilon * \psi_j T)(g) = (\widetilde{\phi_\epsilon} * \psi_j T)(g) = \psi_j T(\phi_\epsilon * g) = T(\psi_j(\phi_\epsilon * g)) \tag{1}$$

By the first para above, $\operatorname{supp}(\phi_\epsilon * g) = L^\epsilon$. The support of ψ_j is contained in V_j. Since $L \cap V_j^\epsilon = \phi$, we have $L^\epsilon \cap V_j = \phi$, so the function of y given by $\psi_j(\phi_\epsilon * g)$ above is the identically zero function, and so T applied to it is therefore zero. This shows that $\phi_\epsilon * \psi_j T$ is a distribution compactly supported in V_j^ϵ, and hence a smooth function of compact support.

We now claim that for a fixed j, the family of distributions (compactly supported smooth functions by the above) $\phi_\epsilon * \psi_j T$ converge to $\psi_j T$ in $\mathcal{D}'$ as $\epsilon \to 0$. By the Lemma 1.2.3 and the fact that $\operatorname{supp}\psi_j = V_j$ a compact set, we have $\psi_j(\phi_\epsilon * g) \to \psi_j g$ uniformly on V_j as $\epsilon \to 0$. Hence the right hand side of the equation (1) above converges to $T(\psi_j g)$ as $\epsilon \to 0$ by the continuity of T, and the claim follows.

Since $\psi_j T \to T$ in $\mathcal{D}'$ as $j \to \infty$, and the compactly supported smooth functions $\phi_\epsilon * \psi_j T \to \psi_j T$ as $\epsilon \to 0$, it follows that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}'$. □

Now we come to one of the chief reasons why the Schwartz space $\mathcal{S}$ and tempered distributions were introduced. We have already observed in (ii) of the Proposition 1.2.5 that

$$\xi^\alpha D_\xi^\beta \widehat{f} = \pm (D_x^\alpha (x^\beta f))^\wedge \quad \text{for} \quad f \in \mathcal{S}$$

Hence if $f_n \to 0$ as $n \to \infty$ in $\mathcal{S}$, $x^\beta D_x^\alpha f_n \to 0$ uniformly on $\mathbb{R}^n$, for all $\alpha,\ \beta$. By Leibnitz's rule for the derivatives of products, it follows that $D_x^\alpha(x^\beta f_n) \to 0$ uniformly on $\mathbb{R}^n$. Thus $\left\| D_x^\alpha (x^\beta f_n) \right\|_1 \to 0$. By the fact that $\|\widehat{g}\|_\infty \le \|g\|_1$ for $g \in \mathcal{S}$ and the equation above it follows that

$$\left\| \xi^\alpha D_\xi^\beta \widehat{f}_n \right\|_\infty = \left\| D_x^\alpha (x^\beta f))^\wedge \right\|_\infty = \left\| D_x^\alpha (x^\beta f_n) \right\|_1 \to 0 \quad \text{as} \quad n \to \infty$$

That is, $\widehat{f}_n \to 0$ in $\mathcal{S}$ as $n \to \infty$. Thus, with the topology introduced earlier on $\mathcal{S}$, we have:

$$\widehat{\ }: \mathcal{S} \to \mathcal{S}$$

is a continuous linear map of topological vector spaces. Hence it makes sense to make the following:

Definition 1.4.11 (Fourier transform of a tempered distribution)**.** Let $T \in \mathcal{S}'$ be a tempered distribution. Define the *Fourier transform* $\widehat{T}$ by the formula:

$$\widehat{T}(g) = T(\widehat{g}) \quad \text{for} \quad g \in \mathcal{S}$$

By the remarks above, $\widehat{T}$ is also a tempered distribution. We leave it as an easy exercise to check that this definition is consistent with the definition for functions, i.e. for an L^1-function (which defines the tempered distribution T_f via integration as indicated in the Example 1.3.3), then we have $T_{\widehat{f}} = \widehat{T_f}$. (Just mimic the proof of (iv) of Proposition 1.2.5 without the complex conjugation).

Analogously, since the inverse Fourier transform and transform differ by reflection of the function, we define the *inverse Fourier transform* $T^\vee$ of a tempered distribution T by the formula:

$$T^\vee(g) = T(g^\vee) \quad \text{for} \quad g \in \mathcal{S}$$

We have the following proposition about the distributional Fourier transform. .

Proposition 1.4.12. Let $T \in \mathcal{S}'$ be a tempered distribution. Then the Fourier transform $\widehat{\ }: \mathcal{S}' \to \mathcal{S}'$ satisfies:

(i): The map $\widehat{\ }$ is a continuous linear isomorphism of $\mathcal{S}'$ of period 4, and we have

$$(\widehat{T})^{\vee} = T \quad \text{for all } T \in \mathcal{S}'$$

(ii): For a polynomial $P = P(X_1, .., X_n)$, we have:

$$(P(D)T)^{\wedge} = P(\xi)\widehat{T}, \quad (P(x)T)^{\wedge} = P(-D)\widehat{T}$$

(iii): If $f \in L_1(\mathbb{R}^n)$, then $T_{\widehat{f}} = \widehat{T_f}$.

(iv): $(\phi * T)^{\wedge} = \widehat{\phi}\widehat{T}$ for $\phi \in \mathcal{S}$.

Proof: If $T_n \to T$ in $\mathcal{S}'$, we have by definition that $T_n(\widehat{g}) \to T(\widehat{g})$ for each $g \in \mathcal{S}$. That is, $\widehat{T}_n(g) \to \widehat{T}(g)$, for each $g \in \mathcal{S}$, which again, by definition, implies $\widehat{T}_n \to \widehat{T}$ in $\mathcal{S}'$. This shows that the Fourier transform is a continuous map. The rest of (i),(ii) and (iii) follow immediately by applying the relevant parts of the Proposition 1.2.5.

To see (iv), note that:

$$(\phi * T)^{\wedge}(g) = (\phi * T)(\widehat{g}) = T(\widetilde{\phi} * \widehat{g})$$

by (ii) of Exercise 1.4.8. But by (iii) of Proposition 1.2.5, we have $\widetilde{\phi} * \widehat{g} = \widehat{\widehat{\phi}} * \widehat{g} = (\widehat{\phi}g)^{\wedge}$, so the last expression above is precisely $(\widehat{T})(\widehat{\phi}\widehat{g}) = (\widehat{\phi}\widehat{T})(g)$. The proposition follows. □

To deduce some more crucial facts about $\widehat{T}$, we need an elementary but very useful lemma about "locally convex topological vector spaces".

Lemma 1.4.13. Let V be a topological vector spaces whose topology is defined by a "family of seminorms" $\{p_\alpha\}_{\alpha\in\Lambda}$, viz., a sequence $x_n \in V$ converges to zero iff $p_\alpha(x_n) \to 0$ for all $\alpha \in \Lambda$. Then a linear map $T : V \to \mathbb{C}$ is continuous iff there exists a constant $C > 0$ and a finite subfamily $p_{\alpha_1}, .., p_{\alpha_k}$ of seminorms such that:

$$| Tx |\leq C \sum_{i=1}^{k} p_{\alpha_j}(x) \quad \text{for all } x \in V$$

Proof: We define the "semiball" (?) in V around 0 with respect to the seminorm p_α in the obvious manner:

$$B_\alpha(0, \epsilon) = \{x \in V : p_\alpha(x) < \epsilon\}$$

and note that by the definition of a seminorm all of these are convex sets, and since each p_α is continuous, they are open. Hence their finite intersections are also convex, open, and contain x. Define a new topology $\mathcal{T}$ on V by declaring a neighbourhood base of 0 to be the family of finite intersections:

$$\mathcal{N}(0) := \{\cap_{i=1}^{k} B_{\alpha_i}(0, \epsilon_i) : \epsilon_i > 0, \ \{\alpha_1, .., \alpha_k\} \subset \Lambda, \ k = 1, 2, ..., \}$$

and the neighbourhood base around x by $\mathcal{N}(x) := x + \mathcal{N}(0)$. It is clear that if $p_\alpha(x_n) \to 0$ for all $\alpha \in \Lambda$, then $x_n \to 0$ in the topology $\mathcal{T}$, because x_n will eventually lie in every basic neighbourhood. On the other hand, if there exists an $\alpha \in \Lambda$ such that $p_\alpha(x_n)$ does not converge to zero, then there exists an $\epsilon > 0$ and some subsequence x_{n_k} such that $p_\alpha(x_{n_k}) > \epsilon$ for all k. That is, $x_{n_k} \notin B_\alpha(x, \epsilon)$ for all k, so the sequence $\{x_n\}$ will fail to eventually belong to this open neighbourhood $B_\alpha(0, \epsilon)$, and hence does not converge to 0 in the topology $\mathcal{T}$. Thus $\mathcal{T}$ is exactly the topology defined by the family of seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$.

Since T is continuous, there is an open neighbourhood U of 0 such that $| Tx |< 1$ for all $x \in U$. Since $\mathcal{N}(0)$ is a neighbourhood basis of 0, we may assume without loss of generality that $U = \cap_{i=1}^{k} B_{\alpha_i}(0, \epsilon_i)$. Set $\epsilon = \min_{1\leq i\leq k}\{\epsilon_i\}$ and $C = \epsilon^{-1}$.

Let $x \in V$. If $p_{\alpha_j}(x) = 0$ for all $j = 1, .., k$, then by the definition of U, it follows that $\lambda x \in U$ for all $\lambda > 0$, and by the choice of U it follows that $| T(\lambda x) |= \lambda | Tx |< 1$ for all $\lambda > 0$, so that $Tx = 0$, and certainly

$$| Tx |= 0 \leq C \sum_{i=1}^{k} p_{\alpha_i}(x)$$

for C as above. On the other hand, if $p_{\alpha_j}(x) > 0$ for some j, observe that $| y |= \epsilon x / \sum_{i=1}^{k} p_{\alpha_i}(x)$ satisfies $p_{\alpha_j}(y) < \epsilon$ for all $j = 1, .., k$, so that $y \in U$ and $| Ty |< 1$. Which is the same as saying that:

$$| Tx |< (\epsilon)^{-1} \sum_{i=1}^{k} p_{\alpha_i}(x)$$

So, since $C = \epsilon^{-1}$, we have the desired inequality in both cases, i.e. for all $x \in V$, and the lemma follows. □

Remark 1.4.14 (: Caution!). The topology of the topological vector space $\mathcal{E}$ is determined by the family of seminorms

$$\{p_{\alpha,K} : \ p_{\alpha,K}(f) := \sup_{x \in K} | D_x^\alpha f |, \ \alpha \text{ a multi-index}, \ K \text{ a compact subset of } \mathbb{R}^n\}$$

Similarly, the Schwartz space $\mathcal{S}$ is defined by the family of seminorms:

$$\{p_{\alpha,\beta} : \ p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} | x^\alpha D_x^\beta f |, \ \alpha, \ \beta \text{ multi-indices}\}$$

However, the topology of $\mathcal{D}$ is not given by the family of seminorms which define $\mathcal{E}$. As we noted in (iii) of the Definition 1.1.1, if we take a fixed compactly supported function $\psi \neq 0$ supported on $[-1, 1] \subset \mathbb{R}$, then define $f_n(x) = \psi(x - n)$ by translating, we have $p_{\alpha,K} f_n \to 0$ for each α and K, but $f_n \not\to 0$ in $\mathcal{D}$.

Proposition 1.4.15. Let $T \in \mathcal{E}'$ be a compactly supported distribution. For $\xi \in \mathbb{R}^n$, denote $e_\xi(x) := e^{i\xi.x} = e_x(\xi)$. Then the tempered distribution $\widehat{T}$ is the function:

$$\widehat{T}(\xi) = T(e_{-\xi})$$

It is a smooth function of slow growth (see the Definition 1.4.4).

Proof: Let us first check the identity above, whose right side, viz the function $\nu(\xi) := T(e_{-\xi})$ makes sense because T is in $\mathcal{E}'$. Then let $K := \operatorname{supp} T$ be the compact support of T (in view of the Lemma 1.3.10), and let $\psi(\xi)$ be a compactly supported function which is identically 1 on K. It is trivial to check that $\psi T = T$. Now let $g \in \mathcal{D}$ be a compactly supported function. Then

$$\begin{aligned} \widehat{T}(g) &= \psi T(\widehat{g}) = T\left[\psi(\xi) \int g(x) e_{-\xi}(x) dx\right] = T\left[\int \psi(\xi) g(x) e_{-x}(\xi) dx\right] \\ &= \int g(x) T(\psi e_{-x}) dx = \int g(x)(\psi T)(e_{-x}) dx \\ &= \int g(x) T(e_{-x}) dx = \int g(x) \nu(x) dx = T_\nu(g) \end{aligned}$$

by using (i) of the Exercise 1.4.8 applied to the compactly supported function $\rho(x, \xi) = \psi(\xi) g(x) e_{-x}(\xi)$.

Now the smoothness of the function $\nu(\xi)$ easily follows by applying continuity of T, and that T is compactly supported so acts on all smooth functions. To check slow growth, we first note that for the family of seminorms

$$\{p_{\alpha,L} : \ \alpha \text{ a multi-index and } L \text{ a compact subset of } \mathbb{R}^n\}$$

which define the topology of $\mathcal{E}$, we have:

$$p_{\alpha,L}(x^\beta e_{-\xi}) = \sup_{x \in L} | D_x^\alpha (x^\beta e^{-i\xi.x}) | \leq C(L)(1+ | \xi |)^{N(\alpha,\beta)}$$

By the Lemma 1.4.13, since T is continuous, there exists a finite subfamily p_{α_j,L_j} such that

$$| D_\xi^\beta \nu(\xi) |=| \pm T(D_\xi^\beta e_{-\xi}) |=| T(x^\beta e_{-\xi}) |\leq C \sum_{j=1}^{k} p_{\alpha_j,L_j}(x^\beta e_{-\xi}) \leq C \sum_{j=1}^{k} C(L_j)(1+ | \xi |)^{N(\alpha_j,\beta)}$$

which is clearly bounded by $C(1+\mid \xi \mid)^N$ for $N = \max_j N(\alpha_j, \beta)$. The proposition follows. □

Since the constant function 1 is a locally L_1 function satisfying $(1+ \mid x \mid)^{-N}.1 \in L_1(\mathbb{R}^n)$ for all $N > n$, by the Example 1.3.3 it is a tempered distribution. The Dirac distribution is in fact a compactly supported distribution, and hence a tempered distribution. Thus it makes sense to take the Fourier transforms of these distributions. Indeed we have the:

Corollary 1.4.16. The Dirac distribution δ_0 and the constant function 1 are Fourier transforms of each other.

Proof: By the above Proposition 1.4.15, we have

$$\widehat{\delta_0}(\xi) = \delta_0(e_{-\xi}) = 1$$

for all ξ. The fact that $\widehat{1} = \delta_0$ then follows from the Fourier inversion formula in (i) of the Proposition 1.4.12, since δ and 1 are invariant under the reflection $x \mapsto -x$. One can check it directly as well, for if $g \in \mathcal{S}$, we have:

$$\widehat{1}(g) = 1(\widehat{g}) = \int \widehat{g}dx = \widehat{\widehat{g}}(0) = g(-0) = g(0) = \delta_0(g)$$

□

Exercise 1.4.17. Using (ii) of the Proposition 1.4.12, show that polynomials in $\xi = (\xi_1, .., \xi_n)$ are exactly the Fourier transforms of tempered distribution defined as finite linear combinations of derivatives of the Dirac distribution δ_0, namely distributions T of the form:

$$T = \sum_{i=1}^{k} c_k D_x^{\alpha_i} \delta_0$$

where α_i are some multi-indices, and $c_i \in \mathbb{C}$.

Indeed, we have the following interesting characterisation of distributions whose support is a point.

Proposition 1.4.18 (Distributions with point support)**.** Let $a \in \mathbb{R}^n$, and let $T \in \mathcal{D}'$ with $\operatorname{supp} T = \{a\}$. Then

$$T = \sum_{i=1}^{k} c_k D_x^{\alpha_i} \delta_a$$

where δ_a is the Dirac distribution at a, α_i are some multi-indices, and $c_i \in \mathbb{C}$.

Proof: By translation, we may assume that $a = 0$. Let $\psi \in \mathcal{D}$ be a cutoff function such that $\psi \geq 0$, $\psi \equiv 1$ on $B(0, \frac{1}{2})$ and $\psi \equiv 0$ outside $B(0, 1)$. Since T is supported in the point $\{0\}$, it follows that $T((1-\psi)\phi) = 0$ for all $\phi \in \mathcal{E}$, and hence $T(\phi) = T(\psi\phi)$ for all $\phi \in \mathcal{E}$. Note that by Leibnitz's formula for the derivatives of a product, we have for a compact set K and α a multi-index, the inequality:

$$p_{\alpha,K}(\psi\phi) = \sup_{x \in K} \mid D^\alpha(\psi\phi) \mid \leq C \sum_{|\beta| \leq |\alpha|} \sup_{\|x\| \leq 1} \mid D^\beta \phi \mid$$

where C depends on $\sup_{\|x\| \leq 1} \mid D^\gamma \psi \mid$ for various $\mid \gamma \mid \leq \mid \alpha \mid$.

Combining the above fact with the Lemma 1.4.13, we have an inequality:

$$\mid T(\phi) \mid = \mid T(\psi\phi) \mid \leq C \sum_{i=1}^{k} p_{\alpha_i, K_i}(\psi\phi) \leq C \sum_{|\alpha| \leq N} \sup_{\|x\| \leq 1} \mid D^\alpha \phi(x) \mid \quad \text{for all } \phi \in \mathcal{E} \tag{2}$$

where $N = \max_{1 \leq i \leq k} \mid \alpha_i \mid$. Now we make the following:

Claim: Let $\phi \in \mathcal{E}$ such that $D^\alpha \phi(0) = 0$ for all $\mid \alpha \mid \leq N$. Then $T(\phi) = 0$.

Consider the sequence of functions $\phi_k \in \mathcal{E}$ defined by

$$\phi_k(x) = \phi(x)(1-\psi(kx)), \qquad x \in \mathbb{R}^n$$

where ψ is the cut-off function defined in the first paragraph. The fact that $D^\alpha\phi(0) = 0$ for $|\alpha| \leq N$ will imply by Taylor's formula for $D^\beta\phi$ around the origin that there exists a $\delta > 0$ such that

$$| D^\beta\phi(x) | \leq C \|x\|^{N+1-|\beta|} \quad \text{for all} \quad \|x\| < \delta, \quad |\beta| \leq N$$

Note that the function $\psi(kx)$ and all its derivatives are supported in the ball $B(0, 1/k)$, and for k large enough, this ball is contained in $B(0, \delta)$.

Now let α be a multi-index such that $|\alpha| \leq N$. Then we have

$$| D^\alpha(\phi(x)\psi(kx)) | = 0 \quad \text{for} \quad \|x\| > \frac{1}{k}, \quad \text{and all} \quad k$$

On the other hand, for $\|x\| < \frac{1}{k}$, and k large enough so that $\frac{1}{k} < \delta$, we have:

$$\begin{aligned} | D^\alpha(\phi(x)\psi(kx)) | &\leq C \sum_{|\beta|\leq|\alpha|} | D^\beta\phi(x) || D^{\alpha-\beta}\psi(kx) | \\ &\leq C \sum_{|\beta|\leq|\alpha|} \|x\|^{N+1-|\beta|} k^{|\alpha|-|\beta|} \left\|D^{\alpha-\beta}\psi\right\|_\infty \\ &\leq C \sum_{|\beta|\leq|\alpha|} (k^{-1})^{N+1-|\beta|} k^{|\alpha|-|\beta|} \\ &\leq Ck^{|\alpha|-N-1} \leq Ck^{-1} \end{aligned}$$

Summing up, we have:

$$\sup_{x\in\mathbb{R}^n} | D^\alpha(\phi_k(x) - \phi(x)) | = \sup_{x\in\mathbb{R}^n} | D^\alpha(\phi(x)\psi(kx)) | \leq Ck^{-1} \quad \text{for} \quad |\alpha| \leq N \quad \text{and} \quad k >> 0$$

Plugging this fact into the inequality (2) above, we find that: $\lim_{k\to\infty} | T\phi_k - T\phi | = 0$, i.e.

$$\lim_{k\to\infty} T(\phi_k) = T(\phi)$$

Now note that ϕ_k are compactly supported in the region $\{1/2k \leq \|x\| < \infty\}$, and hence compactly supported in $\mathbb{R}^n \setminus \{0\}$. Since $\operatorname{supp} T = \{0\}$, $T(\phi_k) = 0$ for all k. Thus $T(\phi) = \lim_{k\to\infty} T(\phi_k) = 0$ and our claim follows.

Now, to show that $T = \sum_{|\alpha|\leq N} c_\alpha D^\alpha \delta_0$, it is enough to show that the Fourier transform $\widehat{T}$ is a polynomial (see the preceding Exercise 1.4.17). By the Proposition 1.4.15, we have for the N chosen above that

$$\begin{aligned} \widehat{T}(\xi) = T(e_{-\xi}) &= T\left(\sum_{0\leq k\leq N} \frac{[-i(\xi.x)]^k}{k!} + \phi\right) \\ &= \sum_{0\leq|\alpha|\leq N} (-i)^{|\alpha|} \frac{\xi^\alpha T(x^\alpha)}{\alpha\,!} = \sum_{0\leq|\alpha|\leq N} c_\alpha \xi^\alpha \end{aligned}$$

since $T(\phi) = 0$ by the Claim above (all its derivatives of order $\leq N$ vanish at 0). This proves the proposition. □

Finally, we have the following description of compactly supported distributions.

Proposition 1.4.19. Let $T \in \mathcal{E}'$ be a compactly supported distribution. Then there exists a continuous function $g \in C_0(\mathbb{R}^n)$ such that:

$$T = \sum_{i=1}^{k} c_i D_x^{\alpha_i} g$$

Proof: By the Proposition 1.4.15, the Fourier transform $\widehat{T}$ is a smooth function, say F, of slow growth. That is, there exist $C > 0$ and N such that:

$$| F(\xi) | \leq C(1+ | \xi |)^N$$

Since $(1+ | \xi |^2)^{-s}$ is in $L^1(\mathbb{R}^n)$ for any $s > n/2$, it follows that the function:

$$G(\xi) = (1+ | \xi |^2)^{-M} F(\xi)$$

is in $L^1(\mathbb{R}^n)$ for $M = N + n$, say. So, by the Riemann-Lebesgue lemma (v) of Proposition 1.2.5 above, the function $g = G^\vee = \widetilde{G}^\wedge$ is in $C_0(\mathbb{R}^n)$, and $G = \widehat{g}$. Also

$$F(\xi) = \left(1 + \sum_{i=1}^n \xi_i^2\right)^M G(\xi) = P(\xi)\widehat{g}(\xi) = (P(D)g)^\wedge(\xi)$$

where $P(\xi) = (1+ | \xi |^2)^M$ is a polynomial, by (ii) of Proposition 1.2.5. But then:

$$T = (\widehat{T})^\vee = F^\vee = ((P(D)g)^\wedge)^\vee = P(D)g$$

which proves the proposition. □

Remark 1.4.20 (Tempered distributions given by non-negative $L_{1,loc}$ functions)**.** We saw with the example of $e^x \cos e^x$ in the Remark 1.3.7 that an $L_{1,loc}$ function which is a tempered distribution is not necessarily a tempered function in the sense of Example 1.3.3. However, if $f \in L_{1,loc}(\mathbb{R}^n)$, *and f is non-negative*, then the distribution T_f defined by f is a tempered distribution implies that the function f is a tempered function in the sense of Example 1.3.3. For it is enough to prove, for example, that for some N, $(1+ | x |)^{-N} f$ is integrable on say $\{| x |\geq 2\}$, because every locally integrable function is integrable on $\overline{B(0,2)}$. Let $\psi \in C_c^\infty(\mathbb{R})$ be a real valued non-negative function with $\psi \equiv 1$ on the interval $[-1/2, 1/2]$, and $\equiv 0$ outside the interval $[-1, 1]$. For $a \geq 2$, define on $\mathbb{R}^n$ the radially symmetric non-negative function:

$$\psi_a(x) = \psi(| x | -a)$$

which is compactly supported in the annulus $\{a - 1 \leq| x |\leq a + 1\}$. Since ψ_a are radially symmetric, it is easy to check that the Schwartz seminorms of these functions are majorised as:

$$p_{\alpha\beta}(\psi_a) = \sup_x | x^\alpha D_x^\beta \psi_a |\leq (a + 1)^{|\alpha|} | \partial_r^\beta \psi |\leq C_\beta (a + 1)^{|\alpha|}$$

where C_β is independent of a. Now, since f is a tempered distribution, we apply the Lemma 1.4.13 above to conclude that for $a \geq 2$ we have:

$$\int_{a-\frac{1}{2}\leq|x|\leq a+\frac{1}{2}} f(x)dx \leq \int_{\mathbb{R}^n} f(x)\psi_a(x)dx \leq \sum_{i=1}^k p_{\alpha_i,\beta_i}(\psi_a) \leq C(a + 1)^N$$

for some N, and C independent of a. That is, the integral of f on the annulus $\{a - \frac{1}{2} \leq| x |\leq a + \frac{1}{2}\}$ is of polynomial growth in a. From this it is easy to check that f is a tempered function.

As a consequence of the above discussion, a function $f \in L_{1,loc}$ is a tempered function iff $| f |$ is a tempered distribution.

2. Distributions and Partial Differential Equations

2.1. **Motivation from Electrostatics.** We recall that in electromagnetism, the Maxwell equations imply that for a smooth charge distribution $g \in C^\infty$, the scalar electrostatic potential is given by a function ϕ, where ϕ is a solution to the *inhomogeneous Laplace equation*

$$\Delta\phi = -\sum_{i=1}^3 \partial_i^2 \phi = g$$

Classically, it was known that the potential due to a unit point charge at the origin was given by $\phi(x) = C | x |^{-1}$ by the inverse square law, so the potential at x due to the "infinitesimal" charge element $g(y)dy$ situated at y

would be $C \mid x-y \mid^{-1} g(y)dy$. Since the scalar potential is additive, the total potential at x due to the entire charge distribution would be the integral:

$$\phi(x) = C \int \mid x-y \mid^{-1} g(y)dy \tag{3}$$

This looks like the convolution of the "function" $C \mid x \mid^{-1}$ and g. Only $C \mid x \mid^{-1}$ is not a function. It is however, a tempered distribution, indeed it is a tempered function as is easily checked by using polar coordinates. So although the expression in (3) above doesn't quite make sense unless we justify the convergence of the integral above, we can try to see if it can be recast as a convolution of a the tempered distribution $C \mid x \mid^{-1}$ which will (a) rigorise and (b) generalise the above heuristic argument.

2.2. **Fundamental solutions.**

Definition 2.2.1. Let L be a linear differential operator with constant coefficients on $\mathbb{R}^n$. That is, $L = P(D)$ where P is an n-variable polynomial. We say that the distribution T is a *fundamental solution of* L if

$$LT = \delta_0$$

as an identity of distributions. We are not suggesting that they exist in general. If, however, T is a *tempered* distribution, then its Fourier transform $\widehat{T}$ must also be a tempered distribution, so also LT. By taking the Fourier transform of $LT = \delta_0$ and applying (ii) of Proposition 1.4.12 and Corollary 1.4.16 we see that $\widehat{T}$ must satisfy the identity:

$$P(\xi)\widehat{T} = 1$$

of tempered distributions. More on this later.

The reason to look for fundamental solutions is the following proposition.

Proposition 2.2.2. Let $g \in \mathcal{S}$ be a rapidly decreasing function, and assume that T is a tempered distribution which is a fundamental solution of $L = P(D)$. Then the smooth function $\phi := g * T$ is a smooth solution of $L\phi = g$. Similarly, if $g \in \mathcal{D}$ is a compactly supported function and T is any distributional fundamental solution to L.

Proof: By the Lemma 1.4.7, we have in all the cases cited above that ϕ is a smooth function. Furthermore, by the same lemma, and the Example 1.4.6, we have:

$$L(\phi) = P(D)(g * T) = g * P(D)T = g * \delta_0 = g$$

This proves our proposition. □

This is a "soft analysis" method of solving the inhomogeneous equation $L\phi = g$, *given* a fundamental solution. Finding a fundamental solution, however, is not a "soft" activity. We illustrate with a few examples below.

Let us define the following linear first order differential operators on $\mathbb{R}^2$:

$$\overline{\partial} := \frac{1}{2}\left(\partial_x + i\partial_y\right), \quad \partial := \frac{1}{2}\left(\partial_x - i\partial_y\right)$$

(The operator $\overline{\partial}$ is called the *Cauchy-Riemann operator*). Note that $4\partial\overline{\partial} = 4\overline{\partial}\partial = -\Delta$ where $\Delta = -\partial_x^2 - \partial_y^2$ is the Laplace operator on the plane.

Proposition 2.2.3 (Cauchy Problem)**.** On $\mathbb{R}^2$, the tempered distribution $2(x+iy)^{-1} = 2/z$ is a fundamental solution to $\overline{\partial}$. The tempered distribution $-\log \mid z \mid$ is a fundamental solution to Δ.

Proof: We recall that our volume element on $\mathbb{R}^2$ is $dV = (2\pi)^{-1}dxdy$. By using polar coordinates, for example, $dV = (2\pi)^{-1}rdrd\theta$, and it is readily verified that $2/z$ and $\log \mid z \mid$ are tempered functions, and hence tempered distributions by Example 1.3.3.

For $f = P + iQ$ a complex valued function, the 1-form fdz on $\mathbb{R}^2$ denotes $(P + iQ)(dx + idy) = (Pdx - Qdy) + i(Qdx + Pdy)$. If $W \subset \mathbb{R}^2$ is any open set, and $\Omega \subset W$ is a compact domain with smooth boundary $\partial\Omega$, then we have the *Green Formulas:*

$$\int_{\partial\Omega} Qdx + Pdy = 2\pi \int_{\Omega} (\partial_x P - \partial_y Q)\, dV, \quad \int_{\partial\Omega} Pdx - Qdy = -2\pi \int_{\Omega} (\partial_x Q + \partial_y P)\, dQ$$

Since $\overline{\partial} f = \frac{1}{2}(\partial_x + i\partial_y)(P + iQ) = \frac{1}{2}(\partial_x P - \partial_y Q) + i(\partial_y P + \partial_x Q)$, we can write the two Green formulas above for the 1-form fdz as the single formula:

$$\int_{\partial\Omega} fdz = 4\pi i \int_{\Omega} (\overline{\partial} f) dV \tag{4}$$

Now we claim that the tempered distribution $\frac{2}{z}$ is a fundamental solution of $\overline{\partial}$ on $\mathbb{R}^2$.

For, let $\phi \in \mathcal{S}$ be a smooth function. Then note that on $\mathbb{R}^2 \setminus \{0\}$, the smooth function $\frac{2}{z}$ is holomorphic, so that on $\mathbb{R}^2 \setminus \{0\}$ we have by the Leibnitz formula that $\overline{\partial}(2\phi/z) = (2/z)\overline{\partial}\phi$. For $\epsilon > 0, R > 0$, let $\Omega_{\epsilon,R} \subset \mathbb{R}^2 \setminus \{0\}$ denote the annulus $\epsilon \leq\mid z \mid\leq R$. Choose $R >> 0$ so that the support of ϕ is contained in $B(0, R)$. We apply Green's formula (4) to the function $f(z) = 2\phi/z$, $W = \mathbb{R}^2 - \{0\}$ and $\Omega = \Omega_{\epsilon,R}$, to obtain:

$$\begin{aligned} \int_{\Omega_{\epsilon,R}} \left(\frac{2}{z}\right) \overline{\partial}\phi dV &= \int_{\Omega_{\epsilon,R}} \overline{\partial}(2\phi/z) dV = \int_{\Omega_{\epsilon,R}} \overline{\partial} f dV \\ &= \frac{1}{4\pi i} \left[\int_{S(R)} f(z)dz - \int_{S(\epsilon)} f(z)dz \right] \\ &= \frac{i}{2\pi} \int_{S(\epsilon)} \frac{\phi(z)}{z} dz \end{aligned}$$

where $S(r)$ denotes the circle of radius r centred at the origin and oriented counterclockwise, and the integral over $S(R)$ vanishes because $\phi \equiv 0$ on $S(R)$ by the choice of R. From the fact that $\int_{S(r)} dz/z = 2\pi i$, it follows that:

$$\int_{\mathbb{R}^2} \frac{2}{z} \overline{\partial}\phi dV = \lim_{\epsilon \to 0} \int_{\Omega_{\epsilon,R}} \frac{2}{z} \overline{\partial}\phi dV = \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{S(\epsilon)} \frac{\phi(z)}{z} dz = -\phi(0)$$

From this it follows that:

$$\overline{\partial}(2/z)(\phi) = -(2/z)(\overline{\partial}\phi) = -\int_{\mathbb{R}^2} \left(\frac{2}{z}\right) \overline{\partial}\phi dV = \phi(0) = \delta_0(\phi)$$

for all $\phi \in \mathcal{S}$. Thus $\overline{\partial}(2/z) = \delta_0$, and the assertion for the Cauchy Riemann operator follows.

The statement for the Laplacian follows by first checking that $\log \mid z \mid$ is a tempered distribution, and obeys the distributional identity:

$$4\partial \log \mid z \mid= (\partial_x - i\partial_y)(\log(x^2 + y^2)) = \frac{2}{z}$$

as distributions on $\mathbb{R}^2$. This is clear enough as an identity of functions on $\mathbb{R}^2 \setminus \{0\}$, but has to be verified as an identity of distributions on $\mathbb{R}^2$, which involves using the annuli $\Omega_{\epsilon,R}$ etc., and writing down a ∂ analogue of the $\overline{\partial}$ Green's formula that we had in (4) above. We leave these details to the reader.

Then it follows that:

$$\Delta(-\log \mid z \mid) = 4\overline{\partial}\partial(-\log \mid z \mid) = \overline{\partial}(4\partial \log \mid z \mid) = \overline{\partial}(2/z) = \delta_0$$

by the fact that $(2/z)$ is a fundamental solution to $\overline{\partial}$ proved above. The proposition follows. □

Proposition 2.2.4 (Fundamental solutions to Δ on $\mathbb{R}^n$, $n \neq 2$)**.** Let $n \neq 2$. A fundamental solution to Δ on $\mathbb{R}^n$ is given by the tempered distribution:

$$T = \frac{(2\pi)^{n/2} r^{-n+2}}{(2 - n)\omega_{n-1}}$$

where $\omega_{n-1} := \operatorname{Vol} S^{n-1}$.

Proof: There is the following special case of Stokes's Theorem (=Gauss's divergence theorem) for a domain Ω with smooth boundary $\partial\Omega$ contained in an open subset $U \subset \mathbb{R}^n$, and a smooth vector field $\mathbf{v}(x) = (v_1(x), .., v_n(x))$ on U.

$$\int_\Omega \left(\sum_i \partial_i v_i \right) dV = (2\pi)^{-n/2} \int_{\partial\Omega} (\mathbf{v}.\nu) dS$$

where dS is the induced surface measure on $\partial\Omega$ from the Euclidean measure $dx_1...dx_n$ on $\mathbb{R}^n$, and ν denotes the outward normal vector field on $\partial\Omega$. (The factor of $(2\pi)^{-n/2}$ comes because for us $dV = (2\pi)^{-n/2} dx_1...dx_n$.)

If we substitute for $\mathbf{v}$ the particular vector field $\mathbf{v} = f\nabla g$, where f, g are smooth functions on U, we have the formula:

$$\int_\Omega (\nabla f.\nabla g) dV - \int_\Omega (f\Delta g) dV = (2\pi)^{-n/2} \int_{\partial\Omega} (f\partial_\nu g) dS$$

where $\partial_\nu g := \nabla g.\nu$ is the normal derivative vector field of g on $\partial\Omega$. (Remember that $\Delta = -\sum_i \partial_i^2$).

Interchanging the roles of f and g, and subtracting, we have the *Green Formula*:

$$\int_\Omega (f\Delta g - g\Delta f) dV = (2\pi)^{-n/2} \int_{\partial\Omega} (g\partial_\nu f - f\partial_\nu g) dS \tag{5}$$

Now note that the function $f(x) := \|x\|^{-n+2} = r^{-n+2}$ is tempered, and on a radially symmetric function it is easily checked by using polar coordinates that:

$$\Delta = -\partial_r^2 - (n-1) r^{-1} \partial_r$$

so that $\Delta f = \Delta(r^{-n+2}) \equiv 0$ on $\mathbb{R}^n \setminus \{0\}$. Now let $\Omega_{\epsilon,R} = \{x : \epsilon \leq \|x\| \leq R\}$. If g is a smooth function of compact support, and $\operatorname{supp} g \subset B(0, R)$, we will have $g = \partial_\nu g \equiv 0$ on the sphere $S(R)$ of radius R. Thus from the Green formula (5) it follows that:

$$\begin{aligned} \int_{\Omega_{\epsilon,R}} f\Delta g &= -(2\pi)^{-n/2} \int_{S(\epsilon)} g\partial_\nu f dS = (2\pi)^{-n/2} \int_{S(\epsilon)} g\partial_r f dS = (2\pi)^{-n/2}(2-n) \int_{S(\epsilon)} g.r^{-n+1} dS \\ &= (2\pi)^{-n/2}(2-n) \int_{S(\epsilon)} g.\epsilon^{1-n} dS = (2\pi)^{-n/2}(2-n) \int_{S(1)} g(\epsilon x) dS \end{aligned}$$

It is clear that as $\epsilon \to 0$, the expression above converges to

$$(2\pi)^{-n/2}(2-n)\omega_{n-1} g(0)$$

where $\omega_{n-1} := \operatorname{Vol} S(1)$ is the volume of the unit sphere in $\mathbb{R}^n$. Thus we have:

$$(\Delta f)(g) = \int_{\mathbb{R}^n} f\Delta g dV = (2\pi)^{-n/2}(2-n)\omega_{n-1}\delta_0(g)$$

which shows that a fundamental solution is as asserted. □

Remark 2.2.5. In general there is nothing unique about a fundamental solution. For example, since the Cauchy-Riemann operator $\overline{\partial}$ annihilates every holomorphic function f, the distribution $2/z + f(z)$ is also a fundamental solution for $\overline{\partial}$. Likewise for the Laplacian Δ, adding on a harmonic function (i.e. a function annihilated by Δ) will also provide a fundamental solution. For a general constant coefficient linear partial differential operator L, *all* the fundamental solutions constitute the affine space:

$$\phi + \ker\{L : \mathcal{D}' \to \mathcal{D}'\}$$

where ϕ is one fundamental solution. It is a consequence of the *elliptic regularity theorem* to be proved later that any distribution in the kernel of $\overline{\partial}$ or Δ is actually a function, a smooth function in fact.

3. Sobolev Theory

We will define certain Hilbert spaces which provide the ideal ones for studying differential operators, and more generally the "pseudodifferential operators" to be introduced later.

3.1. **Sobolev Spaces.**

Definition 3.1.1. Let $s \in \mathbb{R}$. The *Sobolev space* $H_s(\mathbb{R}^n)$ is defined as:

$$H_s(\mathbb{R}^n) = \{f \in \mathcal{S}' : \widehat{f} \text{ is a measurable function and } \int_{\mathbb{R}^n} (1+ \mid \xi \mid^2)^s \mid \widehat{f}(\xi) \mid^2 d\xi < \infty\}$$

For $f, g \in H_s(\mathbb{R}^n)$, their *Sobolev inner-product* is defined by

$$(f,g)_s := \int_{\mathbb{R}^n} (1+ \mid \xi \mid^2)^s \widehat{f}(\xi)\overline{\widehat{g}}(\xi) d\xi$$

which is finite by applying the Cauchy-Schwartz inequality.

Remark 3.1.2.

(i): By the Plancherel Theorem in (iv) of 1.2.5, we have $H_0(\mathbb{R}^n) = L_2(\mathbb{R}^n)$.

(ii): Note that for any $s \in \mathbb{R}$, we have that the function $\rho_s(\xi) := (1+ \mid \xi \mid^2)^{s/2}$ is a slowly increasing function. Thus mlutiplication by this function is an isomorphism $\mathcal{S} \to \mathcal{S}$. Then if we define the linear operator:

$$\begin{aligned} \Lambda_s : \mathcal{S}' &\to \mathcal{S}' \\ f &\mapsto (\rho_s \widehat{f})^\vee \end{aligned}$$

it follows that this operator is a continuous isomorphism (with inverse Λ_{-s}). Hence, in view of the Plancherel Theorem (iv) of 1.2.5, we have the description:

$$H_s = \{f \in \mathcal{S}' : \Lambda_s f \in L_2(\mathbb{R}^n)\}$$

and since Λ_s is an isomorphism, it follows that H_s is isomorphic to L_2 as a Hilbert space. In particular it is a separable Hilbert space.

(iii): For each $t \leq s$, we have $(1+ \mid \xi \mid^2)^t \leq (1+ \mid \xi \mid^2)^s$, so $H_s \subset H_t$ for all $s \geq t$.

(iv): If $T \in \mathcal{E}'$ is a *compactly supported* distribution,then by the Proposition 1.4.15, it follows that $\widehat{T}$ is a function (namely $T(e_{-\xi})$) which is of slow growth. That means its modulus square is also a function of slow growth, and it will be integrable against $(1+ \mid \xi \mid^2)^s$ for some s. Hence T will be in the corresponding $H_s(\mathbb{R}^n)$. On the other hand every non-compactly supported Schwartz class function in in each H_s, but not in $\mathcal{E}'$.

(v): Not every tempered distribution is in some H_s. For, the constant function 1 is a tempered distribution, but since its Fourier transform is δ_0, which is not a function, 1 does not belong to any H_s. Thus in view of (iv) above, we have strict containments:

$$\mathcal{E}' \subset H_{-\infty} := \cup_{s\in\mathbb{R}} H_s \subset \mathcal{S}'$$

(vi): Since Λ_s is an isomorphism on $\mathcal{S}$, and so $\Lambda_s f \in L_2$ for every $f \in \mathcal{S}$ and every s, it follows that:

$$\mathcal{S} \subset H_\infty := \cap_{s\in\mathbb{R}} H_s$$

Also since Λ_s is a Hilbert space isometry of H_s to $H_0 = L_2$, $\Lambda_s(\mathcal{S}) = \mathcal{S}$, and $\mathcal{S}$ is dense in L_2, it follows that $\mathcal{S}$ is dense in each H_s.

(vii): However, the containment:

$$\mathcal{S} \subset H_\infty := \cap_s H_s$$

is also strict. For example, the function $f(x) = (1+x^2)^{-1}$ on $\mathbb{R}$ has the Fourier transform $e^{-|\xi|}$, which is integrable against all powers of $(1+ \mid \xi \mid^2)^s$, so $f \in H_\infty$. However, $f \notin \mathcal{S}$, because its Fourier transform $e^{-|\xi|}$ is not smooth, so not in $\mathcal{S}$.

Exercise 3.1.3. Prove the slightly stronger statement than (vi) above, viz. that $\mathcal{D} = C_c^\infty$ is dense in each H_s. Thus in view of (v) and this statement, one could also define H_s as the completion of the inner product spaces $\mathcal{D}$ or $\mathcal{S}$ with respect to the Sobolev inner product $(-,-)_s$.

We now have a very elementary proposition about these Sobolev Spaces.

Proposition 3.1.4 (Some Facts on Sobolev Spaces)**.**

(i): The inclusion $H_s \hookrightarrow H_t$ for $s \geq t$, defined in (iii) of 3.1.2 is a continuous (=bounded) operator. If $f \in \mathcal{S}$, the multiplication operator $u \mapsto fu$ is a continuous (=bounded) operator.

(ii): If $m \geq 0$ is a non-negative integer, then on the vector subspace $\mathcal{E} \cap H_m(\mathbb{R}^n)$ of H_m, the Sobolev m-norm is equivalent to the norm defined by:

$$\|f\|^2 = \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |\ D_x^\alpha f\ |^2\ dx$$

Thus for such an m, H_m can be described as the completion of $\mathcal{S}$ or $\mathcal{D}$ with respect to this norm.

(iii): If P is a polynomial of degree k, then for the linear constant coefficient differential operator $P(D)$, we have:

$$P(D) : H_s \to H_{s-k}$$

is a continuous(=bounded) operator of Hilbert spaces.

(iv): The sesquilinear pairing:

$$\begin{array}{rcl} \mathcal{S} \times \mathcal{S} & \to & \mathbb{C} \\ f, g & \mapsto & \displaystyle\int_{\mathbb{R}^n} f(x)\overline{g}(x)dx = \langle f, g \rangle \end{array}$$

extends to a sesquilinear pairing of $H_s \times H_{-s}$, also denoted $\langle -,- \rangle$ which satisfies:

$$|\ \langle f, g \rangle\ | \leq \|f\|_s \|g\|_{-s}\ ; \qquad \|f\|_s = \sup_{0 \neq g \in H_s} \frac{|\ \langle f, g \rangle\ |}{\|g\|_{-s}}$$

$\langle -,- \rangle$ is therefore a perfect pairing and identifies H_{-s} with the Hilbert space dual $(H_s)^*$ of H_s.

Proof:

(i) is trivial from the fact for $t \leq s$ we have $(1+ |\ \xi\ |^2)^t \leq (1+ |\ \xi\ |^2)^s$ and all ξ. The second statement is also straightforward, and left as an exercise.

For (ii), we note that there exists a constant C such that:

$$\frac{1}{C}(1+ |\ \xi\ |^2)^m \leq \sum_{\alpha \leq m} |\ \xi^{2\alpha}\ | \leq C(1+ |\ \xi\ |^2)^m, \quad \xi \in \mathbb{R}^n$$

from which it follows by (ii) of the Proposition 1.2.5 that:

$$\frac{1}{C}(1+ |\ \xi\ |^2)^m\ |\ \widehat{f}(\xi)\ |^2 \leq \sum_{\alpha \leq m} |\ (D^\alpha f)^\wedge(\xi)\ |^2 \leq C(1+ |\ \xi\ |^2)^m\ |\ \widehat{f}(\xi)\ |^2$$

and all $\xi \in \mathbb{R}^m$. The result follows by integrating the above two inequalities over $\mathbb{R}^n$, and the Plancherel Theorem (iv), 1.2.5.

(iii) is also clear from the fact that $(P(D)f)^\wedge(\xi) = P(\xi)\widehat{f}(\xi)$, and that $|\ P(\xi)\ |^2 \leq C(1+ |\ \xi\ |^2)^k$ for some $C > 0$ and all $\xi \in \mathbb{R}^n$, if $k = \deg P$.

To see (iv), note that for $f, g \in \mathcal{S}$, we have by Plancherel:

$$\langle f, g \rangle = \left\langle \widehat{f}, \widehat{\overline{g}} \right\rangle = \int \widehat{f}(\xi)(1+ |\ \xi\ |^2)^{s/2}\ \widehat{\overline{g}}(\xi)(1+ |\ \xi\ |^2)^{-s/2} d\xi \leq \|f\|_s \|\overline{g}\|_{-s} = \|f\|_s \|g\|_{-s}$$

by using the Cauchy-Schwartz inequality. To see that equality is achieved in the inequality, choose g such that $\widehat{\widehat{g}} = \widehat{f}(1+ \mid \xi \mid^2)^s$. This yields the rest of (iv), and the proposition follows. □

Remark 3.1.5. (iii) of the Proposition 3.1.4 above is the reason for introducing Sobolev spaces, i.e. in order to view differential operators as being bounded operators between Hilbert spaces.

3.2. **Sobolev Embedding Theorem.** There is a criterion for a function to be a k times continuously differentiable function which can be stated in terms of Sobolev spaces.

Proposition 3.2.1 (Sobolev Embedding Theorem or Sobolev Lemma)**.** Let $k \geq 0$ be an integer. If $s > k + \frac{n}{2}$, then:

- (i): $H_s(\mathbb{R}^n) \subset C_0^k(\mathbb{R}^n)$, where the right hand space denotes the space of k times continuously differentiable functions f with $D_x^\alpha f$ vanishing at ∞ for all $\mid \alpha \mid \leq k$.
- (ii): $\|D_x^\alpha f\|_\infty \leq C_\alpha \|f\|_s$. Indeed if we define the norm
$$\|f\|_{\infty,k} = \sup_{|\alpha|\leq k} \|D_x^\alpha f\|_\infty$$
on $C_k^\infty(\mathbb{R}^n)$, then the inclusion $H_s \subset C_0^k$ of (i) above is continuous.

Proof: We first make the following:

Claim: If $f \in \mathcal{S}'$ is a tempered distribution such that $(D_x^\alpha f)^\wedge$ is a function in $L_1(\mathbb{R}^n)$, then $f \in C_0^k(\mathbb{R}^n)$. Also, $\|D_x^\alpha f\|_\infty = \sup_{x\in\mathbb{R}^n} |D_x^\alpha f| \leq \|(D_x^\alpha f)^\wedge\|_1$.

If $g := (D_x^\alpha f)^\wedge \in L^1$, then by the Riemann-Lebesgue Lemma (v) of Proposition 1.2.5, we have $D_x^\alpha f = g^\vee$ is in $C_0(\mathbb{R}^n)$. The last statement is clear from the fact that $\|g^\vee\|_\infty \leq \|g\|_1$.

In view of the above claim, all we have to do is show that if $f \in H_s(\mathbb{R}^n)$ for $s \geq k + \frac{n}{2}$, then $(D_x^\alpha f)^\wedge$ is an L_1 function. But then:

$$\begin{aligned}\int |(D_x^\alpha f)^\wedge(\xi)|\, d\xi &= \int \mid \xi \mid^\alpha \mid \widehat{f}(\xi) \mid \xi = \int \mid \xi \mid^\alpha (1+ \mid \xi \mid^2)^{-s/2}(1+ \mid \xi \mid^2)^{s/2} \mid \widehat{f}(\xi) \mid d\xi \\ &\leq \left(\int \mid \xi \mid^{2\alpha} (1+ \mid \xi \mid^2)^{-s} d\xi\right)^{1/2} \|f\|_s\end{aligned}$$

by the Cauchy-Schwartz inequality. Since:

$$\mid \xi \mid^{2|\alpha|} (1+ \mid \xi \mid^2)^{-s} \leq (1+ \mid \xi \mid^2)^{k-s}$$

and $s - k > n/2$, the integral $\int \mid \xi \mid^{2\alpha} (1+ \mid \xi \mid^2)^{-s} d\xi$ is finite, and we therefore have:

$$\|(D_x^\alpha f)^\wedge\|_1 \leq C \|f\|_s$$

which implies by the Claim above that:

$$\|D_x^\alpha f\|_\infty \leq C \|f\|_s \quad \text{for all} \quad \mid \alpha \mid \leq k$$

This proves both (i) and (ii) and the proposition follows. □

Corollary 3.2.2. As a consequence of the entire subsection,we have the following chain of inclusions:

$$\mathcal{S} \subset H_\infty \subset H_{-\infty} \subset \mathcal{S}'$$

each of which is strict. Note also that by the Sobolev Lemma above,

$$H_\infty \subset C_0^\infty$$

Note also that the Dirac distribution δ_0 belongs to H_{-s} for all $s > \frac{n}{2}$ since $\widehat{\delta_0} = 1$, and

$$\|\delta_0\|_{-s} = \int (1+ \mid \xi \mid^2)^{-s} d\xi < \infty$$

for all $s > \frac{n}{2}$. In general, the more negative the s, the more singular the tempered distributions that will be included in H_s.

Remark 3.2.3. The Sobolev Lemma above is crucial for proving regularity (smoothness) of distributional solutions to elliptic differential operators.

Exercise 3.2.4. By the Sobolev Lemma, $H_\infty \subset C_0^\infty$. Is C_0^∞ a subset of $H_{-\infty}$?

3.3. **Rellich's Lemma.** The other crucial lemma about the Sobolev spaces is a statement about the inclusion $H_s \subset H_t$ for $s > t$. Before we prove it we state the following lemma about locally compact metric spaces.

Proposition 3.3.1 (Arzela-Ascoli Theorem)**.** Let X be a locally compact σ-compact metric space (σ-compactness means X is a countable union of compact subsets). Let $\{f_k\}$ be a sequence of complex valued functions satisfying:

- (i): $\{f_k\}$ is *equicontinuous*. That is for each $x \in X$ and each $\epsilon > 0$, there is a neighbourhood U_x of x such that $\mid f_k(x) - f_k(y) \mid < \epsilon$ for all $y \in U_x$ and all k.
- (ii): $\{f_k\}$ is *pointwise bounded*, i.e. the set $\{f_k(x) : k \in \mathbb{N}\}$ is a bounded set for each $x \in X$.

Then there exists a function $f \in C(X)$ and a subsequence $\{f_{k_m}\}$ of $\{f_k\}$ such that $f_{k_m} \to f$ uniformly on compact sets.

Proof: See Rudin's *Real and Complex Analysis*, or Folland's *Real Analysis: Modern Techniques and their Applications.* □

Proposition 3.3.2 (Rellich's Lemma)**.** Let $s > t$, so that $H_s \subset H_t$. Let $\{f_k\}$ be a sequence in H_s such that:

- (i): There exists a compact set K such that for all k, the support of (the tempered distribution) f_k is contained in K.
- (ii): $\{f_k\}_{k\in\mathbb{N}}$ is a bounded set in H_s.

Then there is a subsequence of $\{f_k\}$ which converges in H_t.

Proof: First note that for $\xi,\ \eta \in \mathbb{R}^n$, we have by the triangle inequality that:

$$\mid \xi \mid^2 \leq 2(\mid \xi - \eta \mid^2 + \mid \eta \mid^2)$$

which implies that:

$$(1+ \mid \xi \mid^2) \leq 2(1+ \mid \xi - \eta \mid^2)(1+ \mid \eta \mid^2)$$

Thus if $s \geq 0$, we have:

$$(1+ \mid \xi \mid^2)^{s/2} \leq C(1+ \mid \xi - \eta \mid^2)^{s/2}(1+ \mid \eta \mid^2)^{s/2}$$

where C is a constant depending on s. Similarly, if $s < 0$, we can apply the above inequality to $\mid s \mid = -s$ and interchange the roles of ξ and η to obtain the so called *Peetre inequality* for all s

$$(1+ \mid \xi \mid^2)^{s/2} \leq C(1+ \mid \xi - \eta \mid^2)^{|s|/2}(1+ \mid \eta \mid^2)^{s/2}$$

Since $f_k \in H_s$, by definition $\widehat{f_k}$ is a function for each k. Let $\phi \in \mathcal{D}$ be a smooth compactly supported function which is $\equiv 1$ on K. Then $f_k = \phi f_k$ as distributions, and by (iv) of the Proposition 1.4.12 we have $\widehat{f_k} = \widehat{\phi} * \widehat{f_k}$. Thus:

$$\left|\widehat{f_k}(\xi)\right| = \mid (\widehat{\phi} * \widehat{f_k})(\xi) \mid = \left|\int \widehat{\phi}(\xi - \eta)\widehat{f_k}(\eta)d\eta\right| \leq \int \left|\widehat{\phi}(\xi - \eta)\widehat{f_k}(\eta)\right| d\eta$$

which together with Peetre's inequality above implies that:

$$(1+ \mid \xi \mid^2)^{s/2} \mid \widehat{f_k}(\xi) \mid \leq C \int \mid \widehat{\phi}(\xi - \eta)(1+ \mid \xi - \eta \mid^2) \mid^{|s|/2} \mid (1+ \mid \eta \mid^2)^{s/2}\widehat{f_k}(\eta) \mid d\eta$$

for all s. Applying the Cauchy-Schwartz inequality to the integral on the right, we have:

$$(1+ \mid \xi \mid^2)^{s/2} \mid \widehat{f_k}(\xi) \mid \leq C \left\|\phi\right\|_{|s|} \left\|f_k\right\|_s \leq C' \quad \text{for all } \ k \tag{6}$$

since $\{f_k\}$ is a bounded sequence in H_s.

We note that since $\widehat{f_k}$ are compactly supported distributions, by the Proposition 1.4.15 they are *smooth* functions. So, similarly, we have $d_j\widehat{f_k} = d_j(\widehat{\phi} * \widehat{f_k}) = d_j\widehat{\phi} * \widehat{f_k}$, and again a corresponding argument shows that:

$$(1+ \mid \xi \mid^2)^{s/2} \mid d_j\widehat{f_k}(\xi) \mid \leq C'' \quad \text{for all} \quad k$$

This shows that $\widehat{f_k}$ and $d_j\widehat{f_k}$ are both uniformly bounded sequences of functions on each compact $L \subset \mathbb{R}^n$. In particular, the sequence $\widehat{f_k}$ is pointwise bounded, and the condition (ii) of the Arzela-Ascoli Theorem 3.3.1 is satisfied.

The uniform boundedness of all $d_j\widehat{f_k}$ implies by the Mean Value Theorem that on each compact $L \subset \mathbb{R}^n$, we have a uniform Lipschitz constant C satisfying:

$$\mid \widehat{f_k}(x) - \widehat{f_k}(y) \mid \leq C \, \|x - y\|$$

for all $x, y \in L$ and all k. This shows that the sequence of functions $\{\widehat{f_k}\}$ is equicontinuous, and condition (i) of Arzela-Ascoli is satisfied. Thus there is a subsequence of $\{\widehat{f_k}\}$ which converges uniformly on compact subsets of $\mathbb{R}^n$. For notational convenience, denote this subsequence by $\{\widehat{f_k}\}$ as well.

Thus, for $t < s$, we have:

$$\begin{aligned} \|f_j - f_k\|_t^2 &= \int \mid \widehat{f_j}(\xi) - \widehat{f_k}(\xi) \mid^2 (1+ \mid \xi \mid^2)^t d\xi \\ &= \int_{|\xi| \geq r} \mid \widehat{f_j}(\xi) - \widehat{f_k}(\xi) \mid^2 (1+ \mid \xi \mid^2)^t d\xi + \int_{|\xi| \leq r} \mid \widehat{f_j}(\xi) - \widehat{f_k}(\xi) \mid^2 (1+ \mid \xi \mid^2)^t d\xi \end{aligned} \tag{7}$$

for all $r > 0$.

Since $t - s < 0$, we have:

$$(1+ \mid \xi \mid^2)^t = (1+ \mid \xi \mid^2)^{t-s}(1+ \mid \xi \mid^2)^s \leq (1 + r^2)^{t-s}(1+ \mid \xi \mid^2)^s \quad \text{for} \quad \xi \geq r$$

Thus, by the equation (6) (applied to $\widehat{f_j} - \widehat{f_k}$ replacing $\widehat{f_k}$), we get that the first integral in (7) is majorised by $C(1 + r^2)^{t-s}$ for some $C > 0$.

Given $\epsilon > 0$, choose r large enough that $C(1 + r^2)^{t-s} < \epsilon$ then the first integral is $< \epsilon$. The second integral is $< \epsilon$ by choosing k and j large enough, since $\widehat{f_k}$ converges uniformly on the compact set $\{\xi \leq r\}$. This shows that $\{f_k\}$ is a Cauchy sequence in H_t, which is complete, so it converges. The proposition follows. □

Exercise 3.3.3. Again show by considering a sequence of translates of a fixed function of compact support (whose supports thus march off to infinity) that the condition (i) of Rellich's Lemma cannot be dropped.

4. Globalisation to Compact Manifolds

In the sequel, M will denote a paracompact, 2nd countable, Hausdorff, oriented, C^∞-manifold of dimension n. It is well known (by using partitions of unity) that such a manifold has a Riemannian metric on its tangent bundle TM, and by duality, on its cotangent bundle T^*M. We will very soon specialise to M compact.

4.1. Smooth vector bundles and sections.

Definition 4.1.1 (Smooth vector bundles)**.** A smooth manifold pair $\pi : E \to M$, with π a smooth surjective submersion is called a *smooth (or C^∞) real (resp. complex) vector bundle of rank k* if:

(i): For each $x \in M$, the fibre $E_x := \pi^{-1}(x)$ is a real (resp. complex) vector space of real (resp. complex) dimension k.

(ii): There exists an open covering $\{U_i\}_{i=1}^\infty$ of M and smooth diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{F}^k$ (where $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{C}$)) making the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{F}^k \\ \pi \searrow & & \swarrow \mathrm{pr}_1 \\ & U_i & \end{array}$$

where pr_1 denotes projection into the first factor.

(iii): For each $x \in U_i$, and all i, the composite map:

$$E_x = \pi^{-1}(x) \xrightarrow{\phi_i} \{x\} \times \mathbb{F}^k \to \mathbb{F}^k$$

is a linear isomorphism of real (esp. complex) vector spaces.

The smooth diffeos ϕ_i are called *local charts* or *local trivialisations* for the bundle, E is called the *total space* and M the *base space* of the bundle. The conditions (ii) and (iii) above simply say that the restricted bundles $E_{|U_i} : \pi^{-1}(U_i) \to U_i$ are trivial (i.e. product) bundles. When no confusion is likely, one simply writes E to denote the bundle, instead of $\pi : E \to M$.

A smooth map $s : M \to E$ is called a *smooth section* of E if $\pi \circ s = \mathrm{id}_M$. Using local trivialisations, it is easy to see that sections of the restricted bundles $E_{|U_i} \to U_i$ are in bijective correspondence with $\mathbb{F}^k$-valued smooth functions on U_i.

Example 4.1.2 (Some important bundles)**.** Important examples of natural vector bundles on a smooth real (resp. complex) n-dimensional manifold M are its real (resp. holomorphic) tangent bundle TM (resp. $T_{hol}M$) and cotangent bundle T^*M (resp. $T^*_{hol}M$. The local trivialsiations of these bundles arise naturally from a smooth atlas (resp. holomorphic atlas). We will usually be taking a real manifold of dimension n and complexifying its real tangent and cotangent bundles, which will then become complex vector bundles of rank n denoted respectively by $T_\mathbb{C}M$ and $T^*_\mathbb{C}M$. When M happens to a complex manifold of complex dimension n, it can be viewed as a real manifold of dimension $2n$, and $T_\mathbb{C}M = T_{hol}M \oplus \overline{T_{hol}M}$ and $T^*_\mathbb{C}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}$ is the complex dual of $T_{hol}M$ and $T^{0,1}$ the complex dual of $\overline{T_{hol}M}$ (the conjugate bundle to $T_{hol}M$).

When one takes tensor or exterior powers of these bundles, one obtains other smooth bundles: $\otimes^k T_\mathbb{C}M$, the bundle of *contravariant k-tensors*, or $\otimes^k T^*_\mathbb{C}M$ the bundle of *covariant k-tensors*, or $\wedge^p T^*_\mathbb{C}M$, the bundle of complex valued *differential k-forms*. These associated bundles have natural trivialisations arising from the trivialisations of the tangent and cotangent bundles. For further details the reader may consult any standard differential topology or differential geometry text.

By proceeding componentwise, one easily defines the function spaces of $\mathbb{C}^k$-valued smooth functions:

$$\mathcal{E}^k(\mathbb{R}^n) = \oplus_{i=1}^k \mathcal{E}(\mathbb{R}^n)$$

and likewise $\mathcal{D}^k(\mathbb{R}^n)$, or $\mathcal{S}^k(\mathbb{R})^n$. So also the spaces of vector valued distributions $\mathcal{D}'^k(\mathbb{R}^n)$, tempered distributions $\mathcal{S}'^k(\mathbb{R}^n)$ and compactly supported distributions $\mathcal{E}'^k(\mathbb{R}^n)$.

Now let $E \to M$ be a smooth complex vector bundle on a paracompact real manifold M of dimension n. We can choose, by refining if necessary, a covering $\mathcal{U}$ of M by open sets $\{U_i\}_{i=1}^{\infty}$ such that:

(i): U_i is diffeomorphic to $\mathbb{R}^n$, and $\overline{U}_i$ is compact, for each i.

(ii): $E_{|U_i}$ is a trivial bundle for each i.

Choose a partition of unity λ_i subordinate to the open cover $\mathcal{U}$, so that $\operatorname{supp}\lambda_i$ is a compact subset of U_i for each i, which is possible since M is paracompact. Then, if we denote the space of smooth sections of E by $C^\infty(M,E)$, in view of (i) and (ii) above we have a natural inclusion:

$$\begin{aligned} C^\infty(M,E) &\hookrightarrow \prod_{i=1}^{\infty} \mathcal{E}^k(\mathbb{R}^n) \\ s &\mapsto (\lambda_i(\phi_i \circ s))_{i=1}^{\infty} \end{aligned}$$

Note that at each $x \in M$, only finitely many entries on the right have a non-zero value. Indeed, each compact subset $K \subset M$ meets at most finitely many $U_i's$, so that $K \cap (\operatorname{supp}\lambda_i) = \phi$ for all but finitely many i. If we denote $s_i := \lambda_i(\phi_i \circ s)$, we may define seminorms:

$$p^E_{\alpha,K}(s) = \sup_i \left(\sup_{K \cap (\operatorname{supp}\lambda_i)} \mid D^\alpha_x(s_i) \mid \right)$$

where the quantiy in brackets on the right is the usual seminorm introduced earlier for $\mathcal{E}^k(\mathbb{R}^n)$. This defines a topology on $C^\infty(M,E)$.

Exercise 4.1.3. Verify that taking $M = \mathbb{R}^n$, E a trivial real rank k vector bundle (so that $C^\infty(M,E) = \mathcal{E}^k$) and a locally finite covering $\mathcal{U}$ by open balls $\{U_i\}_{i=1}^{\infty}$ (which are diffeomorphic to $\mathbb{R}^n$), and with λ_i being a partition of unity subordinate to $\mathcal{U}$, the topology that is defined as above on $C^\infty(M,E)$ is the same as the topology introduced earlier on $\mathcal{E}^k$. (One needs to fix bounds on derivatives of λ_i on their compact supports etc.)

Similarly for $C^\infty_c(M,E)$, the space of compactly supported smooth sections of E, we have the restriction of the above inclusion:

$$\begin{aligned} C^\infty_c(M,E) &\hookrightarrow \oplus_{i=1}^{\infty} \mathcal{E}^k(\mathbb{R}^n) \\ s &\mapsto \sum_i s_i \end{aligned}$$

where the s_i are as above. We leave it as an exercise for the reader to define the topology on this space in a manner that is consistent (in the sense of the exercise above). We just remark that if $\{s_n\}$ is a sequence of smooth compactly supported sections all having support in some fixed compact set $K \subset M$, then $s_{n,i}$ above will be identically zero for all i such that $i \notin F$, where $F = \{i : U_i \cap K \neq \phi\}$ is a finite set independent of n, and for each $i \in F$, all the $s_{n,i}$ will have support inside the compact set $\operatorname{supp}\lambda_i \cap K$.

Definition 4.1.4 (Distributions on manifolds)**.** A continuous linear functional on $C^\infty_c(M,E)$ is called an *E-valued distribution* on M, and the space of these is denoted as $\mathcal{D}'(M,E)$. Similarly, a continuous linear functional on $C^\infty(M,E)$ is called a *compactly supported E-valued distribution* on M, and their space denoted $\mathcal{E}'(M,E)$. When E is the trivial rank 1 (line) bundle on M, we just write $\mathcal{D}'(M)$ (resp. $\mathcal{E}'(M)$) for the respective spaces of distributions.

When M is *compact*, $C^\infty_c(M,E) = C^\infty(M,E)$, and compactly supported E-valued distributions are exactly the same as E-valued distributions. One doesn't really need the space of tempered distributions on a manifold, their main use on $\mathbb{R}^n$ being the availability of Fourier transform, an operation that doesn't make global sense on a general manifold M.

Example 4.1.5 (Currents on a smooth manifold)**.** In the particular case when $E = \wedge^{n-p} T^*_{\mathbb{C}} M$, the space of its smooth sections $C^\infty(M, E)$ is denoted $\bigwedge^p(M, \mathbb{C})$, and such a section is called a *differential* $(n-p)$*-form.* E-valued distributions on M are known as p*-currents* on M. Likewise, *compactly supported* p*-currents* are elements of $\mathcal{E}'(M, \wedge^{n-p} T^*_{\mathbb{C}} M)$. The reason for the indexing is that one may think of a differential p-form ω as a contnuous linear functional acting on the space $\bigwedge^{n-p}_c(M)$ via integration:

$$T_\omega(\tau) := \int_M \omega \wedge \tau \quad \tau \in \wedge^p_c(M)$$

where integration of an n-form on a singular n-cube is defined for the oriented manifold M as usual, and where the support of τ can be covered by a finite union of k-cubes with the right orientations (i.e. a k-chain) etc. Clearly then, a differential p-form is a p-current by this indexing convention. Using the Stokes formula for a singular k-chain:

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

and the facts that (i) $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$, and (ii) $\tau \in \bigwedge^{n-p}_c(M)$ implies that $\tau \equiv 0$ on the boundary of a sufficiently large k-chain covering the support of τ, the reader can easily check that by defining the *distributional exterior derivative* of a p-current T by $dT(\omega) = (-1)^{p+1} T(d\omega)$ for $\omega \in \wedge^{n-p}_c(M)$ leads to the consistency formula: $dT_\omega = T_{d\omega}$.

Indeed, if we denote the space of p-currents by $\mathcal{C}^p(M, \mathbb{C})$, there is the de-Rham complex of currents:

$$... \to \mathcal{C}^p(M, \mathbb{C}) \xrightarrow{d} \mathcal{C}^{p+1}(M, \mathbb{C}) \to ....$$

with $d \circ d \equiv 0$, and the usual de-Rham complex is a subcomplex of this complex via the chain map $\omega \mapsto T_\omega$. It is a fact (using an approximation theorem analogous to the Proposition 1.4.10 proved for $\mathbb{R}^n$) that this chain map is a chain homotopy equivalence.

Similarly, the singular $(n-p)$-chain σ may be regarded as a *compactly supported* p-current via integration:

$$T_\sigma(\tau) := \int_\sigma \tau \quad \text{for} \ \ \tau \in \wedge^{n-p}(M, \mathbb{C})$$

By Stokes's theorem, the distributional derivative ∂T_σ defined by $\partial T_\sigma(\tau) = T_\sigma(d\tau)$ leads to the usual boundary operator on singular $(n-p)$-chains. In particular, an orientable $(n-p)$-dimensional submanifold N of M is an $(n-p)$ chain in M, and defines a p-current.

Analogously an infinite (Borel-Moore) locally finite $(n-p)$-chain maybe regarded as a p-current, acting on $\wedge^{n-p}_c(M, \mathbb{C})$ via the same integration formula as above. Again, the distributional derivative defined as above leads via Stokes to the usual geometric boundary. Thus p-currents (resp. compactly supported p-currents) are general enough to include both $(n-p)$-Borel-Moore chains (resp. singular p-chains) and differential p-forms (resp. compactly supported p-forms). One then shows that the cohomology of the complex of p-currents $\mathcal{C}^*(M, \mathbb{C})$ is the same as that of the Borel-Moore chain complex Δ^{BM}_{n-*}, as well as the de Rham complex $\bigwedge^*(M, \mathbb{C})$. Similarly for compactly supported currents. Thus follow the standard Poincare duality isomorphisms of the Borel-Moore homology $H^{BM}_{n-p}(M, \mathbb{C})$ and the de-Rham cohomology $H^p_{dR}(M, \mathbb{C}))$ (resp. singular homology $H_{n-p}(M, \mathbb{C})$ and compactly supported de Rham cohomology $H^p_{dR,c}(M, \mathbb{C}))$

Remark 4.1.6. In all of the above, one has chosen a particular partition of unity, and a particular kind of open covering. One needs to check that everything defined above for M is independent of these choices. One can actually define $\mathcal{E}(U)$ and $\mathcal{D}(U)$ for any open subset $U \subset \mathbb{R}^n$. Then one shows that if U is a further locally finite union of U_i, an analogue of the exercise 4.1.3 will imply that the "patching definition" of $\mathcal{D}'(U)$ or $\mathcal{E}'(U)$ is the same as the a priori definition. Then one uses common refinements, the partition of unity $\lambda_i \mu_j$ arising from different partitions of unity λ_i and μ_j, etc. to prove that these various choices are immaterial.

4.2. **Sobolev spaces on a compact manifold.** *In this section M is assumed to be compact throughout*

Definition 4.2.1. Let M be a compact manifold, and E a smooth rank k complex vector bundle on M. Again find a *finite open covering* $\{U_i\}_{i=1}^N$ satisfying:

(i): U_i is diffeomorphic to $\mathbb{R}^n$ for each i via a smooth diffeo ψ_i, and $\overline{U}_i$ is compact in M.

(ii): $E_{|U_i}$ is a trivial bundle for each i.

and let λ_i be a partition of unity subordinate to this open covering. Via (i) and (ii) above, identify U_i with $\mathbb{R}^n$, $E_{|U_i}$ with $U_i \times \mathbb{C}^k$, and using pushforward and pullback under these identifications, identify the Sobolev space $H_s(U_i, E)$ as $[H_s(\mathbb{R}^n)]^k := \oplus_{i=1}^k H_s(\mathbb{R}^n)$. There is a natural Sobolev (direct sum) inner product on this last space, and the resulting Sobolev inner product on $H_s(U_i, E)$ is denoted $(-,-)_{i,s}$.

We now define:

$$H_s(M,E) := \{f \in \mathcal{E}'(M,E) = \mathcal{D}'(M,E) : \lambda_i f \in H_s(U_i,E) \quad \text{for each} \quad i = 1,2,..,N\}$$

In fact, we can define the *Sobolev inner product* on $H_s(M,E)$ by the formula:

$$(f,g)_s := \sum_i (\lambda_i f, \lambda_i g)_{i,s}$$

Equip M with a Riemannian metric g, which will be fixed once and for all. By the orientability of M there results the global non-vanishing smooth section in $\bigwedge^n(M,\mathbb{C})$ called the *Riemannian volume form*, defined in a local coordinate system by:

$$dV(x) := \sqrt{\det\, g_{ij}(x)}dx_1 \wedge dx_2 \wedge .... \wedge dx_n$$

where $g_{ij} := g(\partial_i, \partial_j)$ is the Gramm matrix of the metric. It is readily checked that the expression above for dV is independent of the coordinate chart.

Similarly, one may equip the complex vector bundle E with a Hermitian bundle metric denoted $\langle -,- \rangle$. If f, g are sections in $C^\infty(M,E)$, the function $\langle f(x), g(x) \rangle$ is a smooth $\mathbb{C}$ valued function of $x \in M$, and we may define the *global inner product:*

$$(f,g) := \int_M \langle f(x), g(x) \rangle \, dV(x)$$

which is finite since M is compact. This makes $C^\infty(M,\mathbb{C})$ a complex inner-product space, and we denote its completion by $L_2(M,E)$, the space of all measurable square integrable sections of E.

We can apply the results of the previous subsection and easily deduce the following:

Proposition 4.2.2 (Facts on Sobolev spaces on manifolds)**.**

(i): $H_0(M,E) \equiv L_2(M,E)$ as Hilbert spaces.

(ii): $C^\infty(M,E)$ is dense in $H_s(M,E)$ for each $s \in \mathbb{R}$.

(iii): The sesquilinear pairing:

$$\begin{array}{rcl} C^\infty(M,E) \times C^\infty(M,E) & \to & \mathbb{C} \\ f,g & \mapsto & (f,g) = \int_M \langle f(x), g(x) \rangle \, dV(x) \end{array}$$

extends to a sesquilinear pairing $H_s(M,E) \times H_{-s}(M,E) \to \mathbb{C}$ and identifies $H_{-s}(M,E)$ as the Hilbert space dual $[H_s(M,E)]^*$.

(iv): (Sobolev Embedding Theorem) There is a continuous inclusion $H_s \hookrightarrow C^k(M,\mathbb{C})$ for $s > k + n/2$. This implies that $H_\infty(M,E) := \cap_{s\in\mathbb{R}} H_s(M,E) \subset C^\infty(M,E)$. Since $C^\infty(M,E) \subset H_s(M,E)$ for all s, we have the *equality* $H_\infty(M,E) = C^\infty(M,E)$.

(v): $H_{-\infty}(M,E) := \cup_{s\in\mathbb{R}} H_s(M,E) = \mathcal{D}'(M,E)$

(vi): (Rellich's Lemma) For $s > t$, the inclusion:

$$H_s(M, E) \to H_t(M, E)$$

is a compact operator, viz. every bounded sequence in H_s has a convergent subsequence in H_t.

Proof: Let $\{U_i\}_{i=1}^N$ and λ_i be as in the beginning of this subsection. Since $K_i := \operatorname{supp} \lambda_i$ are compact subsets of U_i, the measure $dV(x)$ and the Lebesgue measure on $U_i \simeq \mathbb{R}^n$ are equivalent on K_i. Similarly, the Hermitian bundle metric $\| \;\|$ on E and the Euclidean metric on $\mathbb{C}^k$ are equivalent on K_i. Hence, for a smooth section $f \in C^\infty(M, E)$, we see that the L_2-norm squared $\int_M \langle \lambda_i f, \lambda_i f \rangle \, dV(x)$ is equivalent to the Euclidean L_2-norm squared of $\lambda_i f$ regarded as an element of $\mathcal{E}^k$. Since $i = 1, .., N$, the first statement follows.

For (v), let $T = \sum_i \lambda_i T \in \mathcal{D}'(M, E)$, and apply (iv) of Remark 3.1.2 to the compactly supported distributions $\lambda_i T$, for $i = 1, .., N$. The remaining statements are direct consequences of corresponding statements of the Propositions 3.1.4, 3.2.1 and 3.3.2 of the last subsection, combined with the remarks of the last paragraph. We leave them as an exercise. □.

5. Pseudodifferential Operators on $\mathbb{R}^n$

5.1. **Motivation.** When one wants to solve a differential equation on a manifold, one basically wants to "invert" a differential operator. This "inverse" is usually *not* a differential operator. For example, if one wants to solve the equation $\overline{\partial} f = g$ on the plane, for say $g \in \mathcal{S}$, one found in the Propositions 2.2.2 and 2.2.3 that the solution was $g * (2/z)$, which is given by the integral:

$$\int_{\mathbb{R}^2} \frac{g(w)}{w - z} dV(w)$$

which is an integral operator acting on g. Thus, one needs to enlarge the class of differential operators to include more general operators. The key to this generalisation is the observation that if $P = \sum_{|\alpha| \le d} a_\alpha(x) D_x^\alpha$ is a differential operator of degree d, a_α smooth functions, then for $f \in \mathcal{S}$ say, we have:

$$Pf(x) = \sum_{|\alpha| \le d} a_\alpha(x) D_x^\alpha f = \sum_{|\alpha| \le d} a_\alpha(x) (\widehat{D_x^\alpha f})^\vee(x) = \sum_{|\alpha| \le d} a_\alpha(x) (\xi^\alpha \widehat{f})^\vee(x) = \int_{\mathbb{R}^n} e^{ix.\xi} p(x, \xi) \widehat{f}(\xi) d\xi$$

where $p(x, \xi) = \sum_{|\alpha| \le d} a_\alpha(x) \xi^\alpha$ is called the *symbol* of the differential operator P. If the function f was vector valued, taking values in $\mathbb{R}^k$, and Pf is $\mathbb{R}^m$-valued, then the $a_\alpha(x)$ would be $m \times k$ matrices, and the symbol $p(x, \xi)$ would be $m \times k$ matrix-valued.

5.2. **Pseudodifferential operators.**

Definition 5.2.1 (Pseudodifferential operators)**.** Let $d \in \mathbb{Z}$. A matrix valued function:

$$\begin{aligned} p : \mathbb{R}^n \times \mathbb{R}^n &\to \hom_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^m) \\ (x, \xi) &\mapsto p(x, \xi) \end{aligned}$$

is called a *symbol of order d* if:

(i): p is a smooth map.

(ii): For each pair of multi-indices α, β, there exists a constant $C_{\alpha\beta} > 0$ such that:

$$\left| D_x^\alpha D_\xi^\beta p(x, \xi) \right| \le C_{\alpha\beta} (1 + |\, \xi \,|)^{d - |\beta|} \quad \text{for all } \; x, \; \xi \in \mathbb{R}^n$$

(Note the norm on the left hand side of the inequality in (ii) is the *Hilbert- Schmidt norm* on $\hom_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^m)$, defined by $|\, A \,|^2 = \operatorname{tr} AA^* = \operatorname{tr} A^* A$.)

It is easily checked that the space of symbols of order d form a $\mathbb{C}$- vector space, which is denoted S^d. Clearly $S^d \subset S^e$ if $d \le e$, and we denote $S^\infty = \cup_{d \in \mathbb{Z}} S^d$ and $S^{-\infty} = \cap_{d \in \mathbb{Z}} S^d$.

For a symbol $p(x, \xi)$ of order d, we define the corresponding *pseudodifferential operator of order d*, or ψDO for short, by the formula:

$$Pf = \int_M e^{ix.\xi} p(x,\xi) f(\xi) d\xi$$

which makes sense at least for $f \in \mathcal{D}^k$ of compact support. The space of ψDO's of order d is denoted Ψ^d. If P is a pseudodifferential operator of order d, we denote its symbol $p(x,\xi)$ of order d by $\sigma(P)$.

Example 5.2.2 (Linear Differential Operators)**.** Clearly a linear differential operator $P = \sum_{|\alpha| \leq d} a_\alpha(x) D_x^\alpha$ of order d is a ΨDO of order d.

Example 5.2.3 (Convolutions)**.** Let $g \in \mathcal{S}$. Then by Proposition 1.4.15, its Fourier transform $\widehat{g}(\xi)$ is also in $\mathcal{S}$. We also have $D_\xi^\beta \widehat{g(\xi)} \in \mathcal{S}$ for each β, and by the rapid decay condition:

$$\left\| | D_\xi^\beta \widehat{g(\xi)} \right\|_\infty \leq C_\beta (1+ | \xi |^2)^d$$

for each $d \geq 0$ and some $C_\beta > 0$. Also $D_x^\alpha D_\xi^\beta \widehat{g(\xi)} \equiv 0$ for all $| \alpha |> 0$, so that $\widehat{g}(\xi)$ is a symbol of every order d, and hence belongs to $S^{-\infty}$.

The corresponding ψDO is defined by:

$$Pf = \int e^{ix.\xi} \widehat{g}(\xi) \widehat{f}(\xi) d\xi = \left(\widehat{g}\widehat{f}\right)^\vee = g * f \qquad \text{for} \quad f \in \mathcal{D}$$

which is just convolution by g. It is a ψDO in $\Psi^{-\infty}$. Thus, in particular, convolution by a smooth compactly supported function is a pseudodifferential operator of infinite order. Convolution is *not* a differential operator. Hence ψDO's are general enough to include both differential operators and integral operators like convolution.

Remark 5.2.4.

(i): The foregoing example showed how the integral operator of convolution by a rapidly decreasing function defined a pseudodifferential operator. There is a converse to this, namely if P is a ψDO in $\Psi^{-\infty}$, with symbol $\sigma(P) = p(x,\xi) \in S^{-\infty}$, (that is, the symbol is rapidly decreasing in the ξ direction), then the ψDO P is an integral operator with smooth kernel. For, let $f \in \mathcal{D}$, then,

$$\begin{aligned} Pf &= \int e^{ix.\xi} p(x,\xi) \widehat{f}(\xi) d\xi = \int e^{ix.\xi} p(x,\xi) \int e^{-iy.\xi} f(y) dy d\xi \\ &= \int \left(\int e^{i.(x-y)} p(x,\xi) d\xi \right) f(y) dy = \int K(x,y) f(y) dy \end{aligned}$$

where the compact y-support of f and the rapid decay of $p(x,\xi)$ in ξ allows the interchange of the integrals, and where

$$K(x,y) := \int e^{i(x-y)} p(x,\xi) d\xi = p^\vee(x, x-y)$$

$p^\vee(x,-)$ being the partial inverse Fourier transform of p in the ξ variable. p is rapidly decreasing in ξ, and smooth in x, so that $p^\vee$ is smooth in both variables, and $K(x,y)$ is smooth. Thus P is an integral operator with smooth kernel K. Loosely speaking, a general ψDO is an integral operator with "distributional" kernel $K(x,y) = p^\vee(x, x-y)$, since $p^\vee$ is in general a distribution.

(ii): Not every integral operator $f \mapsto \int K(x,y) f(y) dy$ with $K(x,y)$ smooth leads to a smoothing operator. For example, taking the smooth kernel $K(\xi, y) = e^{-i\xi.y}$ leads to the integral operator $f \mapsto \widehat{f}$, and say the C^∞ function $f(x) = (1+x^2)^{-1} \in \mathcal{E}(\mathbb{R})$ has Fourier transform $\widehat{f} = e^{-|x|}$, which is not even C^1. However, the next proposition will show that *pseudodifferential operators* of order d "reduce smoothness" by at most d, like constant coefficient differential operators of order d (see (iii) of Proposition 3.1.4).

The way the definition of ψDO's is set up, i.e. using the Fourier transform, it behaves well with respect to Schwartz spaces and tempered distributions. More precisely:

Proposition 5.2.5. For $P \in \Psi^d$ a ψDO of order d, we have that P is a continuous linear operator of $\mathcal{S}^k$ to $\mathcal{S}^m$, and hence defines a continuous map of tempered k-vector valued distributions $\mathcal{S}'^k$ to $\mathcal{S}'^m$. If the x-support of p is compact, (i.e. there exists a $K \subset \mathbb{R}^n$ such that $p(x,-) \equiv 0$ for all $x \notin K$), then P is a bounded operator from $H_{s+d}(\mathbb{R}^n, \mathbb{C}^k)$ to $H_s(\mathbb{R}^n, \mathbb{C}^k)$.

Proof: For simplicity, we will take $k = m = 1$, since it is the same argument, with moduli replaced by Hilbert Schmidt norms etc. Let $\Delta_\xi = -\sum_i \partial_i^2$ denote the Laplacian in the ξ-variable, whose symbol is $p(\xi, x) = \mid x \mid^2$. Then, for $f \in \mathcal{S}$, we have that $\widehat{f}$ is also in $\mathcal{S}$, and so by the definition of a symbol of order d, $p(x,\xi)\widehat{f}(\xi)$ is in $\mathcal{S}$ in the ξ variable, by using Leibnitz formula. Thus the integral defining $Pf(x)$ is finite for each x, and also we have the inequality:

$$\left|\Delta_\xi^N \left[p(x,\xi)\widehat{f}(\xi)\right]\right| \leq C_{r,N}(1+\mid \xi \mid^2)^{-r}$$

for any $r > 0$. Hence:

$$\begin{aligned} \mid x^{2N} Pf(x) \mid &= \left|\int (\Delta_\xi^N e^{ix.\xi}) p(x,\xi)\widehat{f}(\xi) d\xi\right| = \left|\int e^{ix.\xi} \Delta_\xi^N \left(p(x,\xi)\widehat{f}(\xi)\right) d\xi\right| \\ &\leq C_{r,N} \int (1+\mid \xi \mid^2)^{-r} d\xi \end{aligned}$$

where we have used integration by parts for the last equality of the first line, since $p(x,\xi)\widehat{f}(\xi)$ is rapidly decreasing (Schwartz class) in ξ. Choosing $r > n/2$ shows that $\mid x \mid^{2N} Pf$ is bounded for all n. For the higher derivatives D_x^α with respect to x, we differentiate under the integral sign with respect to x and note that $D_x^\alpha(e^{ix.\xi}p(x,\xi))$ is a sum of terms of the kind $\xi^\gamma D_x^{\alpha-\gamma} p(x,\xi)$. But if $p(x,\xi)$ is a symbol of order d, so is $D_x^{\alpha-\gamma} p(x,\xi)$ by definition, and if $\widehat{f}(\xi) \in \mathcal{S}$, so is $\mid \xi \mid^\gamma \widehat{f}(\xi)$, so the same argument as above applies to each term in this sum, and we have $Pf \in \mathcal{S}$.

To prove the second statement, let K denote the x-support of $p(x,\xi)$. For $f \in \mathcal{S}$, we have:

$$(Pf)^\wedge(\eta) = \int e^{-ix.\eta} e^{ix.\xi} p(x,\xi)\widehat{f}(\xi) d\xi dx = \int q(\eta - \xi, \xi)\widehat{f}(\xi) d\xi$$

where the compact x-support and rapid decay in ξ of $p(x,\xi)\widehat{f}(\xi)$ (since $f \in \mathcal{S}$ implies $\widehat{f} \in \mathcal{S}$ as well) justifies the change of integrals above. Here

$$q(\eta, \xi) := \int e^{-ix.\eta} p(x,\xi) dx$$

is the partial Fourier transform of p in the x-direction, which is a Schwartz class function in η since p has compact x-support. In the ξ variable, $q(\eta,\xi)$ has the same growth properties as $p(x,\xi)$. Putting these two facts together, we have:

$$\mid q(\eta,\xi) \mid \leq C_k (1+\mid \xi \mid^2)^{d/2} (1+\mid \eta \mid^2)^{-k/2}$$

which implies that:

$$\mid q(\eta - \xi, \xi) \mid \leq C_k (1+\mid \xi \mid^2)^{d/2} (1+\mid \eta - \xi \mid^2)^{-k/2} \tag{8}$$

where we will conveniently choose k to be large enough later on.

Now, let $g \in \mathcal{S}$. Then by the Plancherel theorem (iv) of Proposition 1.2.5 and the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} \mid (Pf, g)_0 \mid &= \mid (\widehat{Pf}, \widehat{g})_0 \mid = \left|\int q(\eta - \xi, \xi)\widehat{f}(\xi)\overline{\widehat{g}}(\eta) d\xi d\eta\right| \\ &\leq \int \mid K(\eta,\xi) \mid^{1/2} \mid (1+\mid \xi \mid^2)^{s/2} \widehat{f}(\xi) \mid\mid K(\eta,\xi) \mid^{1/2} \mid (1+\mid \eta \mid^2)^{\frac{d-s}{2}} \overline{\widehat{g}}(\eta) \mid d\xi d\eta \\ &\leq \left(\int \mid K(\eta,\xi) \mid d\eta (1+\mid \xi \mid^2)^s \mid \widehat{f}(\xi) \mid^2 d\xi\right)^{1/2} \left(\int \mid K(\eta,\xi) \mid d\xi (1+\mid \eta \mid^2)^{d-s} \mid \widehat{g}(\eta) \mid^2 d\eta\right)^{1/2} \end{aligned} \tag{9}$$

where

$$K(\eta,\xi) := q(\eta - \xi, \xi)(1+\mid \xi \mid^2)^{-s/2} (1+\mid \eta \mid^2)^{\frac{s-d}{2}}$$

Because of the inequality (8) above, and Peetre's inequality, we have:

$$\begin{aligned} \mid K(\eta,\xi) \mid &= \mid q(\eta-\xi,\xi)(1+\mid \xi \mid^2)^{-s/2}(1+\mid \eta \mid^2)^{\frac{s-d}{2}} \\ &\leq C_k(1+\mid \xi \mid^2)^{\frac{d-s}{2}}(1+\mid \eta \mid^2)^{\frac{s-d}{2}}(1+\mid \eta-\xi \mid^2)^{-k/2} \\ &\leq C_k(1+\mid \eta-\xi \mid^2)^{\frac{|d-s|-k}{2}} \end{aligned}$$

This shows that by choosing k so that $\mid d-s \mid -k < -n$, or $k >\mid d-s \mid +n$, the integrals:

$$\int \mid K(\eta,\xi) \mid d\eta \leq A; \quad \int \mid K(\eta,\xi) \mid d\xi \leq A$$

where $A < \infty$ is independent of ξ, η, so that from the inequality (9) above, we have for $f, g \in \mathcal{S}$:

$$\mid (Pf,g)_0 \mid \leq AC_k \left\|f\right\|_s \left\|g\right\|_{d-s}$$

By the density of $\mathcal{S}$ in H_s and H_{d-s}, we have the same inequality for all $f \in H_s$ and all $g \in H_{d-s}$. Then, by (iv) of Proposition 3.1.4, we have for $f \in \mathcal{S}$ that:

$$\left\|Pf\right\|_{s-d} = \sup_{g\in H_{d-s};\, g\neq 0} \frac{\mid (Pf,g)_0 \mid}{\left\|g\right\|_{d-s}} \leq C \left\|f\right\|_s$$

which proves that $P : H_s \to H_{s-d}$ is bounded, and the proposition follows. □

Remark 5.2.6. Like the spaces $L_{p,loc}$, one can define the *localised Sobolev spaces*:

$$H_{s,loc}(\mathbb{R}^n) = \{f \in \mathcal{S}' : \psi f \in H_s(\mathbb{R}^n) \text{ for all } \psi \in C_c^\infty(\mathbb{R}^n)\}$$

Then if one drops the compact x-support condition on $\sigma(P) = p(x,\xi)$, one observes that the pseudodifferential operator ψP defined by $(\psi P)f(x) := \psi(x)Pf(x)$ will have the symbol $\sigma(\psi P) = \psi(x)p(x,\xi)$, which *will* have compact x- support, so that the previous proposition applied to ψP will yield the fact that $\left\|(\psi P)f\right\|_{s-d} < \infty$ for $f \in H_s(\mathbb{R}^n)$. That is, $Pf \in H_{s-d,loc}$ for $f \in \mathcal{H}_s$. In fact, if one defines a topology on $H_{s-d,loc}$ by $f_n \to 0$ iff $\psi f_n \to 0$ for each $\psi \in C_c^\infty(\mathbb{R}^n)$, then the argument above shows that for a general ψDO P we have $P : H_s \to H_{s-d,loc}$ a continuous linear map.

Exercise 5.2.7.

(i): Show that the obvious containment $H_s \subset H_{s,loc}$ is strict for each s. In fact, find a function which is in $H_{s,loc}$ for every s, but not in H_s for any s.

(ii): Show that the localised analogue of the Sobolev lemma holds. That is, if a tempered distribution $f \in \mathcal{S}'$ is in $H_{\infty,loc} := \cap_{s\in\mathbb{R}} H_{s,loc}$ then $f \in C^\infty$. One can no longer conclude, of course, that f or its derivatives vanish at ∞, i.e. in general f won't be in C_0^∞.

Corollary 5.2.8 (Infinitely smoothing operators)**.** If P is in $\Psi^{-\infty} = \cap_d \Psi^d$, then $P(H_s) \subset C^\infty$ for every s. In particular, $P(H_{-\infty}) \subset C^\infty$. (Such operators are called *infinitely smoothing*. Thus convolutions with $g \in \mathcal{S}$ are infinitely smoothing, by Example 5.2.3.)

Proof: Apply the Remark 5.2.6 and (ii) of the Exercise 5.2.7 above. □

5.3. **Some Technical Lemmas on ψDO's.** We will need a few lemmas to perform operations with $\psi DO's$. We make a a couple of definitions first.

Definition 5.3.1. Let $p(x,\xi) \in S^d$ be a symbol of order d with compact x-support K. For an open subset $U \subset \mathbb{R}^n$, we will say that $p \in S^d(U)$ if $K \subset U$. Clearly $S^d(U) \subset S^d(V)$ for $U \subset V$.

Definition 5.3.2. Let $p, q \in S^d(U)$. We will say $p \sim q$ if $p - q \in S^{-\infty}(U) := \cap_{d \in \mathbb{R}} S^d(U)$. If $d_1 > d_2 > ... > d_j < ...$ is a sequence of real numbers with $d_j \to -\infty$, and $p_j \in S^{d_j}(U)$ for $j = 1, 2, ..,$, we will say $p \sim \sum_j p_j$ if $p - \sum_{j=1}^{k-1} p_j \in S^{d_k}(U)$ for all k.

Lemma 5.3.3. Let U be a relatively compact open set in $\mathbb{R}^n$, and let $d_1 > d_2 > ... > d_j > ...$ be a sequence of real numbers with $d_j \to -\infty$. Let $p_j \in S^{d_j}(U)$ for $j = 1, 2, ...,$. Then for any V containing $\overline{U}$, there is a symbol $p \in S^{d_1}(V)$ and such that $p \sim \sum_j p_j$ in $S^{d_1}(V)$.

Proof: By definition, there are constants $C^j_{\alpha,\beta}$ satisfying:

$$\left| D^\alpha_x D^\beta_\xi p_j(x,\xi) \right| \leq C^j_{\alpha,\beta} (1+ \mid \xi \mid)^{d_j - |\beta|}$$

for all α, β, j.

Let $\psi \geq 0$ be a smooth function in $C^\infty_c(\mathbb{R}^n)$ with $\psi(x) \equiv 0$ for $\mid \xi \mid \leq 1$ and $\psi \equiv 1$ for $\mid \xi \mid \geq 2$. Let $1 \leq r_1 \leq r_2 ... < r_j < ...$ be a sequence of positive real numbers with $\lim_{j \to \infty} r_j = \infty$. We define the symbol:

$$p(x,\xi) = \sum_{k=1}^{\infty} \psi(r_k^{-1}\xi)\, p_k(x,\xi)$$

For a fixed ξ, $\mid r_j^{-1}\xi \mid \leq 1$ for j large enough, so $\psi(r_j^{-1}\xi) \equiv 0$ for j large enough, and the sum on the right is finite, and makes sense. Also, since the x-support of each p_j is contained in U, the x-support of p is contained in $\overline{U}$, which is compact. Thus the x-support of p is contained in every open set $V \supset \overline{U}$.

To make p a symbol in $S^{d_1}(V)$, we need to make a careful choice of r_j. For each multi-index γ, let $A_\gamma > 0$ be a constant so that:

$$\mid D^\gamma_\xi \psi(\xi) \mid \leq A_\gamma \quad \text{for all } \ \xi$$

Then, since $r_j \geq 1$ for all j, it follows that:

$$\mid D^\gamma_\xi \psi(r_j^{-1}\xi) \mid \leq A_\gamma r_j^{-|\gamma|} \quad \text{for each multi-index } \gamma \ \text{ and all } \ \mid \xi \mid \leq 2r_j; \quad \equiv 0 \ \text{ for } \quad \mid \gamma \mid > 0, \ \mid \xi \mid > 2r_j$$

Thus, for any choice of $1 \leq r_1 < r_2 < ... < r_j < ...$, we have:

$$\begin{aligned} \mid D^\alpha_x D^\beta_\xi (\psi(r_j^{-1})\, p_j(x,\xi) \mid &\leq \sum_{\gamma \leq \beta} \left| \frac{\beta!\, D^\gamma_\xi \psi(r_j^{-1}\xi) D^\alpha_x D^{\beta-\gamma}_\xi p(x,\xi)}{(\gamma)!(\beta-\gamma)!} \right| \\ \leq \sum_{\gamma \leq \beta} \beta! A_\gamma C^j_{\alpha,\beta-\gamma} (1+ \mid \xi \mid)^{d_j - |\beta| + |\gamma|} r_j^{-|\gamma|} &\leq \sum_{\gamma \leq \beta} \beta! A_j C^j_{\alpha,\beta-\gamma} (1+ \mid \xi \mid)^{d_j - |\beta|} (1+2r_j)^{|\gamma|} r_j^{-|\gamma|} \\ &\leq M^j_{\alpha,\beta} (1+ \mid \xi \mid)^{d_j - |\beta|} \end{aligned} \tag{10}$$

where:

$$M^j_{\alpha,\beta} := \beta! \sum_{\gamma \leq \beta} 3^{|\gamma|} A_\gamma C_{\alpha,\beta-\gamma}$$

is a positive constant independent of any choice of the sequence $1 < r_1 < r_2 .. < r_j < ....$ This shows that $\psi(r_j^{-1}\xi) p_j(x,\xi)$ is also a symbol of order d_j, and lies in $S^{d_j}(U) \subset S^{d_j}(V)$.

For each $k \in \mathbb{Z}_+$, define:

$$M_k = \sup\{M^k_{\alpha,\beta} : \mid \alpha \mid \leq k, \ \mid \beta \mid \leq k\}$$

Now choose a sequence of numbers $r_k > 0$ such that $r_k \to \infty$ and:

$$\sum_{k=2}^{\infty} \frac{M_k}{(1+r_k)^{d_{k-1}-d_k}} = C < \infty \tag{11}$$

We need to check that p is a symbol of order d_1. In fact, we make the more general:

Claim:

$$q_j := \sum_{k \geq j} \psi(r_k^{-1}\xi)\, p_k(x,\xi)$$

is a symbol of order d_j.

Let α, β be multi-indices with $|\alpha|, |\beta| \leq m$. It is clearly enough to check the decay condition for $D_x^\alpha D_\xi^\beta q_j$ on the set $|\xi| \geq r_m$. Also, since $\psi(r_i^{-1}\xi)p_i(x,\xi)$ is in $S^{d_i}(U)$, the finite sum:

$$\psi(r_j^{-1}\xi)\, p_j(x,\xi) + \ldots.. + \psi(r_{m-1}^{-1}\xi)\, p_{m-1}(x,\xi)$$

is clearly a symbol of order $\max\{d_j, d_{j+1}, .., d_{m-1}\} = d_j$. Thus we just need to verify that:

$$|D_x^\alpha D_\xi^\beta(q_m)| \leq (\text{const})(1+|\xi|)^{d_j-|\beta|} \quad \text{for all} \quad |\xi| \geq r_m$$

We have from (10) that:

$$|D_x^\alpha D_\xi^\beta(q_m)| \leq \sum_{s\geq 0}\left| D_x^\alpha D_\xi^\beta \left[\psi(r_{m+s}^{-1}\xi)\, p_{m+s}(x,\xi)\right]\right| \leq \sum_{s\in F(\xi)} M_{\alpha,\beta}^{m+s}(1+|\xi|)^{d_{m+s}-|\beta|} \tag{12}$$

where:

$$F(\xi) = \{s : s \geq 0 \ \text{ and } \ r_{m+s} < |\xi|\}$$

since $\psi(r_{m+s}^{-1}\xi) \equiv 0$ for $r_{m+s}^{-1}|\xi| \leq 1$, i.e. for all s such that $r_{m+s} \geq |\xi|$.

Since $|\alpha|, |\beta| \leq m \leq m+s$, we have $M_{\alpha,\beta}^{m+s} \leq M_{m+s}$ for all $s \geq 0$. Also, for an $s \in F(\xi)$, because $d_{m+s} - d_m < 0$, and $|\xi| > r_{m+s}$, we have the inequality:

$$(1+|\xi|)^{d_{m+s}-|\beta|} = (1+|\xi|)^{d_{m+s}-d_m}(1+|\xi|)^{d_m-|\beta|} \leq (1+r_{m+s})^{d_{m+s}-d_m}(1+|\xi|)^{d_m-|\beta|}$$

Plugging these two facts into the inequality (12), and noting that $d_m - d_{m+s} \geq d_{m+s-1} - d_{m+s}$ for $s \geq 1$, we have:

$$\begin{aligned}
|D_x^\alpha D_\xi^\beta(q_m)| &\leq \left[\sum_{s\in F(\xi)} \frac{M_{m+s}}{(1+r_{m+s})^{d_m-d_{m+s}}}\right](1+|\xi|)^{d_m-|\beta|} \\
&\leq \left[M_m + \sum_{s\in F(\xi), s\geq 1} \frac{M_{m+s}}{(1+r_{m+s})^{d_{m+s-1}-d_{m+s}}}\right](1+|\xi|)^{d_m-|\beta|} \\
&\leq \left[M_m + \sum_{k=2}^{\infty} \frac{M_k}{(1+r_k)^{d_{k-1}-d_k}}\right](1+|\xi|)^{d_m-|\beta|} \\
&\leq (M_m + C)(1+|\xi|)^{d_j-|\beta|}
\end{aligned}$$

by the equation (11) and the fact that $d_m \leq d_j$. This proves the Claim that $q_j \in S^{d_j}(V)$, and in particular $p = q_1 \in S^{d_1}(V)$.

Also note that for each j, $p_j(x,\xi) - \psi(r_j^{-1}\xi)\, p_j(x,\xi)$ has compact support in both x and ξ, so is a symbol in $S^{-\infty}(U) \subset S^{-\infty}(V)$. Hence $p_j(x,\xi) \sim \psi(r_j^{-1}\xi)\, p_j(x,\xi)$ in $S^{d_j}(V)$, so that:

$$p(x,\xi) - \sum_{j=1}^{k-1} p_j(x,\xi) \sim p(x,\xi) - \sum_{j=1}^{k-1} \psi(r_j^{-1}\xi)\, p_j(x,\xi) = q_j(x,\xi)$$

and since $q_j \in S^{d_j}(V)$, it follows that $p \sim \sum_{j=1}^{\infty} p_j$ in $S^{d_1}(V)$ and the proposition follows. □

The other technical lemma one needs stems from the following observation. Let P be a ψDO given by the symbol $p(x,\xi)$. Suppose $f \in \mathcal{D}$, and we write the formula for Pf, viz.,

$$\begin{aligned} Pf &= \int e^{ix.\xi} p(x,\xi) \widehat{f}(\xi) d\xi \\ &= \int e^{ix.\xi} p(x,\xi) \int e^{-iy.\xi} f(y) dy d\xi \end{aligned}$$

which can be viewed (by interchanging the orders of integration) as a special case of:

$$Kf := \int e^{i(x-y).\xi} a(x,y,\xi)\, f(y) d\xi dy \tag{13}$$

where $a(x,y,\xi) = a(x,x,\xi) = p(x,\xi)$ for all y. The natural question is: do we enlarge the class of ψDO's by using the formula (13) instead of the formula for Pf in terms of $p(x,\xi)$ in the first line above ?

This is answered by the following definition and lemma.

Definition 5.3.4. A *bi-symbol* $a(x,y,\xi)$ of order d is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \hom_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^m)$ which satisfies:

(i): The x-support of a is compact.

(ii): $\left| D_y^\alpha D_x^\beta D_\xi^\gamma a(x,y,\xi) \right| \leq C_{\alpha,\beta,\gamma} (1+ \mid \xi \mid)^{d-|\gamma|}$, where $|\ |$ on the left hand side denotes Hilbert-Schmidt norm, as usual.

By this definition, a symbol $p(x,\xi) \in S^d$ with compact x-support is a bi-symbol of order d, with $a(x,y,\xi) := a(x,x,\xi) := p(x,\xi)$ for all y.

Now we have the answer to our earlier question, in the following:

Lemma 5.3.5. Let $a(x,y,\xi)$ be a bi-symbol of order d, and define the operator K by:

$$Kf(x) = \int e^{i(x-y).\xi} a(x,y,\xi) f(y) dy d\xi \quad \text{for} \quad f \in \mathcal{D}$$

Then K is a ψDO of order d whose symbol k has the asymptotic expansion (i.e. upto a symbol in $S^{-\infty}$) given by:

$$k(x,\xi) \sim \sum_\alpha \frac{d_\xi^\alpha D_y^\alpha a(x,y,\xi)_{|y=x}}{\alpha!} \tag{14}$$

Note that in the special case when $a(x,y,\xi) = a(x,x,\xi) = p(x,\xi)$ for all y, i.e. the bi-symbol is actually a symbol in disguise, we have $D_y^\alpha \equiv 0$ for all $\mid \alpha \mid > 0$, and the expansion above reduces to just its first $\alpha = 0$ term, viz. $a(x,x,\xi)$, and this is as it should be.

Proof: We will as usual simplify by assuming that $k = m = 1$, because the proof is the same. Since the formula for Kf in the statement of this lemma is being defined on $f \in \mathcal{D} = C_c^\infty(\mathbb{R}^n)$, we can write $f = \psi(y) f(y)$ where $\psi \equiv 1$ on supp f, so that in the formula above, $a(x,y,\xi)$ is replaced by $a(x,y,\xi)\psi(y)$, and we lose no generality in assuming that the y-support of $a(x,y,\xi)$ is also compact.

Define the function:

$$q(x,\rho,\eta) := \int e^{-iy\cdot\rho}\, a(x,y,\eta) dy \tag{15}$$

which is the Fourier transform of $a(x,y,\eta)$ in the y-direction. From this, it follows that:

$$D_y^\alpha a(x,y,\eta)_{|y=x} = \int e^{iy.\rho} \rho^\alpha q(x,\rho,\eta) d\rho_{|y=x} = \int e^{ix.\rho} q(x,\rho,\eta) d\rho \tag{16}$$

Now we do some formal manipulations to express Kf in the form of a pseudodifferential operator with some symbol, and then check that the alleged symbol is actually a symbol. First note that by the Fourier inversion formula:

$$\begin{aligned} \int e^{-iy.\xi}\, a(x,y,\xi) f(y) dy &= \int e^{-iy.\xi}\, a(x,y,\xi) \int e^{iy.\eta} \widehat{f}(\eta) d\eta dy \\ = \int \left(\int e^{-iy.(\xi-\eta)}\, a(x,y,\xi) dy \right) \widehat{f}(\eta) d\eta &= \int q(x, \xi-\eta, \xi) \widehat{f}(\eta) d\eta \end{aligned}$$

where the interchange of integrals is allowed since $a(x,y,\xi)$ has compact y-support, and $f \in \mathcal{D}$ implies $\widehat{f}(\eta)$ has rapid decay in η. We also need a precise estimate on the decay of $q(x,\xi-\eta,\xi)\widehat{f}(\eta)$. Since $q(x,\eta,\xi)$ has rapid decay in η as stated above, and the same decay as $a(x,y,\xi)$ in ξ, we have, for each $k \geq 0$

$$| q(x,\eta,\xi) | \leq C_k (1+ | \xi |)^d (1+ | \eta |)^{-k} \tag{17}$$

Since $\widehat{f}(\eta)$ is rapidly decreasing, we also have, for the same k:

$$| \widehat{f}(\eta) | \leq C_k (1+ | \eta |)^{-k}$$

where C_k above (and below) is a generic constant depending on k. Hence:

$$\begin{aligned} | q(x,\xi-\eta,\xi)\widehat{f}(\eta) | &\leq C_k (1+ | \xi |)^d (1+ | \xi-\eta |)^{-k} (1+ | \eta |)^{-k} \\ &\leq C_k (1+ | \xi |)^d (1+ | \xi |)^{-k} = C_k (1+ | \xi |)^{d-k} \end{aligned}$$

by using the Peetre inequality (see the proof of Proposition 3.3.2) for $k/2 > 0$ and the fact that the ratio of $(1+r^2)^{k/2}$ and $(1+r)^k$ is bounded above and below by strictly positive constants independent of $r \geq 0$. By choosing k large enough, we see that $| q(x,\xi-\eta,\xi)\widehat{f}(\eta) |$ is integrable over $\mathbb{R}^n$ in ξ, as well as η (since it is rapidly decreasing in the middle variable).

Now,

$$\begin{aligned} Kf(x) = \int e^{i(x-y).\xi} a(x,y,\xi) f(y) dy d\xi &= \int e^{ix.\xi} \left(\int e^{-iy.\xi}\, a(x,y,\xi) f(y) dy) \right) d\xi \\ = \int e^{ix.\xi} \left(\int q(x,\xi-\eta,\xi) \widehat{f}(\eta) d\eta \right) d\xi &= \int e^{ix.\eta} \left(\int e^{ix.(\xi-\eta)} q(x,\xi-\eta,\xi) d\xi \right) \widehat{f}(\eta) d\eta \\ &= \int e^{ix.\eta} p(x,\eta) \widehat{f}(\eta) d\eta \end{aligned}$$

where the interchange of ξ and η variables is allowed because of the last paragraph, and where we have introduced the function:

$$p(x,\eta) := \int e^{ix.(\xi-\eta)} q(x,\xi-\eta,\xi) d\xi$$

Now we check the decay of the derivatives of $p(x,\eta)$ in both variables. This is easily done by changing variables $\rho := \xi - \eta$, so that:

$$p(x,\eta) = \int e^{ix.\rho} q(x,\rho,\eta+\rho) d\rho$$

Applying the estimate (17) above for q, we have:

$$\begin{aligned} | p(x,\eta) | &\leq C_k \int (1+ | \eta+\rho |)^d (1+ | \rho |)^{-k} d\rho \leq C_k \int (1+ | \rho |)^{|d|} (1+ | \eta |)^d (1+ | \rho |)^{-k} d\rho \\ &\leq C_k (1+ | \eta |)^d \left[\int (1+ | \rho |)^{|d|-k} d\rho \right] \leq C_k (1+ | \eta |)^d \end{aligned}$$

where we have used Peetre's inequality in the first line above, and chosen $k >| d | +n$. Similarly, by writing down the corresponding estimates for $D_x^\alpha D_\xi^\beta q(x,\eta,\xi)$ analogous to (17), one can deduce the estimates for $D_x^\alpha D_\eta^\beta p(x,\eta)$ using exactly the same arguments.

To get the asymptotic formula for $p(x, \eta)$, first expand the function $q(x, \rho, \eta + \mu)$ by Taylor's theorem in the third variable, to obtain:

$$q(x, \rho, \eta + \mu) = \sum_{|\alpha| \le k} \frac{d_\eta^\alpha q(x, \rho, \eta)}{\alpha!} \mu^\alpha + q_k(x, \rho, \eta; \mu) \tag{18}$$

where $q_k(x, \rho, \eta; \mu)$ is a constant times integral of the derivative $d_\eta^{k+1} q(x, \rho, \eta + t\mu)$ over $0 \le t \le 1$. Analogous to the inequality (17), since $a(x, y, \eta)$ is compactly supported (hence rapidly decreasing) in the middle variable, that, for all $p \ge 0$:

$$\begin{aligned} |\, q_k(x, \rho, \eta; \mu) \,| &\le C_k \sup_{0 \le t \le 1} |\, d_\eta^{k+1} q(x, \rho, \eta + t\mu) \,| \le C_k \sup_{0 \le t \le 1} (1 + |\, \eta + t\mu \,|)^{d-k-1} (1 + |\, \rho \,|)^{-p} \\ &\le \sup_{0 \le t \le 1} C_k (1 + |\, t\mu \,|)^{k+1-d} (1 + |\, \eta \,|)^{d-k-1} (1 + |\, \rho \,|)^{-p} \\ &\le C_k (1 + |\, \mu \,|)^{k+1-d} (1 + |\, \eta \,|)^{d-k-1} (1 + |\, \rho \,|)^{-p} \end{aligned}$$

by Peetre's inequality, for if $k >> 0$, we have $|\, d - k - 1 \,| = k + 1 - d \ge 0$. Hence:

$$|\, q_k(x, \rho, \eta; \rho) \,| \le C_k (1 + |\, \rho \,|)^{k+1-d-p} (1 + |\, \eta \,|)^{d-k-1}$$

Thus, by choosing $p > k + 1 - d + n$, we see that:

$$|\, p_k(x, \eta) \,| := \left| \int e^{ix.\rho} q_k(x, \rho, \eta; \rho) d\rho \right| \le C_k (1 + |\, \eta \,|)^{d-k-1} \tag{19}$$

and is a symbol in S^{d-k-1}. Hence, putting together the equations (16) and (18) we have:

$$\begin{aligned} p(x, \eta) &= \int e^{ix.\rho} q(x, \rho, \eta + \rho) d\rho \\ &= \int e^{ix.\rho} \left(\sum_{|\alpha| \le k} \frac{d_\eta^\alpha q(x, \rho, \eta)}{\alpha!} \rho^\alpha \right) d\rho + p_k(x, \eta) \\ &= \sum_{|\alpha| \le k} \frac{d_\eta^\alpha D_y^\alpha a(x, y, \eta)_{y=x}}{\alpha!} + p_k(x, \eta) \end{aligned}$$

which proves the proposition, in view of the fact that $p_k(x, \eta) \in S^{d-k-1}$ for all k. □

Corollary 5.3.6. Let a and K be as in the previous Proposition 5.3.5. If $a(x, y, \xi)$ vanishes in a neighbourhood of the diagonal $\Delta := \{(x, x, \xi)\}$, then the ψDO is infinitely smoothing.

Proof: By hypothesis, $D_y^\alpha a(x, y, \xi)_{|x=y} \equiv 0$, and the asymptotic series of the previous proposition implies the symbol $\sigma(K) = k(x, \xi)$ is equivalent to 0, i.e. is a symbol in $S^{-\infty}$. □

The next corollary is the key to many patching arguments for ΨDO's that are going to be used on compact manifolds.

Corollary 5.3.7. Let $\psi := (\psi_1, \psi_2) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ be a pair of compactly supported smooth functions, and let $P \in \Psi^d$ be a pseudodifferential operator of order d. Then the operator defined by:

$$(P^\psi f)(x) := \psi_1(x) P(\psi_2 f) \quad \text{for} \quad f \in \mathcal{S}$$

is also a ψDO in Ψ^d.

Proof: By definition, for $f \in \mathcal{S}$, we have:

$$(P^\psi f)(x) = \psi_1(x) \int e^{ix.\xi} p(x, \xi) (\psi_2 f)^\wedge(\xi) d\xi = \int e^{i(x-y).\xi} \psi_1(x) p(x, \xi) \psi_2(y) f(y) dy d\xi$$

which, by Proposition 5.3.5, implies that it is a ψDO of order d, because the bisymbol

$$a(x, y, \xi) = \psi_1(x) p(x, \xi) \psi_2(y)$$

is a bi-symbol of order d, with compact x and y support. □

If $L = \sum_{|\alpha|\leq d} a_\alpha(x) D_x^\alpha$ is a linear differential operator, then we have the obvious fact that Lf vanishes identically on any neighbourhood on which f vanishes identically. i.e.

$$\text{supp}\,(Lf) \subset \text{supp}\, f$$

for $f \in C^\infty$. This property is expressed by saying that linear differential operators are *local*, they read only the local behaviour of f. This is clearly false for pseudodifferential operators, because for example we can take $f \in C_c^\infty$, which is everywhere ≥ 0, and convolve it with an everywhere > 0 Schwartz class function like $e^{-|x|^2}$ (which is an infinitely smoothing ψDO by the example 5.2.3, and note that $g * f$ will be strictly positive at all points. However, ψDO's have the property that they diminish *singular support*, i.e.

Proposition 5.3.8 (ψDO's are pseudolocal)**.** If $f \in H_s$ for some $s \in \mathbb{R}$, and if $f_{|U}$ is a smooth function on some open set $U \subset \mathbb{R}^n$, then for every $P \in \Psi^\infty$, we have Pf is smooth on U.

Proof: Let $x \in U$, and let $\psi_1 \in C_c^\infty(U)$ with $\psi_1 \equiv 1$ on a neighbourhood $V \subset U$ of x. Let $\psi_2 \in C_c^\infty(U)$ with $\psi_2 \equiv 1$ on a neighbourhood $W \subset \overline{W} \subset U$ of the support of ψ_1. Clearly, $\psi_2 f \in C_c^\infty(U)$, and hence $\psi_2 f \in \mathcal{S}$. By the Proposition 5.2.5, we have $P\psi_2 f \in \mathcal{S}$.

On the other hand, since $\psi_1 P(1-\psi_2)$ is defined by the bi- symbol:

$$a(x, y, \xi) = \psi_1(x) p(x, \xi)(1 - \psi_2(y))$$

where $p = \sigma(P)$, it is easily checked to be of the same order as p. Also since $(1-\psi_2(y))$ vanishes identically for y contained in the neighbourhood W of suppψ_1, it follows that $a(x, y, \xi)$ vanishes identically on a neighbourhood of the diagonal. Thus the pseudodifferential operator $\psi_1 P(1-\psi_2)$ is infinitely smoothing, by the Corollary 5.3.6 above. Hence, in the neighbourhood V of x, since $\psi_1 \equiv 1$, we have

$$Pf = \psi_1 Pf = \psi_1 P\psi_2 f + \psi_1 P(1-\psi_2)f$$

and both the terms on the right are smooth on V. Hence the proposition. □

5.4. **The algebra of ψDO's.** When $\mathcal{H}$ is a separable Hilbert space, there is the (non- commutative) algebra $L(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$, with multiplication given by composition, and a star operarion given by adjoints. Inside $L(\mathcal{H})$, there is the closed two-sided ideal of compact operators, denoted $K(\mathcal{H})$. Finally, we pass to the quotient, and obtain the *Calkin algebra* $C(\mathcal{H}) := L(\mathcal{H})/K(\mathcal{H})$. The so-called *Fredholm operators* are defined to be the invertible elements in $C\mathcal{H})$, i.e. they are invertible modulo compact operators. (These matters will be delved in a future section).

We would like to mimic all this for pseudodifferential operators, with the role of compact operators being played by infinitely smoothing operators. The first task is to define composition and adjoints of ψDO's.

Definition 5.4.1 (Adjoints)**.** Let P be a ψDO. For $f \in \mathcal{S}$, define the *adjoint* P^* of P by the formula:

$$(P^* f, g) = \int \langle P^* f(x), g(x)\rangle\, dx = (f, Pg) = \int \langle f(x), Pg(x)\rangle\, dx \quad \text{for all } \; g \in \mathcal{S}$$

This certainly defines $P^* f$ as a tempered distribution, for each $f \in \mathcal{S}$. We will eventually check that P^* is also a ψDO of the same order as P.

For P, Q, two $\psi DO's$, one defines the composite PQ by $(PQ)f = P(Qf)$ for all $f \in \mathcal{S}$, which makes sense since $Pf \in \mathcal{S}$ for $f \in \mathcal{S}$ by the Proposition 5.2.5.

Definition 5.4.2 (Support of a ψDO)**.** We will say that a ψDO P *is supported in a compact set* K if:

(i): $\text{supp}\, Pf \subset K$ for all $f \in C_c^\infty(\mathbb{R}^n)$.

(ii): $Pf \equiv 0$ if $f \in C_c^\infty(\mathbb{R}^n)$ and $\text{supp}\, f \cap K = \phi$

In this event we will say $\text{supp}\, P = K$.

Exercise 5.4.3. If $P \in \Psi^d$, and $\operatorname{supp} P \subset K$, then the x-support of $p(x,\xi) = \sigma(P)$ is contained in K. The converse is false in general, but clearly true for differential operators.

Now we can state the main proposition.

Proposition 5.4.4. Let $P \in \Psi^d$ with symbol $\sigma(P) = p$ and $Q \in \Psi^e$ with symbol $\sigma(Q) = q$ be two ψDO's, with $\operatorname{supp} P$, $\operatorname{supp} Q$ in some compact set $K \subset \mathbb{R}^n$. Then:

(i): P^* is a ψDO of order d, supported in K, and its symbol is given by the asymptotic formula:

$$\sigma(P^*) \sim \sum_\alpha \frac{d_x^\alpha D_\xi^\alpha p^*(x,\xi)}{\alpha\,!}$$

where $p^*(x,\xi) = \bar{p}^t(x,\xi)$, the matrix adjoint of p.

(ii): The composite PQ is a ψDO of order $d+e$, supported in K, and its symbol is given by the asymptotic expansion:

$$\sigma(PQ) \sim \sum_\alpha \frac{d_\xi^\alpha p\, D_x^\alpha q}{\alpha\,!}$$

Proof: We have to just write down a suitable bi-symbol for P^* and PQ, and appeal to the Proposition 5.3.5. First, for the adjoint we have for, $f, g \in \mathcal{S}$ and $\langle -, - \rangle$ denoting the Hermitian inner product on $\mathbb{C}^m$, that:

$$\begin{aligned}(f, Pg) &= \int \langle f(y), Pg(y) \rangle\, dy = \int e^{-i\xi.y} \langle f(y), p(y,\xi)\widehat{g}(\xi) \rangle\, d\xi dy \\ &= \int e^{-i\xi.y} \langle p^*(y,\xi) f(y), \widehat{g}(\xi) \rangle\, d\xi dy = \int\int e^{i(x-y).\xi} \langle p^*(y,\xi) f(y), g(x) \rangle\, dx d\xi dy \\ &= (P^* f, g)\end{aligned}$$

where all changes of integrals are allowed by the rapid decay of f and g and compact x-support and rapid ξ-decay of $p(x,\xi)\widehat{g}(\xi)$:

$$P^* f := \int e^{i(x-y).\xi} p^*(y,\xi) f(y) d\xi dy$$

which is the ψDO corresponding to the bisymbol:

$$a(x,y,\xi) = p^*(y,\xi)$$

It is easy to check from the definition $(P^* f, g) = (f, Pg)$ that the support $\operatorname{supp} P^* \subset K$ if $\operatorname{supp} P \subset K$. Also the y-support of $a(x,y,\xi)$ is contained in K, by the previous Exercise 5.4.3, and the ξ-decay is the same as that of p^*, which is the same as that of p. So, by the Proposition 5.3.5, we have that P^* is a ψDO of order d, and its symbol has the asymptotic expansion:

$$\sigma(P^*) \sim \sum_\alpha \frac{d_\xi^\alpha D_x^\alpha p^*(x,\xi)}{\alpha\,!}$$

which proves (i) of our proposition.

To see (ii), let us first note that if $r(y,\xi) := \sigma(Q^*)$, the symbol of Q^*, then by definition we have for $f \in \mathcal{S}$ that:

$$Q^* g(y) = \int e^{iy.\xi} r(y,\xi)\widehat{g}(\xi)(y) dy$$

Now, for $f \in \mathcal{S}$, we have:

$$\begin{aligned}(\widehat{Qf}, \widehat{g}) &= (Qf, g) = (f, Q^* g) = \int \langle f(y), (Q^* g)(y) \rangle\, dy \\ &= \int \left\langle f(y), e^{iy.\xi} r(y,\xi)\widehat{g}(\xi) \right\rangle d\xi dy \\ &= \int \left\langle \int e^{-iy.\xi} r^*(y,\xi) f(y) dy, \widehat{g}(\xi) \right\rangle d\xi\end{aligned}$$

which implies that:

$$\widehat{Qf}(\xi) = \int e^{-iy.\xi} r^*(y,\xi) f(y) dy \tag{20}$$

Now, letting $p(x,\xi) = \sigma(P)$, we have by definition, and substitution from (20) above:

$$PQf(x) = \int e^{ix.\xi} p(x,\xi)(\widehat{Qf})(\xi) d\xi = \int e^{i(x-y).\xi} p(x,y) r^*(y,\xi) f(y) dy d\xi$$

which is in the form required by the Lemma 5.3.5, with the bi-symbol:

$$a(x,y,\xi) = p(x,\xi) r^*(y,\xi)$$

which, by the self-same lemma shows that PQ is a pseudodifferential operator whose symbol has the asympttotic expansion:

$$\begin{aligned} \sigma(PQ) \quad &\sim \quad \sum_{\alpha} \frac{d_\xi^\alpha D_y^\alpha (p(x,\xi) r^*(y,\xi))_{|y=x}}{\alpha\,!} = \sum_{\gamma,\alpha} \frac{d_\xi^{\alpha-\gamma} p(x,\xi) d_\xi^\gamma D_y^\alpha r^*(y,\xi)_{|y=x}}{\gamma\,!(\alpha-\gamma)!} \\ &= \quad \sum_{\rho,\delta} \frac{d_\xi^\rho p(x,\xi) d_\xi^\delta D_y^{\rho+\delta} r^*(y,\xi)_{|y=x}}{\rho!\delta!} = \sum_{\rho} \frac{d_\xi^\rho p(x,\xi) D_x^\rho}{\rho\,!} \left(\sum_{\delta} \frac{d_\xi^\delta D_x^\delta r^*(x,\xi)}{\delta\,!} \right) \end{aligned}$$

Now, since $Q = (Q^*)^*$, and the symbol of Q^* is $r(x,\xi)$, we have by the part (i) above:

$$\sigma(Q) = q(x,\xi) \sim \sum_{\delta} \frac{d_\xi^\delta D_x^\delta r^*(x,\xi)}{\delta\,!}$$

which on substitution into the last equation above yields:

$$\sigma(PQ) \sim \sum_{\rho} \frac{d_\xi^\rho p(x,\xi) D_x^\rho q(x,\xi)}{\rho\,!}$$

and proves (ii) of our proposition. The statements about the supports are readily verified, and left as an exercise. □

Corollary 5.4.5. Denote by Ψ_K^d the space of ψDO's with support in K, and let $\Psi_K^{-\infty} := \cap_d \Psi_K^d$, and $\Psi_K^\infty := \cup_d \Psi_K^d$. Then, by the previous proposition, Ψ_K^∞ is a (non-commutative) algebra with adjoints.

5.5. **Ellipticity.**

Notation : 5.5.1. From this point onwards, the letter "P" will always denote a *linear differential operator* or order d, so that its symbol $p(x,\xi)$ will always be a polynomial in ξ, with coefficients as smooth matrix-valued functions in x. In this situation, $\operatorname{supp} P$ is contained in K iff the x-support of $p(x,\xi)$ is contained in K.

Definition 5.5.2. A *differential operator* P is said to be *elliptic* over an open set $U \subset \mathbb{R}^n$ if:

(i): There exists a constant $C > 0$ such that for some $V \supset \overline{U}$, $p(x,\xi)$ is an invertible linear transformation for all $x \in V$ and all $|\,\xi\,| \geq C$, and furthermore,

(ii): The Hilbert-Schmidt norm of the matrix $p(x,\xi)^{-1}$ for $|\,\xi\,| \geq C$ satisfies:

$$|\, p(x,\xi)^{-1} \,| \leq A(1+ |\,\xi\,|)^{-d} \quad \text{for} \quad x \in V, \quad |\,\xi\,| \geq C$$

In this event, we say that $p(x,\xi)$ is an *elliptic symbol of order d* over U.

Example 5.5.3. It is trivial to check that for any positive integer d, the symbol:

$$p(x,\xi) = (1+ \mid \xi \mid^2)^d$$

is an elliptic symbol of order $2d$. If we take $a(x) \in C_c^\infty$ with support a compact set K, then the symbol:

$$p(x,\xi) = a(x)(1+ \mid \xi \mid^2)^d$$

will be elliptic over any open set U whose closure is contained in K. Thus elliptic symbols of all even orders exist.

Definition 5.5.4 (Leading symbol)**.** For a differential operator $P = \sum_{|\alpha| \leq d} a_\alpha(x) D_x^\alpha$ of order d, we define its *leading symbol* as:

$$\sigma_L(P) := \sum_{|\alpha|=d} a_\alpha(x)\xi^\alpha$$

Here is a simple criterion for checking ellipticity of a linear differential operator.

Lemma 5.5.5. P is elliptic over U iff $\sigma_L(P)$ is elliptic over U.

Proof: Let P be elliptic over U, of order d, with symbol $p(x,\xi)$. By definition, for $\mid \xi \mid \geq C$, $p(x,\xi)$ is invertible for $x \in V \supset \overline{U}$. Let $q(x,\xi) := (p(x,\xi))^{-1}$ for $x \in V$, and $\mid \xi \mid \geq C$. For $t > 1$, we have by (ii) of the Definition 5.5.2 that for $x \in V$ and $\mid \xi \mid \geq C$.

$$\text{Id} = p(x,t\xi)q(x,t\xi) = t^{-d}p(x,t\xi).t^d q(x,t\xi)$$

On taking limits, we find that $\lim_{t\to\infty} t^{-d}p(x,t\xi) = \sigma_L(P)(x,\xi)$, for all x, ξ. This implies that

$$\lim_{t\to\infty} t^d q(x,t\xi)$$

exists and is finite for $x \in V$ and $\mid \xi \mid \geq C$. Call this limit $r(x,\xi)$. It follows that $r(x,\xi)$ is the inverse of $\sigma_L(P)(x,\xi)$.

Since

$$\mid q(x,t\xi) \mid \leq A(1+t \mid \xi \mid)^{-d} \quad \text{for } \; x \in V, \mid \xi \mid \geq C$$

we clearly have:

$$\mid r(x,\xi) \mid \leq B(1+ \mid \xi \mid)^{-d} \quad \text{for } \; x \in V, \; \mid \xi \mid \geq C$$

Thus it follows that $r(x,\xi) = (\sigma_L(P)(x,\xi))^{-1}$ for $x \in V$ and $\mid \xi \mid \geq C$, and that $\sigma_L(P)$ fulfils both (i) and (ii) of 5.5.2, and hence is an elliptic symbol.

To check the converse, one merely writes:

$$p(x,\xi) = \sigma_L(P)(x,\xi)(I - k(x,\xi))$$

where $\mid k(x,\xi) \mid < 1$ for $\mid \xi \mid$ large enough. Then one uses the geometric series expansion to get

$$p(x,\xi)^{-1} = (\sigma_L(P)(x,\xi)^{-1}(I + k(x,\xi) + k(x,\xi)^2 + ... + ...)$$

for $\mid \xi \mid$ large enough. We leave the estimate for $\mid p(x,\xi)^{-1} \mid$ as an exercise, it follows from the corresponding estimate for $\sigma_L(P)^{-1}$. □

Example 5.5.6. If M is a Riemannian manifold, then in a local coordinate chart U, we can write the Laplacian of M as:

$$\Delta = -\sum_{i,j} g^{ij}\partial_i\partial_j + \text{(lower order terms)}$$

so that on the coordinate chart U, its leading symbol is $-\sum_{i,j} g^{ij}(x)\xi_i\xi_j$, which is certainly elliptic of order 2 all over U, since $[g^{ij}(x)]$ is a positive definite quadratic form for each x.

Definition 5.5.7. Let $P \in \Psi^d$. We say that the ψDO Q is a *parametrix* for P if $Q \in \Psi^{-d}$, and $PQ - I$ and $QP - I$ are infinitely smoothing operators (i.e. are elements of $\Psi^{-\infty}$).

Remark 5.5.8. Note that if P is elliptic of order d over U, $\sigma(P) = p(x,\xi)^{-1}$ exists for all $x \in V$ and $|\xi| \geq C$. It follows that $p(x,\xi)$ is everywhere non-vanishing for $x \in V \supset \overline{U}$ and $|\xi| \geq 2C$. Thus, if the support of p is a compact set K ($\Leftrightarrow$ $\operatorname{supp} P = K$, since P is a differential operator) we must have $K \supset \overline{V}$.

Definition 5.5.9. Let us say a symbol $s(x,\xi)$ is *infinitely smoothing over* V if $\psi(x)s(x,\xi) \in S^{-d}(V)$ for all $\psi \in C_c^\infty(V)$, and all d. (See the Definition 5.3.1). A ψDO P is said to be *infinitely smoothing over* V if its symbol $p(x,\xi)$ is infinitely smoothing over V.

Clearly, since $C_c^\infty(U) \subset C_c^\infty(V)$ for $U \subset V$, we have s is infinitely smoothing over U if it is infinitely smoothing over $V \supset U$.

Lemma 5.5.10. Let $p(x,\xi)$ be an elliptic symbol over U, of order d, and let V, C be as in the Definition 5.5.2, with $\overline{U} \subset V$. Then there exists a symbol $q_0 \in S^{-d}$ such that:

(i): $pq_0 - I$ and $q_0 p - I$ are infinitely smoothing over V_1, where V_1 is any open set satisfying $\overline{U} \subset V_1 \subset \overline{V}_1 \subset V$.

(ii): If p has compact x-support, with $\operatorname{supp}_x p = K$, then the x-support of q_0 satisfies:

$$\operatorname{supp}_x q_0(x,\xi) \subset V \subset \overline{V} \subset K$$

Proof: By hypothesis,

$$|\, p(x,\xi)^{-1} \,| \leq A(1 + |\,\xi\,|)^{-d} \ \text{ for } \ x \in V,\ |\,\xi\,| \geq C$$

Let $\phi(t) \in C_c^\infty(\mathbb{R})$ such that $\phi \equiv 0$ for $t \leq C$ and $\phi \equiv 1$ for $t \geq 2C$. Define:

$$q_0(x,\xi) = \phi(|\,\xi\,|)p(x,\xi)^{-1} \ \text{ for } \ \ x \in V$$

Multiplying q_0 above with a function $\psi \in C_c^\infty(V)$ which is $\equiv 1$ on the subset $V_1 \subset V$, we can assume that $q_0(x,\xi)$ is defined for all $x \in \mathbb{R}^n$, and the above equation defining q_0 holds good for all $x \in V_1$.

Thus $pq_0 - I$ and $q_0 p - I$ are equal to $(\phi(|\,\xi\,|) - 1)I$ for all $x \in V_1$ and all ξ. Since $\phi(|\,\xi\,|) - 1) \equiv 0$ for $|\,\xi\,| \geq 2C$, the operator $(\phi(|\,\xi\,|) - 1)I$ is infinitely smoothing over V_1. The proof that q_0 obeys the decay conditions for a ψDO of order $(-d)$ follows from the decay condition for $|\, p(x,\xi)^{-1} \,|$ in (ii) of the Definition 5.5.2, and formulas like:

$$D_{x_j}(p^{-1}) = -p^{-1}(D_{x_j}p)p^{-1}, \ \ D_{\xi_j}(p^{-1}) = -p^{-1}(D_{\xi_j}p)p^{-1}$$

combined with Leibnitz's rule. This proves (i).

For the statement about x-supports, note that if x-support of p is K, then by the Remark 5.5.8, we have $\overline{V} \subset K$, the x-support of p. Since we have multiplied by the compactly supported function $\psi \in C_c^\infty(V)$ right after the definition of q_0, we have that the support of q_0 is a compact subset of V, and (ii) follows. □

Proposition 5.5.11 (Parametrices for elliptic operators)**.** Let P be an elliptic differential operator of order d, elliptic over U. Assume that $\operatorname{supp} P \subset K$. Let $V \supset \overline{U}$ as in the Definition 5.5.2. Then, there exists a ψDO Q of order $-d$ such that for any open set V_1 satisfying:

$$\overline{U} \subset V_1 \subset \overline{V}_1 \subset V$$

$PQ - I$ and $QP - I$ are infinitely smoothing over V_1.

Proof: Note that by the Remark 5.5.8 above, we must have $K \supset \overline{V} \supset U$.

Define $q_0 \in S^{-d}$ by Lemma 5.5.10 above, so that $pq_0 - I$ and $q_0 p - I$ are infinitely smoothing over V_1. For $k > 0$, we would like to satisfy the formula $\sigma(PQ - I) \sim 0$ and $\sigma(QP - I) \sim 0$, and we would like $q = \sigma(Q)$ to be a sum:

$$q \sim q_0 + q_1 + ... q_j + ...$$

with $q_j \in S^{-d-j}$, in accordance with the Lemma 5.3.3. From (ii) of the Proposition 5.4.4, we see that $\sigma(PQ-I) \sim 0$ results in the requirements:

$$pq_0 - I \sim 0; \quad \text{and} \quad \sum_{0\le|\alpha|\le k} \frac{d_\xi^\alpha p\, D_x^\alpha q_{k-|\alpha|}}{\alpha\,!} \sim 0 \ \text{ for } \ k>0$$

where the sum on the right is the homogeneous component of $\sigma(PQ-I)$ which lies in S^{-k} for $k>0$ (Note that $D_\xi^\alpha p \in S^{d-|\alpha|}$ and $D_x^\alpha q_{k-|\alpha|} \in S^{-d-k+|\alpha|}$). The first is already satisfied by the definition of q_0 and the Lemma 5.5.10, and the second may be rewritten as:

$$pq_k \sim -\sum_{0<|\alpha|\le k} \frac{d_\xi p^\alpha\, D_x^\alpha q_{k-|\alpha|}}{\alpha\,!} \quad \text{for} \ \ k>0$$

where the right hand side involves only $q_0, ..., q_{k-1}$. Since $q_0 p \sim I$, we might as well multiply both sides on the left by q_0, and *define* q_k by the inductive formula:

$$q_k = -q_0 \sum_{0<|\alpha|\le k} \frac{d_\xi^\alpha p\, D_x^\alpha q_{k-|\alpha|}}{\alpha\,!} \quad \text{for} \ \ k>0$$

Indeed, from this inductive definition, it inductively follows that $q_k \in S^{-d-k}$ for all k.

If $\text{supp}_x p = K$, then by (ii) of the Lemma 5.5.10 above, we have $\text{supp}_x q_0 \subset V \subset \overline{V} \subset K$, and by the definition of q_k, we also have $\text{supp}_x q_k \subset V \subset \overline{V} \subset K$ for all k. Then, if one defines $q \sim \sum_j q_j$ by the Lemma 5.3.3, q will be supported in a subset of $\overline{V}$. At any rate, since the inductive definition forces $PQ - I \sim 0$ on V_1, we have that $PQ - I$ is infinitely smoothing on V_1.

By a similar procedure, one may define $Q' \in S^{-d}$ such that $Q'P - I$ is infinitely smoothing on V_1. But then since pre or post-composing an infinitely smoothing operator with any ψDO leads to an infinitely smoothing operator (by (ii) of Proposition 5.4.4), we have:

$$Q' \sim Q'.I \sim Q'PQ \sim I.Q \sim Q$$

on V_1. The proposition follows. □

6. ψDO's and Elliptic Operators on Compact Manifolds

We revert to the setup of §4. Let E, F be smooth complex vector bundles on a compact manifold M, and let $\{U_i\}_{i=1}^N$ be an open covering of M such that U_i is diffeomorphic to $\mathbb{R}^n$ for each i, and the restricted bundles $E_{|U_i}$ and $F_{|U_i}$ are both trivial (of ranks k and m respectively). $\{\lambda_i\}$ is a smooth partition of unity subordinate to $\{U_i\}$.

6.1. **Basic definitions and lemmas.**

Definition 6.1.1. Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a $\mathbb{C}$-linear operator. We say P is a *ψDO or pseudo-differential operator on M* of order d if for all $i,j \in 1,2,...,N$, and all $\psi \in C_c^\infty(U_j)$ and $\phi \in C_c^\infty(U_i)$, the "localised operators"

$$\psi P \phi : C^\infty(U_i, E_{|U_i}) \to C^\infty(U_j, F_{|U_j})$$

are $\psi DO's$ of order d, where (the domain and target are identified with $\mathcal{E}^k$ and $\mathcal{E}^m$ respectively). That this definition makes sense follows from the Corollary 5.3.7. The $\mathbb{C}$-vector space of these ψDO's of order d is denoted $\Psi^d(M)$, where we have suppressed $E,\ F$ from the notation for brevity.

Furthermore, we will call P as above a *linear differential operator* of order d if all the localisations above are differential operators of order d. We will call it an *elliptic* differential operator if each of these localisations $\psi P \phi$ are elliptic over each open set U satisfying $\overline{U} \subset \{x : \phi(x)\psi(x) \neq 0\} \subset U_i \cap U_j$.

We now have an analogue of the Proposition 5.2.5.

Proposition 6.1.2. Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a ψDO of order d. Then P extends to a continuous (=bounded) linear operator of Hilbert Spaces:

$$P : H_{s+d}(M,E) \to H_s(M,F)$$

where the Sobolev spaces $H_{s+d}(M,E)$ and $H_s(M,F)$ are as defined in Definition 4.2.1.

Proof: Let $\{\lambda_i\}$ be the partition of unity as described above at the beginning of this section (i.e. as in §4.2), subordinate to the open covering $\{U_i\}_{i=1}^N$. By the foregoing definition, we have $\lambda_i P \lambda_j$ a ψDO, with symbol of compact support. Now, for $f \in C^\infty(M,E)$, we compute, using $f = \sum_{j=1}^N \lambda_j f_{|U_j}$, that:

$$\begin{aligned} \|Pf\|_s^2 &= \left\| \sum_{j=1}^N P\lambda_j f_{|U_j} \right\|_s^2 \leq C \sum_{j=1}^N \left\| P\lambda_j f_{|U_j} \right\|_s^2 \\ &= C \sum_{i,j=1}^N \left\| \lambda_i P \lambda_j f_{|U_j} \right\|_s^2 \leq C \sum_{i,j=1}^N C_{ij} \left\| f_{|U_j} \right\|_{s+d}^2 \\ &\leq C \|f\|_{s+d}^2 \end{aligned}$$

where we have used the Definition 4.2.1, Proposition 5.2.5 applied to $\lambda_i P \lambda_j$ and $\left\| f_{|U_j} \right\|_{s+d}^2 \leq \|f\|_{s+d}^2$ to arrive at the last line. The proposition follows. □

Similarly, one can deduce the pseudolocal property of ψDO's on M by appealing to the Proposition 5.3.8, whose statement and proof we leave as an exercise.

Proposition 6.1.3. Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a ψDO of order d. Using Hermitian metrics on E and F, gives global L_2-inner products on $C^\infty(M,E)$ and $C^\infty(M,F)$ (which we called $(-,-)$ in (iii) of Proposition 4.2.2), call them $(-,-)_E$ and $(-,-)_F$ respectively. Define the L_2-adjoint of P by the formula:

$$(P^* f, g)_E = (f, Pg)_F \quad \text{for } f \in C^\infty(M,F), \;\; g \in C^\infty(M,E)$$

Then P^* is a ψDO of order d.

If $P : C^\infty(M,E) \to C^\infty(M,F)$ is a ψDO of order d, and $Q : C^\infty(M,F) \to C^\infty(M,G)$ is a ψDO of order e, the composite $QP : C^\infty(M,E) \to C^\infty(M,G)$ is a ψDO of order $d+e$.

Proof: Let ϕ, ψ be as in Definition 6.1.1. Then, by the definition of P^*, we have:

$$(\phi P^* \psi f, g)_E = (P^* \psi f, \overline{\phi} g)_E = (\psi f, P\overline{\phi} g)_F = (f, \overline{\psi} P \overline{\phi} g)_F$$

which implies that $\phi P^* \psi = (\overline{\psi} P \overline{\phi})^*$. Because the right hand expression is a ψDO of order d by definition 6.1.1 and (i) of Proposition 5.4.4, it follows that P^* is a ψDO of order d.

For the composite, note that if P and Q are ψDO's of orders d and e, and ϕ and ψ are as in the last paragraph, we may write:

$$\phi P Q \psi = \sum_{i=1}^N \phi P \tau_i \lambda_i Q \psi$$

where $\tau_i \in C_c^\infty(U_i)$ is a function which is $\equiv 1$ on the support of λ_i, and therefore satisfies $\tau_i \lambda_i \equiv \lambda_i$ for all i. Now we can appeal to (ii) of the Proposition 5.4.4 to conclude that each term $(\phi P \tau_i)(\lambda_i Q \psi)$ on the right is a ψDO of order $d+e$, and hence so is their sum. □

6.2. **Elliptic operators on manifolds and parametrices.** Now we come to the most crucial proposition about elliptic differential operators on compact manifolds.

Proposition 6.2.1 (Parametrices for elliptic operators on manifolds)**.** Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be an *elliptic* differential operator on the compact manifold M. Then there exists a ψDO $Q : C^\infty(M,F) \to C^\infty(M,E)$ of order $-d$ such that $PQ - I$ and $QP - I$ are infinitely smoothing operators.

Proof: It is enough to construct the "left" parametrix satisfying $QP - I \in \Psi^{-\infty}(M)$, for by the same argument as the last paragraph of Proposition 5.5.11, it serves as the "right" parametrix too.

So let λ_i, U_i be as at the outset of this section. Let us denote:

$$W_i := \{x : \lambda_i(x) \neq 0\} \subset U_i$$

By the choices and definitions made in the past, the closure $\overline{W}_i$ is a compact subset of U_i for all $i = 1, 2., , N$. Let $\psi_i \in C_c^\infty(U_i)$ with $\psi_i \equiv 1$ on $\overline{W}_i$. Let $\rho_i \in C_c^\infty(U_i)$ with $\rho_i \equiv 1$ on the support $\operatorname{supp}\psi_i$, for $i = 1, 2, .., N$.

Consider the localisation $\psi_i P \rho_i$. It is easy to check that $W_i = \{x : \lambda_i(x) \neq 0\}$ is an open subset of

$$\{x : \psi_i(x) \neq 0\} \cap \{x : \rho_i(x) \neq 0\}$$

and indeed $\overline{W}_i$ is contained in the intersection above. Thus, by the Definition 6.1.1, $\psi_i P \rho_i$ is elliptic over W_i.

Since P is a *differential operator*, and $\rho_i \equiv 1$ on $\operatorname{supp}\psi_i$, we have $\psi_i P \rho_i = \psi_i P$ for all $i = 1, 2, .., N$. Thus $\psi_i P$ is elliptic over W_i. Also $\psi_i P$ has support contained in the compact set $K_i = \operatorname{supp}\psi_i$.

Thus, by the Proposition 5.5.11, there exists an open set $V_i \supset \overline{W}_i$ and a parametrix Q_i which is a ψDO of order $-d$ such that $Q_i(\psi_i P) - I$ is infinitely smoothing over V_i. That is, $\lambda(Q_i(\psi_i P - I)$ is infinitely smoothing on M for all $\lambda \in C_c^\infty(V_i)$. In particular, since $\operatorname{supp}\lambda_i = \overline{W}_i$ is a compact subset of V_i, we have $\lambda_i(Q_i(\psi_i P) - I)$ is in $\Psi^{-\infty}(M)$. Hence so is the sum:

$$\sum_i \lambda_i(Q_i(\psi_i P) - I) = \sum_i (\lambda_i Q \psi_i) P - I$$

since $\sum_i \lambda_i \equiv 1$. But this means that $Q := \sum_i \lambda_i Q \psi_i$ is the desired left parametrix. It is of order $-d$ because each term in this sum is of order $-d$. □

One of the deepest consequences of the existence of a parametrix for an elliptic differential operator is the following proposition.

Proposition 6.2.2 (Garding-Friedrichs Inequality)**.** Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be an elliptic differential operator of order d. Then there exists a constant $C > 0$ (depending only on P, M, E and F) such that:

$$\|f\|_{s+d} \leq C\left(\|Pf\|_s + \|f\|_s\right) \quad \text{for all} \quad f \in H_{s+d}(M,E)$$

Proof: Let Q be the parametrix for P from the previous Proposition 6.2.1. Then, by definition:

$$f = QPf + Sf$$

where $S : C^\infty(M,E) \to C^\infty(M,E)$ is infinitely smoothing. Thus

$$\|f\|_{s+d} \leq \|QPf\|_{s+d} + \|Sf\|_{s+d}$$

Since S is in $\Psi^{-\infty}(M)$, it is in $\Psi^{-d}(M)$, so by the Proposition 6.1.2, we have:

$$\|Sf\|_{s+d} \leq C\,\|f\|_s$$

By the same proposition, since $Q \in \Psi^{-d}(M)$, we have:

$$\|QPf\|_{s+d} \leq C\,\|Pf\|_s$$

Thus the desired inequality follows. □

Corollary 6.2.3 (An equivalent Sobolev norm)**.** Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order d. Let $(-,-)$ denote the global L_2 inner product on $C^\infty(M,E)$ as before. Then the norm associated to the inner product:

$$\langle f,g\rangle := (f,g) + (Pf,Pg) \quad f,g \in C^\infty(M,E)$$

is equivalent to the Sobolev norm $\| \ \|_d$ on $C^\infty(M,E)$ defined in the Definition 4.2.1. Hence completing $C^\infty(M,E)$ with respect to the norm defined by $\langle -,-\rangle$ gives exactly the Sobolev space $H_d(M,E)$.

Proof: Let us denote:

$$\|f\|' := \langle f,f\rangle^{\frac{1}{2}}$$

for $f \in C^\infty(M,E)$. Then, noting that $(-,-) = (-,-)_0$, the Sobolev 0-norm, we have

$$\begin{aligned} \|f\|'^2 &= \|Pf\|_0^2 + \|f\|_0^2 \\ &\leq C\|f\|_d^2 + \|f\|_d^2 \leq C\|f\|_d^2 \end{aligned}$$

where we have used the Proposition 6.1.2, and the fact that $\|f\|_0 \leq \|f\|_d$ for $d \geq 0$ in the last line above.

On the other hand, by the Garding-Friedrichs inequality of 6.2.2, we have:

$$\begin{aligned} \|f\|_d &\leq C(\|Pf\|_0 + \|f\|_0) \\ &\leq C(\|f\|' + \|f\|') = 2C\|f\|' \end{aligned}$$

Thus our proposition follows. Since $H_d(M,E)$ is the completion of $C^\infty(M,E)$ with respect to $\| \ \|_d$, and the last norm is equivalent to $\| \ \|'$, the second statement of the proposition follows. □

Proposition 6.2.4 (Elliptic Regularity Theorem)**.** Let $P = \sum_\alpha a_\alpha(x)D_x^\alpha : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of degree $d \geq 1$, so that $P : \mathcal{D}'(M,E) \to \mathcal{D}'(M,E)$ gets defined on distributional sections (see the Definition 4.1.4) by the formula:

$$Pf(g) = f\left(\sum_\alpha (-1)^{|\alpha|} D_x^\alpha a_\alpha(x)^t g\right) \quad \text{for} \quad f \in \mathcal{D}'(M,E),\ g \in C^\infty(M,E) = \mathcal{D}(M,E)$$

Let $f \in \mathcal{D}'(M,E)$ be a distributional solution to:

$$Pf = g$$

where $g \in H_s(M,E)$. Then $f \in H_{d+s}(M,E)$. In particular, if g is smooth, then f is also smooth.

Proof: Since M is compact, we have from (v) of Proposition 4.2.2 that $\mathcal{D}'(M,E) = \cup_k H_k(M,E)$. Thus $f \in H_k(M,E)$ for some k. Let $Q \in \Psi^{-d}(M)$ be a parametrix for P, by the Proposition 6.2.1. Then, by definition, the operator $S := QP - I \in \Psi^{-\infty}(M)$ is infinitely smoothing, and we have:

$$f = QPf + Sf = Qg + Sf$$

But since $g \in H_s(M,E)$ and $Q \in \Psi^{-d}(M)$, we have $Qg \in H_{s+d}(M,E)$, by Proposition 6.1.2. Also $f \in H_k(M,E)$ and $S \in \Psi^{-\infty}(M)$ implies $S \in \Psi^{k-d-s}(M,E)$, so that again by 6.1.2, we have $Sf \in H_{d+s}(M,E)$. Thus $f \in H_{d+s}(M,E)$.

If $g \in C^\infty(M,E)$, we have $g \in H_s(M,E)$ for all s by the Sobolev Embedding Theorem (iv) of Proposition 4.2.2. The last paragraph implies that $f \in H_{s+d}(M,E)$ for all s, i.e. $f \in H_\infty(M,E) = C^\infty(M,E)$ by the same proposition. □

7. Elliptic operators on $\mathbb{R}^n$

7.1. **Parametrices on $\mathbb{R}^n$.** It is quite natural to ask what the analogues of the results obtained in the last section are in the setting of $\mathbb{R}^n$.

Definition 7.1.1. Let $P = \sum_\alpha a_\alpha(x) D_x^\alpha$ be a linear differential operator of order d. Then say that P is *elliptic* if it is elliptic over a neighbourhood U of each point $x \in \mathbb{R}^n$ (in the sense of Definition 5.5.2. (Note that this is weaker than saying that it is elliptic over $\mathbb{R}^n$, because we are not demanding one single constant C for all $x \in \mathbb{R}^n$)

Proposition 7.1.2 (Existence of parametrices)**.** Let P be an elliptic linear differential operator on $\mathbb{R}^n$ of order d. Then there exists a $q(x, \xi)$ of such that:

(i): $\rho(x)q(x,\xi) \in S^{-d}$ for all $\rho \in C_c^\infty(\mathbb{R}^n)$.

(ii): For a relatively compact subset $W \subset \mathbb{R}^n$, let $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\rho(x) \equiv 1$ for all $x \in W$. Then for the ψDO Q corresponding to $\rho(x)q(x,\xi)$, the $\psi DO's$ $PQ - I$ and $QP - I$ are infinitely smoothing over W.

Proof: By definition, we have P elliptic over U_α, for $\{U_\alpha\}$ an open covering of $\mathbb{R}^n$. By appealing to paracompactness and second countability of $\mathbb{R}^n$, we have a countable locally finite open covering $\{U_i\}_{i=1}^\infty$ of $\mathbb{R}^n$ such that P is elliptic over U_i. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\{U_i\}$.

By the Proposition 5.5.11 , there are ψDO's Q_i which satisfy $PQ_i - I$ is infinitely smoothing over $V_{1,i}$ where $V_{1,i} \supset \overline{U_i}$. That is, $\rho_i(PQ_i - I) = S_i$, where $S_i \in \Psi^{-\infty}$ for all $\rho_i \in C_c^\infty(V_{1,i})$. If we take $\rho_i \equiv 1$ on U_i, we have:

$$PQ_i - I = S_i; \quad Q_iP - I = T_i \quad \text{on } U_i$$

where S_i, T_i are the restrictions to U_i of some infinitely smoothing operators in $\Psi^{-\infty}$. Since we can replace P above by $\rho_i P$ on U_i, we can also assume by the last para of the proof of Proposition 5.5.11 that the x-supports of q_i are compact sets for all i, as are the x-supports of $t_i = \sigma(T_i)$ and $s_i = \sigma(S_i)$.

The trouble is that Q_i and Q_j won't generally agree on the overlaps $U_i \cap U_j$. However, we do know that for $x \in U_i \cap U_j$,we have:

$$\begin{aligned} q_i(x,\xi) &= \sigma(Q_i) = \sigma(Q_i.I) = \sigma\left(Q_i(PQ_j - S_j)\right) = \sigma\left(Q_iPQ_j - Q_iS_j\right) = \sigma\left(Q_j + T_iQ_j - Q_jS_i\right) \\ &= q_j(x,\xi) + r_{ij}(x,\xi) \end{aligned}$$

where $r_{ij} = \sigma(R_{ij}) := \sigma(T_iQ_j - Q_iS_j)$. By the formula in (ii) of 5.4.4, the symbols $\sigma(T_iQ_j)$ and $\sigma(Q_iS_j)$ are also compactly supported, and we may as well assume that the support $\text{supp}_x r_{ij}(x,\xi)$ is compact for all i, j.

Finally, by 5.5.11, each R_{ij} is the restriction of an infinitely smoothing operator (the pre and post composition of an infinitely smoothing operator with any ψDO is infinitely smoothing), call it R_{ij} again, to $U_i \cap U_j$. Thus $r_{ij} \in S^{-\infty}$.

Also note that for $x \in U_i \cap U_j$ we have $r_{ij}(x,\xi) = -r_{ji}(x,\xi)$, and on the triple intersection $U_i \cap U_j \cap U_k$ we have the *cocycle condition* on the r_{ij}'s:

$$r_{ij}(x,\xi) + r_{jk}(x,\xi) + r_{ki}(x,\xi) = (q_i - q_j) + (q_j - q_k) + (q_k - q_i) = 0 \quad \text{for } x \in U_i \cap U_j \cap U_k$$

Now we borrow a trick from sheaf theory and define:

$$k_i(x,\xi) = \sum_l \lambda_l r_{il}(x,\xi)$$

since λ_j are a partition of unity , the sum on the right makes sense. Unfortunately, k_i *are no longer compactly supported*, and hence the decay conditions on $r_{ij}(x,\xi)$ will no longer translate into global decay conditions for k_i. However, for *any relatively compact subset* $W \subset \mathbb{R}^n$, W will meet only finitely many of the locally finite collection U_i, say for $i \in F$. Then, since we have conditions:

$$| D_x^\alpha D_\xi^\beta r_{ij}(x,\xi) | \leq C_{\alpha,\beta}^{ij}(1 + | \xi |)^{-k} \quad \text{for all } i, j, \ \alpha, \ \beta, \ k, \ x$$

we will get a corresponding condition:

$$| D_x^\alpha D_\xi^\beta k_i(x,\xi) | \leq C^i(W)_{\alpha,\beta}(1+ | \xi |)^{-k} \quad \text{for all } \ i,j, \ \alpha, \ \beta, \ k, \ x \in \overline{W}$$

by majorising all the derivatives of $\{\lambda_l\}_{l\in F}$ upto order α and the $C^{ij}_{\alpha,\beta}$ over $\overline{W}$. This implies that that k_i is infinitely smoothing over every W which is relatively compact.

Also we have:

$$k_i(x,\xi) - k_j(x,\xi) = \sum_l (\lambda_l r_{il} - \lambda_l r_{jl}) = \sum_l \lambda_l(-r_{li} - r_{jl}) = \sum_l \lambda_l r_{ij} = r_{ij}(x,\xi) \ \text{ for } \ x \in U_i \cap U_j$$

This implies:

$$q_i(x,\xi) - q_j(x,\xi) = k_i(x,\xi) - k_j(x,\xi) \ \text{ for } \ x \in U_i \cap U_j$$

which implies that $q_i(x,\xi) - k_i(x,\xi) = q_j(x,\xi) - k_j(x,\xi)$ for $x \in U_i \cap U_j$. Let us define a global function:

$$q(x,\xi) := q_i(x,\xi) - k_i(x,\xi) \quad \text{for } \ x \in U_i$$

Then q makes sense all over $\mathbb{R}^n$. It may not be a symbol for the simple reason that k_i are no longer globally defined symbols. However, from the decay properties above for k_i on a relatively compact open set $W \subset \mathbb{R}^n$, it is trivial to check that $\rho Q \in S^{-d}$ for all $\rho \in C_c^\infty(\mathbb{R}^n)$. It is also readily verified that if W is a relatively compact subset of $\mathbb{R}^n$ with $\rho \equiv 1$ on W, and Q is the ψDO corresponding to $\rho(x)q(x,\xi)$, we have $\sigma(PQ - I)$ is infinitely smoothing over W. Likewise for $QP - I$. This proves the proposition. □

Definition 7.1.3. Let W be a *relatively compact* (=bounded) open subset of $\mathbb{R}^n$. Define the Sobolev space $H_s^0(W)$ to be the closure of $C_c^\infty(W)$ with respect to the Sobolev s-norm $\| \ \|_s$. Note that it is a closed subspace of $H_s(\mathbb{R}^n)$ by definition.

Proposition 7.1.4. Let $W \subset \mathbb{R}^n$ be a relatively compact open set, and let $P \in \Psi^d$ be a ψDO or order d with symbol $p(x,\xi) \in S^d$. Assume that the support $\text{supp}_x p(x,\xi)$ is a compact. Then

$$P : H^0_{s+d}(W) \to H_s(\mathbb{R}^n)$$

is a bounded operator. If further the compact subset $K = \text{supp}_x p(x,\xi)$ is contained in W, then P is a bounded operator from $H^0_{s+d}(W) \to H^0_s(W)$

Proof: The first statement is clear from the Proposition 5.2.5, because with the compact x-support hypothesis imposed on p, we have $P : H_{s+d}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is a bounded operator, and $H^0_{s+d}(W)$ is a closed subspace of $H_{s+d}(\mathbb{R}^n)$, so the restriction to this subspace is also bounded.

For the second statement, let $f \in H^0_{s+d}(W)$, and let $f_n \in C_c^\infty(W)$ be a sequence of smooth functions with $\|f_n - f\|_{s+d} \to 0$. Since p is compactly supported, and f_n are clearly Schwartz class, the Proposition 5.2.5 implies that Pf_n are smooth Schwartz class functions on $\mathbb{R}^n$. Also, the formula:

$$Pf_n(x) = \int e^{ix.\xi} p(x,\xi) \widehat{f_n}(\xi) d\xi$$

shows that $\text{supp}_x Pf_n \subset \text{supp}_x p(x,\xi) = K \subset W$. Thus $Pf_n \in C_c^\infty(W)$. Also, since the x-support of p is compact, we have by 5.2.5 that:

$$\|Pf_n - Pf_m\|_s \leq C \|f_n - f_m\|_{s+d}$$

Thus $\{Pf_n\}$ is a Cauchy sequence in $H_s^0(W)$, and since it converges to $Pf \in H_s(\mathbb{R}^n)$, and the subspace $H_s^0(\mathbb{R}^n)$ is a closed subspace of $H_s(\mathbb{R}^n)$, it follows that $Pf \in H_s^0(W)$. By the first part, the restricted operator:

$$P : H^0_{s+d}(W) \to H^0_s(W)$$

is also a bounded operator. The proposition follows. □

Proposition 7.1.5 (Garding-Friedrichs Inequality II)**.** Let W be a relatively compact open subset of $\mathbb{R}^n$, and let P be a linear differential operator elliptic over W. Then there exists a constant depending only on W and P such that:

$$\|f\|_{s+d} \leq C\,(\|Pf\|_s + \|f\|_s) \quad \text{for } \; f \in H^0_{s+d}(W)$$

Proof: By hypothesis, there is a open set $V \supset \overline{W}$ and a constant C such that that $p(x,\xi)$ is invertible for $x \in V$ and $\mid \xi \mid \geq C$, and the following estimate holds:

$$\mid p(x,\xi)^{-1} \mid \leq C(1+ \mid \xi \mid)^{-d} \quad \text{for} \quad x \in V, \; \mid \xi \mid \geq C$$

Let $\rho \in C_c^\infty$ be a smooth function which is identically 1 on V, and hence identically 1 on W. Then since P is a differential operator, we have $\rho\, Pf = Pf$ for all $f \in C_c^\infty(W)$. Also $\rho\, P$ is clearly elliptic over W by the above criterion, so without loss of generality, we may assume that $p(x,\xi) := \sigma(P)$ has compact x-support.

By the Proposition 5.5.11, there exists a ψDO Q which is of order $(-d)$, also having compact x-support for its symbol, and satisfying

$$QP - I = S$$

where S is infinitely smoothing over $V_1 \supset \overline{V}$. This means τS is in $\Psi^{-\infty}$ for every $\tau \in C_c^\infty(V_1)$. Let us choose a τ which is identically 1 on W. Then we have:

$$f = \tau f = (\tau I)f = \tau\, QPf - \tau Sf \quad \text{for } \; f \in C_c^\infty(W)$$

Thus

$$\|f\|_{s+d} \leq \|\tau QPf\|_{s+d} + \|(\tau S)f\|_{s+d} \quad \text{for} \quad f \in C_c^\infty(W)$$

Since P is a differential operator, $Pf \in C_c^\infty(W)$ as well, and since τQ is a compactly supported ψDO of order $-d$, we have by the first part of the last Proposition 7.1.4 that:

$$\|\tau QPf\|_{s+d} \leq C\,\|Pf\|_s$$

Because τS is also a compactly supported ψDO in $\Psi^{-\infty} \subset \Psi^{-d}$, and $f \in C_c^\infty(W)$, we have similarly:

$$\|\tau Sf\|_{s+d} \leq C\,\|f\|_s$$

by the same Proposition 7.1.4. Thus we have the desired inequality for all $f \in C_c^\infty(W)$.

Now let $f \in H^0_{s+d}(W)$. Choose a sequence $f_n \in C_c^\infty(W)$ with $f_n \to f$ in $H^0_{s+d}(W)$. Since P has compact x-support, it follows by the first part of 7.1.4 that $Pf_n \to Pf$ in $H_s(\mathbb{R}^n)$. Since the inclusion $H^0_{s+d}(W) \to H_{s+d}(\mathbb{R}^n) \to H_s(R^n)$ is continuous, we also have $f_n \to f$ in $H_s(\mathbb{R}^n)$. Thus the norms $\|Pf_n\|_s \to \|Pf\|_s$ and $\|f_n\|_s \to \|f\|_s$. Thus we have:

$$\|f\|_{s+d} = \lim_{n\to\infty} \|f_n\|_{s+d} \leq C \lim_{n\to\infty} (\|Pf_n\|_s + \|f_n\|_s) = C(\|Pf\|_s + \|f\|_s)$$

which proves the proposition. □

8. Operators on Hilbert Spaces and Fredholm Theory

$\mathcal{H}$ will always denote a separable complex Hilbert space, with inner product denoted $\langle -,-\rangle$, which is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second. $\mathcal{B}(\mathcal{H})$ will denote the algebra of bounded operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, its *adjoint* is the operator $T^* \in \mathcal{B}(\mathcal{H})$, and is the operator defined by $\langle Tx, y\rangle = \langle x, T^*y\rangle$. This defines an involution on $\mathcal{B}(\mathcal{H})$ and makes it C^*-algebra. More generally, for $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ (=the space of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$), the adjoint $T^* \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ is defined by the formula $\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$, where $\langle -,-\rangle_i$ are the inner products in the Hilbert spaces $\mathcal{H}_i$.

8.1. Compact Operators.

Definition 8.1.1 (Compact operator)**.** Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. Then $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be a *compact operator* if for every bounded sequence x_n in $\mathcal{H}_1$, the sequence Tx_n in $\mathcal{H}_2$ contains a convergent subsequence. Because our Hilbert spaces are always assumed separable, this condition is equivalent to saying that the image $T(B)$ of each bounded set $B \subset \mathcal{H}_1$ has compact closure in $\mathcal{H}_2$. The subset of compact operators in $\mathcal{B}(\mathcal{H})$ is denoted $\mathcal{K}(\mathcal{H})$.

Example 8.1.2. Clearly, the identity operator $I \in \mathcal{B}(\mathcal{H})$ is a compact operator if and only if $\mathcal{H}$ is finite dimensional.

Example 8.1.3 (Linear maps of finite rank)**.** If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear map such that $\dim \operatorname{Im} T < \infty$, then T is a compact operator. For compactness, note that the Heine-Borel theorem for $V = \mathbb{C}^n$ implies that every bounded subset of V has compact closure. Thus if T has finite dimensional image, the image $T(B)$ of every bounded subset $B \subset \mathcal{H}$ would be a bounded subset of the finite dimensional space $V = \operatorname{Im} T \subset \mathcal{H}$, and hence have compact closure.

Next, if $T \in \mathcal{B}(\mathcal{H})$ is such that $\ker T$ has finite codimension, i.e. $\dim\, (\ker\, T)^{\perp} < \infty$, then again T is compact. For then, T would induce a linear embedding:

$$\widetilde{T} : (\ker\, T)^{\perp} \to \mathcal{H}$$

whose image is the same as $\operatorname{Im} T$. But since $\operatorname{Im} \widetilde{T}$ is finite dimensional, we have $\dim \operatorname{Im} T < \infty$ as well, so T is a bounded operator of finite rank, and a compact operator by the above discussion. Finally, if $\mathcal{H}$ is itself finite dimensional, then $\operatorname{End}_{\mathbb{C}}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$.

Example 8.1.4 (Diagonal operators)**.** Let $T \in \mathcal{B}(\mathcal{H})$, and $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$ such that $Te_n = \lambda_n e_n$ for every n, where $\lambda_n \in \mathbb{C}$. Then (exercise) T is compact iff $\lim_{n\to\infty} \lambda_n \to 0$.

Example 8.1.5 (The Green Operator on S^1)**.** The Hilbert space $\mathcal{H} := L_2(S^1)$ has an orthonormal basis $\{e_n := e^{int}\}_{n\in\mathbb{Z}}$ where $0 \le t < 2\pi$ is the angle parameter on S^1. The *Green operator* on S^1 is the operator defined by:

$$\begin{aligned} G : \mathcal{H} &\to \mathcal{H} \\ e_n &\mapsto \frac{e_n}{n^2} \ \text{ for } \ n \neq 0 \\ &\mapsto 0 \ \text{ for } \ n = 0 \end{aligned}$$

In view of the previous example 8.1.4, this operator G is compact. It has the following significance. For the Laplace operator $\Delta : C^\infty(S^1) \to C^\infty(S^1)$ on S^1, defined by $\Delta = \frac{-d^2}{dt^2}$ on the circle, we have an extension to the domain of Δ, call it $\mathcal{D} := \operatorname{dom} \Delta \subset \mathcal{H}$. We note that e_n satisfy $\Delta e_n = n^2 e_n$ for $n \in \mathbb{Z}$. Thus $\mathcal{D}$ consists of all $f = \sum_n \widehat{f}(n) e_n \in \mathcal{H}$ such that the series $\sum_{n\in\mathbb{Z}} n^4 |\widehat{f}(n)|^2$ is convergent. That is, the sequence $\{n^2 \widehat{f}(n)\}_{n\in\mathbb{Z}}$ should be in $l_2(\mathbb{Z})$. Note that $\mathcal{D}$ is a proper L_2-dense linear subspace of $\mathcal{H}$ (it contains each e_n!).

In fact, we see that $\mathcal{D}$ is set-theoretically the Sobolev space $H_2(S^1, E)$ for the trivial bundle $E = S^1 \times \mathbb{C}$. This is because Δ is clearly an elliptic operator (its leading symbol is $\equiv -1$ in any chart with coordinate t), and for $f \in C^\infty(S^1)$, we have Δf is given by convergent Fourier series $\sum_{n\in\mathbb{Z}} n^2 \widehat{f}(n) e_n$, so that the $L_2(S^1)$ norm (= Sobolev 0-norm $\| \ \|_0$) of Δf is given by:

$$\|\Delta f\|^2 = \sum_{n\in\mathbb{Z}} n^4 \mid \widehat{f}(n) \mid^2$$

By the Garding-Friedrichs inequality and its Corollary 6.2.3, we have:

$$\|f\|_2^2 = \|f\|^2 + \|\Delta f\|^2$$

and this is finite iff $f \in \mathcal{H}$ and $\Delta f \in \mathcal{H}$, i.e. iff $f \in \mathcal{D}$.

Note that since we are putting the L_2-norm (and *not* the Sobolev 2- norm) on $\mathcal{D}$, the operator $\Delta : \mathcal{D} \to \mathcal{H}$ is an unbounded operator. Indeed, $\|e_n\| = 1$ but $\|\Delta e_n\| = n^2$. However, we claim that Δ has *closed range in* $\mathcal{H}$, and $\operatorname{Im}\Delta = (\mathbb{C}e_0)^\perp$, the closed subspace of all functions in $\mathcal{H}$ which are orthogonal to e_0, i.e.

$$\operatorname{Im}\Delta = \left\{ f \in \mathcal{H} : \int_0^{2\pi} f(t)dt = 0 \right\}$$

This is seen as follows. Note that the Green operator G defined above satisfies the identity:

$$I_{\mathcal{H}} = \pi_0 + \Delta G$$

where π_0 is orthogonal projection onto the space $\mathbb{C}e_0 = \ker\Delta$, and defined by $\pi_0 f = \langle f, e_0\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$. The above identity makes sense since $G(\mathcal{H}) \subset \mathcal{D}$. It is true on all of $\mathcal{H}$ because it is trivially checked to be true for all $e_n,\ n \in \mathbb{Z}$. It follows that the image $\operatorname{Im}\Delta$ is nothing but $\operatorname{Im}(Id - \pi_0) = \operatorname{Im}\pi_1$ where $\pi_1 : \mathcal{H} \to (\mathbb{C}e_0)^\perp$ is the complementary orthogonal projection to π_0. Thus $\operatorname{Im}\Delta$ is $(\mathbb{C}e_0)^\perp$, which is closed. Thus G is an 'inverse' to Δ on $\operatorname{Im}\Delta$, and gives a Hilbert space isomorphism between $(\mathbb{C}e_0)^\perp$ and $\operatorname{Im}\Delta$. Note that $\ker G = \ker\Delta = \mathbb{C}e_0$.

Similarly, we have the other identity:

$$\pi_{1|\mathcal{D}} = I_{\mathcal{D}} - \pi_{0|\mathcal{D}} = G\Delta$$

which holds on $\mathcal{D}$.

Example 8.1.6. Let M be a compact Riemannian manifold. Then, by Rellich's Lemma in (vi) of the Proposition 4.2.2, the inclusion:

$$i : H_s(M, E) \hookrightarrow H_t(M, E)$$

is a compact operator.

Example 8.1.7. If we take a *non-compact* manifold, say $M = \mathbb{R}$. Then as pointed out in the Exercise 3.3.3, take a fixed function $\phi \in H_1(\mathbb{R})$ of $\|\phi\|_1 = 1$, with compact support in say $(-\frac{1}{2}, \frac{1}{2})$, and consider its translates $\phi_n = \phi(x + n)$. Clearly, by (ii) of the Proposition 3.1.4,

$$\|\phi_n\|_1^2 = \|\phi_n\|_0^2 + \|D_x\phi_n\|_0^2 = \|\phi\|_1^2 \ \text{ for all } \ n$$

so that $\{\phi_n\}$ is a bounded sequence in $H_1(\mathbb{R})$. But $\{\phi_n\}$ can have no convergent subsequence in $H_0(\mathbb{R})$. Indeed, since ϕ_n and ϕ_m have disjoint supports for $n \neq m$, we have $\langle\phi_n, \phi_m\rangle = 0$ for $n \neq m$, which implies $\|\phi_n - \phi_m\|_0 = \sqrt{2}\,\|\phi\|_0$ for all $n \neq m$. Hence $\{\phi_n\}$ cannot have a Cauchy subsequence in $H_0(\mathbb{R})$. Thus the inclusion $H_1(\mathbb{R}) \hookrightarrow H_0(\mathbb{R})$ is *not* compact.

Example 8.1.8. Let M be a compact Riemannian manifold, and let:

$$P : H_s(M, E) \to H_{s+d}(M, E)$$

be any pseudo-differential operator of order $-d < 0$ (See the Proposition 6.1.2). Then, the composite:

$$H_s(M, E) \xrightarrow{P} H_{s+d}(M, E) \hookrightarrow H_s(M, E)$$

is a compact operator. This is because P is a bounded operator $H_s(M,E) \to H_{s+d}(M,E)$, and $i : H_{s+d}(M,E) \to H_s(M,E)$ is a compact operator by 8.1.6 above, and it is easy to check that pre or post composing a bounded operator with a compact operator results in a compact operator (See Proposition 8.2.1 below).

In particular, if Q is a parametrix for an elliptic differential operator P on M of order $d > 0$, then the composite

$$H_s(M,E) \stackrel{Q}{\to} H_{s+d}(M,E) \hookrightarrow H_s(M,E)$$

is a compact operator for each s, since Q is of order $-d$.

The Green Operator cited in the Example 8.1.5 above is a particular case for $M = S^1$ and $E = M \times \mathbb{C}$, the trivial bundle. For Δ is clearly an elliptic operator of order 2 on S^1, and by the Proposition 6.2.1, has a parametrix Q, which is precisely the operator G, because as we remarked above $I - \Delta G$ and $I - G\Delta$ give projection to e_0, which is the constant function 1 on S^1, and hence infinitely smoothing. Since G is an operator of order -2, if we view G as the composite operator:

$$\mathcal{H} = L_2(S^1) = H_0(S^1) \to H_2(S^1) \hookrightarrow L_2(S^1) = \mathcal{H}$$

then by the last paragraph, G is a compact operator.

Finally, if S is an *infinitely smoothing operator*, then for any $s, t \in \mathbb{R}$, we choose d so that $d > t - s$, and since $S \in \Psi^d$ for each d, we see that the composite:

$$H_s(M,E) \stackrel{S}{\to} H_{s+d} \to H_t(M,E)$$

is compact for all s, t.

Example 8.1.9. It is natural to wonder what happens for the Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$, which is an elliptic differential operator of order 2 on $\mathbb{R}^n$. To simplify things, let us take the case of $n = 1$, because the sharp contrast with the compact manifold S^1 considered above are already visible for $n = 1$. Indeed, we saw in the Example 8.1.7 above how the inclusion $H_s(\mathbb{R}) \hookrightarrow H_t(\mathbb{R})$ fails to be compact for $s > t$. This affects everything, as we shall soon see.

The first thing to note is that if $f \in H_{-\infty}$ is a tempered distribution, then $\Delta f = 0$ implies that f is smooth. (This is a version of elliptic regularity for $\mathbb{R}^n$, which can be deduced from the existence of local parametrices from 5.5.11 applied to Δ and noting that f is smooth over U iff ρf is smooth for all $\rho \in C_c^\infty(U)$).

Thus, for every s, the space of harmonic distributions inside the Sobolev space H_s is given by:

$$\{ax + b : \ a, b \in \mathbb{C}\} \cap H_s(\mathbb{R})$$

from which it follows that $(\ker \Delta) \cap \mathcal{H} = \{0\}$, where we define $\mathcal{H} := L_2(\mathbb{R}) = H_0(\mathbb{R})$.

The natural domain $\mathcal{D} \subset \mathcal{H}$ for the operator Δ can also be described. Let $\mathcal{D} \subset \mathbb{R} = \{f \in \mathcal{H} : \Delta f \in \mathcal{H}\}$, which makes sense because for $f \in \mathcal{H} = H_0(\mathbb{R})$, Δf is a tempered distribution in $H_{-2}(\mathbb{R})$. We use Plancherel's Theorem (iv) of the Proposition 1.2.5 on $L_2(\mathbb{R})$ and the fact (ii) of the same proposition that $(\Delta f)^\wedge = \xi^2 \widehat{f}(\xi)$ to get the commutative diagram:

$$\begin{array}{ccc} \mathcal{D} & \stackrel{\Delta}{\longrightarrow} & \mathcal{H} \\ {}^\wedge\downarrow & & \downarrow^\wedge \\ \mathcal{D}_1 & \stackrel{\xi^2}{\longrightarrow} & \mathcal{H} \end{array}$$

where $\mathcal{D}_1 := \mathcal{D}^\wedge$, and the lower horizontal arrow is multiplication by ξ^2. Note that since $f \mapsto \widehat{f}$ is an isometry, and $\ker \Delta \cap \mathcal{H} = \{0\}$ as noted above, both horizontal maps are injective linear isomorphisms, though not bounded operators.

Since $g \in \mathcal{H}$ iff $\widehat{g} \in \mathcal{H}$, it follows from the diagram above $\mathcal{D}_1 = \{g \in \mathcal{H} : \xi^2 g \in \mathcal{H}\}$, and hence:

$$\mathcal{D}_1 = L_2(\mathbb{R}, d\mu)$$

where the measure $d\mu = (1+ \mid \xi \mid^2)^2 d\xi$. Hence also the space $\mathcal{D}$ is given by:

$$\mathcal{D} := \operatorname{dom} \Delta = \{f \in \mathcal{H} : \ \xi^2 \widehat{f}(\xi) \in \mathcal{H}\}$$

$\mathcal{D}$ is again an L_2-dense linear subspace of $\mathcal{H}$, for it contains all Schwartz class (rapidly decreasing) functions. This is similar with with the case of S^1 discussed in 8.1.5 above, where the condition for $f \in \mathcal{D}$ was that $\{n^2 \widehat{f}(n)\}$ should be a square-summable sequence, which again included all $f \in C^\infty(S^1)$, a dense subspace. In fact, exactly as in the S^1-case, one immediately checks by using (ii) of the Proposition 3.1.4 that the conditions $f \in \mathcal{H}$ and $\xi^2 \widehat{f} \in \mathcal{H}$ imply that $\mathcal{D}$ is set-theoretically the Sobolev space $H_{(2)}(\mathbb{R}) \subset \mathcal{H} = L_2(\mathbb{R})$.

But here the analogy ends. It is clear that for a Fourier series $\sum_{n\in\mathbb{Z}} \widehat{f}(n) e_n$ on S^1, the finiteness of $\sum_{n\in\mathbb{Z}} |n^2 \widehat{f}(n)|^2$ implies the finiteness of $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2$. On the other hand we have:

Claim 1: $\xi^2 \mathcal{D}_1 \neq \mathcal{H}$, or equivalently, $\Delta(\mathcal{D}) \neq \mathcal{H}$.

Take any $g \in C_c^\infty(\mathbb{R})$ such that $g(0) \neq 0$, then we claim that the function:

$$\begin{aligned} \rho(\xi) &:= \xi^{-2} g(\xi) \ \text{ for } \ \xi \neq 0 \\ &= \ 0 \ \text{ for } \ \xi = 0 \end{aligned}$$

is not in $\mathcal{H}$. For, since $g(0) \neq 0$, we have $\mid g(\xi) \mid^2 \geq C > 0$ for $\xi \in (0, a)$ and some $a > 0$ so that

$$\|\rho\|^2 \geq \int_0^a C\xi^{-4} d\xi = \infty$$

so that $\rho \notin \mathcal{H}$, so $\rho \notin \mathcal{D}_1$, but $\xi^2 \rho(\xi) = g(\xi)$ is in $\mathcal{H}$. Thus the image of $\mathcal{D}_1$ under ξ^2 is not all of $\mathcal{H}$ and excludes, for example, all compactly supported $g \in C_c^\infty(\mathbb{R})$ with $g(0) \neq 0$. Hence, for any such g, $g^\vee \notin \Delta(\mathcal{D})$, and so $\Delta(\mathcal{D})$ is a proper subspace of $\mathcal{H}$ by the commutative diagram above. □

However, we have:

Claim 2: $\xi^2 \mathcal{D}_1$ is dense in $\mathcal{H}$, or equivalently, $\Delta(\mathcal{D})$ is dense in $\mathcal{H}$. (Contrast with S^1, where $\Delta(\mathcal{D})$ was of codimension 1 in $\mathcal{H}$)

Let $g \in C_c^\infty(\mathbb{R})$ be a compactly supported function, then the function:

$$\begin{aligned} g_n(\xi) &:= \ g(\xi) \quad \text{for} \quad \mid \xi \mid \geq \frac{1}{n} \\ &= \ 0 \quad \text{for} \quad \mid x \mid \leq \frac{1}{n} \end{aligned}$$

is in $\mathcal{H}$ for each n. Again, one computes:

$$\|g_n - g\|^2 = \int_{-1/n}^{1/n} \mid g(\xi) \mid^2 d\xi \leq \frac{2}{\sqrt{2\pi} n} \cdot \|g\|_\infty^2$$

so that $g_n \to g$ in $\mathcal{H}$. Now $g_n = \xi^2(\xi^{-2} g_n)$ and $\xi^{-2} g_n \in \mathcal{D}_1$ since it is bounded and compactly supported, so $g_n \in \xi^2(\mathcal{D}_1)$. Thus $\xi^2 \mathcal{D}_1$ is dense in $C_c^\infty(\mathbb{R})$, and since $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{H}$, we have $\xi^2 \mathcal{D}_1$ is dense in $\mathcal{H}$. The commutative diagram above implies $\Delta(\mathcal{D})$ is dense in $\mathcal{H}$. □

Claim 3: $\xi^2 \mathcal{D}_1$ is not closed in $\mathcal{H}$, or equivalently $\Delta(\mathcal{D})$ is not closed in $\mathcal{H}$.

For, if $\xi^2 \mathcal{D}_1$ were closed in $\mathcal{H}$, then Claim 2 above would imply $\xi^2 \mathcal{D}_1 = \mathcal{H}$, which would contradict Claim 1. The commutative diagram implies that $\Delta(\mathcal{D}) \neq \mathcal{H}$. □

An immediate consequence of Claims 2 and 3 above is that the cokernel $\operatorname{Coker} \Delta$ in $\mathcal{H}$ is infinite dimensional. Contrast with S^1, where the cokernel was the 1-dimensional space $\mathbb{C} e_0$.

Also, in sharp contrast to the case of the circle in 8.1.5, if one formally defines the Green operator on the subspace $\Delta(\mathcal{D})$ to be Δ^{-1}, it would fail to be a compact operator. In fact,

Claim 4: $\xi^{-2} : \xi^2 \mathcal{D}_1 \to \mathcal{D}_1$ is an unbounded operator, or equivalently, $G = \Delta^{-1} : \Delta(\mathcal{D}) \to \mathcal{D}$ is an unbounded operator.

For, let $g \in C_c^\infty(\mathbb{R})$ with $g(0) \neq 0$, as in the proof of Claim 1 above. Define $\rho(\xi) = \xi^{-2}g(\xi)$ and $\rho_n(\xi) = \xi^{-2}g_n$, where g_n are as in the proof of Claim 2 above. We saw that $g_n \to g$ in $\mathcal{H}$, so that we have:

$$\left\|\xi^2\rho_n\right\| = \|g_n\| \to \|g\|$$

and hence $\left\|\xi^2\rho_n\right\|$ is a *bounded* sequence. However, letting $n > 1/a$, a as in the proof of Claim 1 above, we have

$$\|\rho_n\|^2 \geq \int_{1/n}^{a} \mid \rho_n(\xi) \mid^2 d\xi = \int_{1/n}^{a} \mid \rho(\xi) \mid^2 d\xi = \int_{1/n}^{a} \xi^{-4} \mid g(\xi) \mid^2 d\xi \geq C \int_{1/n}^{a} \xi^{-4} d\xi \geq A\, n^3$$

for some $A > 0$, which means $\|\rho_n\|$ is an *unbounded* sequence. Since $\rho_n = \xi^{-2}g_n \in \mathcal{D}_1$ from the proof of Claim 2, it follows that the operator $\xi^{-2} : \xi^2(\mathcal{D}_1) \to \mathcal{D}_1$ cannot be a bounded operator. From the commutative diagram above, $G := \Delta^{-1} : \Delta(\mathcal{D}) \to \mathcal{D}$ is also not bounded.

Later, we will see how discreteness of the spectrum of Δ has to do with the compactness of the Green operator, which in turn has to do with the compactness of M. Meanwhile, we state a proposition which is the key to many of the results on spectra of the Laplacian, and more generally any self-adjoint elliptic differential operator.

Proposition 8.1.10 (Spectra of self-adjoint compact operators)**.** Let $\mathcal{H}$ be a separable Hilbert space, and let $G \in \mathcal{B}(\mathcal{H})$ be a compact self-adjoint operator. Then there is an orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ consisting of eigenvectors of G, viz.

$$Ge_n = \mu_n e_n \quad \text{for } \; n = 1, 2, ...$$

with $\mu_n \in \mathbb{R}$. Indeed $\{\mu_n\}_{n=1}^\infty$ is a bounded sequence, and satisfies $\lim_{n\to\infty} \mu_n = 0$.

Proof: That there is an orthonormal basis $\{e_n\}_{n=1}^\infty$ of eigenvectors for G is a consequence of the well-known spectral theorem for a bounded self-adjoint operator. That the set of eigenvalues $\{\mu_n\}_{n=1}^\infty$ is a bounded subset of $\mathbb{R}$ follows from the boundedness and self-adjointness of G.

If $\mu \neq 0$ is a cluster point of $\{\mu_n\}_{n=1}^\infty$, then we can find a subsequence μ_{n_k} satisfying $|\mu_{n_k}| > |\mu|/2$, say. Then, if B is the unit ball in the infinite dimensional subspace $W \subset \mathcal{H}$ spanned by $\{e_{n_k}\}_{k=1}^\infty$, the image $G(B)$ will contain the ball $\frac{|\mu|}{2}B$, which is non-compact. Thus $G(B)$ cannot have compact closure, contradicting that G is a compact operator. Thus $\mu = 0$, and $\lim_{n\to\infty} \mu_n = 0$. □

8.2. **The Calkin Algebra.** Let $\mathcal{H}$ be a complex Hilbert space as above, with inner product $\langle -, - \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$, and let $\mathcal{K}(\mathcal{H})$ denote the complex linear subspace of compact operators (verify that it is a complex subspace). We have the following easy lemma:

Proposition 8.2.1. $\mathcal{K}(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$. Finally $\mathcal{K}(\mathcal{H})$ is closed with respect to the operator norm topology on $\mathcal{B}(\mathcal{H})$.

Proof: Let $\{x_n\}$ be a bounded sequence in $\mathcal{H}$, $T \in \mathcal{K}(\mathcal{H})$, and $S \in \mathcal{B}(\mathcal{H})$. Then, since there is a convergent subsequence $\{Tx_{n_k}\}$, and since S is bounded and hence continuous, the sequence $\{STx_{n_k}\}$ is also convergent, so ST is a compact operator.

Similarly, since S is bounded, $\{Sx_n\}$ is also a bounded sequence in $\mathcal{H}$. By the compactness of T, there exists a convergent subsequence $\{TSx_{n_l}\}$ of $\{TSx_n\}$. Thus TS is also a compact operator.

To show $\mathcal{K}(\mathcal{H})$ is a star ideal, we need to show that T^* is compact if T is compact. Let $\{x_n\}$ be a bounded sequence in $\mathcal{H}$, with $\|x_n\| \leq A$ for all n. Since T^* is a bounded operator, we have from the fact that $\mathcal{K}(\mathcal{H})$ is a right ideal that TT^* is a compact operator, if T is a compact operator. Thus there exists a subsequence $\{TT^*x_{n_k}\}$ which converges. That is, for each $\epsilon > 0$, there exists a $N(\epsilon)$ such that

$$\|TT^*x_{n_k} - TT^*x_{n_l}\| < \epsilon \quad \text{for all } \; k, l \geq N(\epsilon)$$

This implies, since $\|x_{n_k} - x_{n_l}\| \leq 2A$ for all k, l, and Cauchy-Schwartz, that

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = \langle x_{n_k} - x_{n_l}, TT^*x_{n_k} - TT^*x_{n_l}\rangle < 2A\epsilon \quad \text{for all } \; k, l \geq N(\epsilon)$$

which shows that the subsequence $\{T^* x_{n_k}\}$ is a Cauchy sequence, hence convergent. Thus T^* is compact, and $\mathcal{K}(\mathcal{H})$ is a $*$-ideal.

To see that $\mathcal{K}(\mathcal{H})$ is a closed ideal, let $T_n \in \mathcal{K}(\mathcal{H})$ be a sequence of compact operators, with $T_n \to T$, and $T \in \mathcal{B}(\mathcal{H})$. We need to show that T is a compact operator. Let $\{x_n\}$ be a bounded sequence in $\mathcal{H}$, with $\|x_n\| \leq A$ for all n. Let $\epsilon > 0$ be given.

Because of the compactness of all T_n's, we can first find a subsequence $\{x_n^1\}$ of x_n such that $\{T_1 x_n^1\}$ converges, and then a subsequence $\{x_n^2\}$ of $\{x_n^1\}$ such that $\{T_2 x_n^2\}$ converges. Clearly then, both $\{T_1 x_n^2\}$ and $\{T_2 x_n^2\}$ converge. Proceeding inductively, for each $j \geq 1$ we have the following:

(i): $\{x_n^j\}$ is a subsequence of $\{x_n^{j-1}\}$.

(ii): $\{T_m x_n^j\}$ is a convergent sequence for all $m \leq j$.

Now consider the diagonal subsequence $\{x_n^n\}$ by taking the n-th element of the n-th subsequence among the $\{x_n^j\}$. By (i) above, $\{x_n^n\}$ is a subsequence of each of the subsequences $\{x_n^j\}$, so it is a subsequence of $\{x_n\}$.

Claim: The sequence $\{T x_n^n\}$ is convergent.

For, let $\epsilon > 0$. Since $\{x_n^n\}$ is a subsequence of each $\{x_n^j\}$, it follows by (ii) above that $\{T_j x_n^n\}$ is a convergent sequence for each j. Let its limit be y_j.

Since $T_n \to T$, there exists an $N \geq 0$ such that

$$\|T_j - T\| < \epsilon \quad \text{for all} \quad j, k \geq N$$

where the norm is operator norm. This implies that for $j, k \geq N$, $\|T_j - T_k\| < 2\epsilon$, and hence:

$$\|T_j x_n^n - T_k x_n^n\| \leq 2\epsilon \|x_n^n\| \leq 2A\epsilon \quad \text{for all} \quad j, k \geq N \;\text{ and each }\; n$$

Taking the limit $\lim_{n\to\infty}$ of these inequalities, we obtain:

$$\|y_j - y_k\| \leq 2A\epsilon \quad \text{for} \;\; j, k \geq N$$

which shows that $\{y_j\}$ is a Cauchy sequence, and hence converges to $y \in \mathcal{H}$.

Thus there is an $N_1 \geq N > 0$ such that $\|y_j - y\| < \epsilon$ for $j \geq N_1$. Also there is an $N_2 > 0$ such that

$$\|T_{N_1} x_n^n - y_{N_1}\| < \epsilon \;\text{ for }\;\; n \geq N_2$$

Then for $n \geq N_2$, we have:

$$\begin{aligned} \|T x_n^n - y\| &\leq \|T x_n^n - T_{N_1} x_n^n\| + \|T_{N_1} x_n^n - y_{N_1}\| + \|y_{N_1} - y\| \\ &\leq \|T - T_{N_1}\| \, \|x_n^n\| + \epsilon + \epsilon \\ &\leq (A+2)\epsilon \end{aligned}$$

which proves that $\{T x_n^n\}$ converges to y, and hence the claim.

Hence T is compact, and $\mathcal{K}(\mathcal{H})$ is a closed ideal. □

Definition 8.2.2. The quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called the *Calkin Algebra* of $\mathcal{H}$, and denoted $\mathcal{C}(\mathcal{H})$. By the lemma 8.2.1 above, this algebra is a Banach $*$-algebra. The star operation in $\mathcal{C}(\mathcal{H})$ is the one induced from $\mathcal{B}(\mathcal{H})$, viz. $[T]^* := [T^*]$. The norm of an element $[T] \in \mathcal{C}(\mathcal{H})$ is defined as:

$$\|[T]\| = \inf\{\|T + K\| : K \in \mathcal{K}(\mathcal{H})\}$$

which is a bonafide norm because $\mathcal{K}(\mathcal{H})$ is closed. From the fact that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, and the lemma above, it follows (not entirely trivially) that $\mathcal{C}(\mathcal{H})$ is also a C^*-algebra with this norm.

8.3. **Fredholm Operators.**

Definition 8.3.1. We say that $T \in \mathcal{B}(\mathcal{H})$ is a *Fredholm operator* if its image $[T] \in \mathcal{C}(\mathcal{H})$ is an invertible element of $\mathcal{C}(\mathcal{H})$. Since $\mathcal{K}(\mathcal{H})$ is a two-sided ideal, T is Fredholm if and only if there exist operators $S, S_1 \in \mathcal{B}(\mathcal{H})$ such that $ST - I_{\mathcal{H}} \in \mathcal{K}(\mathcal{H})$ and $TS_1 - I_{\mathcal{H}} \in \mathcal{K}(\mathcal{H})$. (Since inverses are unique in $\mathcal{C}(\mathcal{H})$, we see that $[S] = [S_1] = [T]^{-1}$, i.e., $S - S_1 \in \mathcal{K}(\mathcal{H})$)

Remark 8.3.2. Note that T Fredholm implies that T^* is also Fredholm, because $ST - I_{\mathcal{H}}$ (resp. $TS_1 - I_{\mathcal{H}}$) compact implies $T^*S^* - I_{\mathcal{H}}$ (resp. $S_1^*T^* - I_{\mathcal{H}}$) are compact, because $\mathcal{K}(\mathcal{H})$ is a $*$-ideal by lemma 8.2.1.

The definition above is often not very practical, since we have to be lucky enough to hit upon the operators S and S_1, given the operator T. Fortunately, there is a criterion for T to be Fredholm which can be stated entirely in terms of T. More precisely:

Proposition 8.3.3 (Fredholm Theorem)**.** Let $T \in \mathcal{B}(\mathcal{H})$. Then T is Fredholm if and only if all the following three criteria are satisfied:

(i): The image $\operatorname{Im} T$ of $T$is a closed subspace of $\mathcal{H}$.

(ii): The kernel $\ker T$ is a finite dimensional subspace of $\mathcal{H}$.

(iii): The cokernel $\operatorname{Coker} T := (\operatorname{Im} T)^{\perp}$ is finite dimensional.

Proof: First let us prove the sufficiency (i.e. the if) part. Let us denote $N := \ker T$, $R := \operatorname{Im} T$, both closed subspaces of $\mathcal{H}$ by hypothesis. Let $V := N^{\perp}$, and $W := R^{\perp}$. By hypothesis $\dim N < \infty$ and $\dim W < \infty$. Let i_N, i_V, i_W, i_R denote the inclusions of N, V, W, R into $\mathcal{H}$, and similarly let π_N, π_V, π_W, π_R denote the orthogonal projections onto these closed subspaces.

By definition (and the Open Mapping Theorem), there is an induced map:

$$T_1 : V = N^{\perp} \to R$$

(viz. the restriction of T to $V = N^{\perp}$) which is an isomorphism. Note that $T_1\pi_V = \pi_R T$ and $Ti_V = i_R T$.

Let $Q : R \to V$ be the inverse of T_1. Then $QT_1 = I_V$, and $T_1 Q = I_R$. We need to construct maps $S, S_1 \in \mathcal{B}(\mathcal{H})$ with $ST - I_{\mathcal{H}}$ and $TS_1 - I_{\mathcal{H}}$ compact.

Set $S = S_1 := i_V Q \pi_R$. Then $ST = i_V Q \pi_R T = i_V Q T_1 \pi_V = i_V I_V \pi_V = i_V \pi_V = I_{\mathcal{H}} - i_N \pi_N$. But $i_N \pi_N \in \mathcal{B}(\mathcal{H})$ has finite dimensional range, viz. N, so it is compact by the example 8.1.3. Hence $ST - I_{\mathcal{H}}$ is compact. Similarly, one checks that $TS_1 = i_R \pi_R = I_{\mathcal{H}} - i_W \pi_W$, so that $TS_1 - I_{\mathcal{H}}$ is the compact operator $i_W \pi_W$.

To see the necessity part, assume T is Fredholm. To show that $R := \operatorname{Im} T$ is closed, let $y_n = Tx_n$ be a sequence in R, with $\lim_{n\to\infty} y_n = y \in \mathcal{H}$. We need to show that $y \in R$. Without loss of generality, one can assume that $x_n \perp N(T)$ for all n. We first claim that $\{x_n\}$ must then be bounded. For if not, assume there is a subsequence $\{x_{n_k}\}$ such that $\|x_{n_k}\| \geq k$. Then set $z_k = \|x_{n_k}\|^{-1} x_{n_k}$. Then

$$\lim_{k\to\infty} Tz_k = \lim_{k\to\infty} \|x_{n_k}\|^{-1} Tx_{n_k} = 0$$

since $Tx_{n_k} \to y$. Thus $Tz_k \to 0$. By the equation $ST - I = K$ a compact operator, it follows that some subsequence of Kz_k converges (since $\|z_k\| = 1$) and thus z_k contains a convergent subsequence. Let the limit of that subsequence be z. Then $Tz = 0$ by the above. Thus $z \in N(T)$. On the other hand $\|z_k\| = 1$, and $z_k \in (N(T))^{\perp}$ implies $\|z\| = 1$ and $z \in N(T)^{\perp}$. This is a contradiction, and proves the claim.

Since x_n is a bounded sequence, Kx_n contains a convergent subsequence x_{n_k}. Also STx_{n_k} converges to Sy. Thus $x_{n_k} = STx_{n_k} - Kx_{n_k}$ is a convergent sequence, converging to x say. Then clearly $Tx = y$.

To show that $N = \ker T$ is finite dimensional, let $x \in N$ be any vector. Then $STx = 0 = I_{\mathcal{H}}x + Kx = x + Kx$. Thus $x = -Kx$ for all $x \in N$. That is, $I_N = -\pi_N K i_N$, so that I_N is a compact operator. From Example 8.1.2, this implies that N is finite dimensional.

By remark 8.3.2, T^* is also Fredholm, it follows that dim $N(T^*) < \infty$ as well, by replacing T with T^* in the last paragraph. But by the fact that R is closed, it is easy to check that $\operatorname{Coker} T = R^\perp = N(T^*) = \ker T^*$. Thus $\operatorname{Coker} T$ is finite dimensional, and the proposition is proved. □

8.4. **Two Hilbert Spaces for the price of one.** All of the above discussion can be generalised to $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two different separable Hilbert spaces. This is scarcely surprising, since all infinite dimensional separable Hilbert spaces are (non-canonically) isomorphic to one another, but sometimes it helps to see them as distinct objects. For finite dimensional $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}_1$ may not be isomorphic to $\mathcal{H}_2$, but in that case everything is a tautology from elementary linear algebra.

Note that $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is no longer an algebra, but just a Banach space. (If both $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional separable Hilbert spaces, we can fix an isomorphism $\Psi : \mathcal{H}_2 \to \mathcal{H}_1$, then the map $T \mapsto \Psi \circ T$ will be an isomorphism of the Banach space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with the Banach space $\mathcal{B}(\mathcal{H}_1)$, and we can use this isomorphism of Banach spaces to define an algebra structure on the former. But, of course, this algebra structure will be non-canonical, and depend on Ψ.)

We have already seen in definition 8.1.1 what a compact operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ is. The subset of compact operators in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is denoted $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$. It is easily seen to be a closed Banach subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

The adjoint defines a $\mathbb{C}$-antilinear isomorphism

$$\begin{aligned} * : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) &\to \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \\ T &\mapsto T^* \end{aligned}$$

We also have the following proposition, whose proof is a trivial generalisation of the proofs of the corresponding propositions for $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ enunciated in the last two subsections.

Proposition 8.4.1. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ etc. be as above. Then:

- (i): For $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, and $S_1 \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$, $S_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$, $\mathcal{H}_3$ any separable Hilbert space, $T \circ S_1$ and $S_2 \circ T$ are compact operators.
- (ii): $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ is a closed subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.
- (iii): Under the isomorphism $*$ defined above, $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ maps isomorphically onto $\mathcal{K}(\mathcal{H}_2, \mathcal{H}_1)$.
- (iv): An operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be Fredholm if there exist operators $S, S_1 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $ST - I_{\mathcal{H}_1} \in \mathcal{K}(\mathcal{H}_1)$ and $TS_1 - I_{\mathcal{H}_2} \in \mathcal{K}(\mathcal{H}_2)$. T is Fredholm iff ker T is finite dimensional, $\operatorname{Im} T$ is closed and $\operatorname{Coker} T$ is also finite dimensional. The adjoint T^* is also a Fredholm operator if T is a Fredholm operator.
- (v): If $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ are Fredholm, then so is $ST \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$.

We now run through some examples of Fredholm operators.

Example 8.4.2. If $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is invertible, then clearly T is Fredholm. The composite of two Fredholm operators is also clearly Fredholm.

Example 8.4.3. Obviously, any linear map between two *finite dimensional Hilbert spaces* $\mathcal{H}_1$, $\mathcal{H}_2$ is always Fredholm.

Example 8.4.4 (Unilateral shifts)**.** Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Then define the *unilateral right 1-shift* operator:

$$\begin{aligned} T : \mathcal{H} &\rightarrow \mathcal{H} \\ e_i &\mapsto e_{i+1} \quad \text{for all } \ i \geq 1 \end{aligned}$$

This is clearly a Fredholm operator by the proposition 8.3.3, for $\ker T = \{0\}$, and the range $\operatorname{Im} T = (\mathbb{C}e_1)^{\perp}$ is closed, and the cokernel $\operatorname{Coker} T = \mathbb{C}e_1$. The adjoint of this operator is easily checked to be:

$$\begin{aligned} T^* : \mathcal{H} &\rightarrow \mathcal{H} \\ e_i &\mapsto e_{i-1} \quad \text{for all } \ i \geq 2 \\ e_1 &\mapsto 0 \end{aligned}$$

As remarked before, T^* is also Fredholm, and is called the *unilateral (-1)-shift* operator. Now $\ker T^* = \mathbb{C}e_1$, and $\operatorname{Coker} T = \{0\}$. By (v) of the proposition 8.4.1 above, the unilateral k-shift T^k and the unilateral $(-k)$-shift $(T^*)^k$ are also Fredholm, and their kernels (resp. cokernels) are $\{0\}$ and $\oplus_{i=1}^{k}\mathbb{C}e_i$ (resp. $\oplus_{i=1}^{k}\mathbb{C}e_i$ and $\{0\}$) respectively.

Example 8.4.5 (Parametrices)**.** Let M be a compact Riemannian manifold, E and F two complex vector bundles on M, and let:

$$P : C^{\infty}(M, E) \rightarrow C^{\infty}(M, F)$$

be an elliptic differential operator of order $d \geq 1$ (See Definition 6.1.1). Then $P : H_{s+d}(M, E) \rightarrow H_s(M, F)$ is a Fredholm operator.

For, by the Proposition 6.2.1, we have a parametrix

$$Q : H_s(M, F) \rightarrow H_{s+d}(M, E)$$

such that $S := PQ - I$ is an infinitely smoothing operator on $H_s(M, F)$ and $T := QP - I$ is an infinitely smoothing operator on $H_s(M, E)$. By the Example 8.1.8, it follows that both S and T are compact operators. Thus, by definition, both P and Q are Fredholm operators. Hence, by the Fredholm Theorem 8.3.3, $P(H_{s+d}(M, E))$ is closed in $H_s(M, F)$, and $\ker P$ and $\operatorname{Coker} P$ are finite dimensional.

As a particular case, let us look at the Laplacian on S^1 again.

Example 8.4.6 (Green operator on S^1)**.** We recall the example 8.1.5. Let $\mathcal{H} = L_2(S^1)$ as before, and recall

$$\mathcal{D} = \operatorname{dom} \Delta = \{f \in L_2(S^1) : \sum_{n=-\infty}^{\infty} n^4 |\widehat{f}(n)|^2 < \infty\}$$

We also recall that $\Delta e_n = n^2 e_n$, (where $e_n = e^{in\theta}$) so that Δ became an unbounded linear operator from $\mathcal{D} \rightarrow \mathcal{H}$. Then consider the space:

$$\mathcal{H}_2 := H_2(S^1) = \{f \in \mathcal{H} : \sum_{n=-\infty}^{\infty} (1 + n^4)|\widehat{f}(n)|^2 < \infty\}$$

Clearly, $\mathcal{H}_2 = \mathcal{D}$ as a vector space. However, on $\mathcal{H}_2$ we have the *Sobolev inner product* $\langle -, - \rangle_2$, which by the Corollary 6.2.3, can also be defined as:

$$\langle f, g \rangle_2 := \langle f, g \rangle_0 + \langle \Delta f, \Delta g \rangle_0 = \sum_{n=-\infty}^{\infty} (1 + n^4)\widehat{f}(n)\overline{\widehat{g}(n)}$$

which explains the notation $H_2(S^1)$ for the space above, and by earlier considerations makes it into a Hilbert Space. It is clear that $e_n = e^{in\theta}$ continue to be orthogonal, *but not orthonormal* with respect to $\langle -, - \rangle_2$. Indeed, $\|e_n\|_2 = (1 + n^4)^{\frac{1}{2}}$.

Clearly, by definition, we have:

$$\|\Delta f\|^2 = \sum_{n=-\infty}^{\infty} n^4 |\widehat{f}(n)|^2 \leq \|f\|_2^2$$

which makes $\Delta : \mathcal{H}_2 \to \mathcal{H}$ a bounded operator, an element of $\mathcal{B}(\mathcal{H}_2, \mathcal{H})$ (a fact we already know from Proposition 6.1.2) an element of $\mathcal{B}(\mathcal{H}_2, \mathcal{H})$.

Similarly, for the Green operator G introduced in 8.1.5,

$$\|Gf\|_2^2 = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(1+n^4)}{n^4} |\widehat{f}(n)|^2 \leq 2\,\|f\|^2$$

so G is also a bounded operator,and lies in $\mathcal{B}(\mathcal{H}, \mathcal{H}_2)$. The relations $I_{\mathcal{H}_2} - \pi_0 = G\Delta$ and $I_{\mathcal{H}} - \pi_0 = \Delta G$ found in 8.1.5 show that both $\Delta : \mathcal{H}_2 \to \mathcal{H}$ and $G : \mathcal{H} \to \mathcal{H}_2$ are Fredholm operators. Note that ker $\Delta = \mathbb{C}e_0$, and ker $G = \mathbb{C}e_0$ as well.

Exercise 8.4.7. The Green operator can be written explicitly as a convolution with an L_2 function on S^1. Define the function $g \in L_2(S^1) = \mathcal{H}$ by the formula:

$$g = \frac{1}{2\pi} \sum_{n\neq 0,\, n\in\mathbb{Z}} \frac{1}{n^2} e_n$$

where $e_n(e^{it}) = e^{int}$ for $z = e^{it} \in S^1$. Verify that:

$$(Gf)(z) = \int_{S^1} g(zw^{-1}) f(w) dw \quad \text{for} \;\; z \in S^1$$

where $w = e^{is}$, and $dw := ds$. Calculate the distribution $\Delta\, g$.

We might as well record a direct consequence of the last few sections in the following:

Proposition 8.4.8 (Green Operator for a Self-adjoint Elliptic Differential Operator)**.** Let M be a *compact* Riemannian manifold, with a smooth complex vector bundle E on it. Let $dV(x)$ be the Riemannian volume form on M and let $\langle -,- \rangle$ be a Hermitian inner product on E. Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order $d > 0$. Assume that P *is formally self-adjoint*, viz.

$$(Pf, g) = \int_M \langle Pf(x), g(x) \rangle_x \, dV(x) = (f, Pg) \quad \text{for all} \;\; f, \, g \in C^\infty(M,E)$$

Consider the bounded operator:

$$P : H_d(M,E) \to H_0(M,E) = L_2(M,E)$$

Then, for this last operator, we have:

(i): $\dim \ker P < \infty$, and this kernel is contained in $C^\infty(M,E)$, and in particular $H_s(M,E)$ for all $s \in \mathbb{R}$.

(ii): $\operatorname{Im} P \subset L_2(M,E)$ is closed, and $\operatorname{Coker} P := (\operatorname{Im} P)^\perp = \ker P$.

(iii): There exists a bounded self-adjoint operator called the *Green Operator*

$$G : L_2(M,E) \to L_2(M,E)$$

for P which satisfies:

(a): $G \equiv 0$ on $\ker P \subset L_2(M,E)$, and $G = P^{-1}$ on $\ker P^\perp = \operatorname{Coker} P^\perp = \operatorname{Im} P \subset L_2(M,E)$. G is a compact operator.

(b): $G(C^\infty(M,E)) \subset C^\infty(M,E)$, and $GP = PG$ on $C^\infty(M,E)$.

(c): $G : L_2(M,E) \to L_2(M,E)$ is a compact self adjoint operator. There is an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $L_2(M,E)$ of eigensections of G, which satisfy

$$Ge_i = \mu_i e_i \quad \text{for all} \;\; i$$

where $\mu_i \in \mathbb{R}$ for all i. 0 is the only cluster point of the set $\{\mu_i\}_{i=1}^\infty$, and $\lim_{i\to\infty} \mu_i = 0$.

(d): The eigensections $\{e_i\}$ of (c) above are all smooth, and are also eigensections for P, satisfying:

$$Pe_i = \lambda_i e_i \quad \text{for all} \quad i$$

where $\lambda_i \in \mathbb{R}$ is a discrete subset of $\mathbb{R}$, and $\lim_{i\to\infty} \mid \lambda_i \mid = \infty$

Proof: The operator:

$$P : H_d(M,E) \to L_2(M,E) = H_0(M,E)$$

is bounded by the Proposition 6.1.2. Its kernel $\ker P$ is finite dimensional by the Example 8.4.5 above, where P was found to be Fredholm, and (ii) of the Fredholm Theorem 8.3.3. That $\ker P \subset C^\infty(M,E)$ is a consequence of the elliptic regularity theorem Proposition 6.2.4. Since $C^\infty(M,E) \subset H_s(M,E)$ for all s, it follows that $\ker P \subset H_s(M,E)$ for all s. This proves (i).

That $\operatorname{Im} P$ is closed in $L_2(M,E)$ follows from (i) of the Fredholm Theorem 8.3.3, and Example 8.4.5. Since $P : H_d \to H_0$ is bounded, $P(C^\infty(M,E))$ is dense in $\operatorname{Im} P$. Hence $f \in L_2(M,E)$ is orthogonal to $\operatorname{Im} P$ iff $(f, Pg) = 0$ for all $g \in H_d(M,E)$. By the formal self-adjointness of P, and the natural duality of H_d and H_{-d} in (iii) of 4.2.2, $(f, Pg) = (Pf, g)$ for $f \in H_0$ and $g \in H_d$. Thus we have $(Pf, g) = 0$ for all $g \in H_d(M,E)$. This is equivalent to $Pf = 0$. Thus $\operatorname{Coker} P = (\operatorname{Im} P)^\perp = \ker P$, and (ii) follows.

By (ii), we have an L_2-orthogonal decomposition:

$$L_2(M,E) = \operatorname{Im} P \oplus \operatorname{Coker} P = \operatorname{Im} P \oplus \ker P$$

We now *define* G by setting $G \equiv 0$ on $\ker P$, and G to be equal θ which is the composite:

$$\operatorname{Im} P \stackrel{P^{-1}}{\to} (\ker P)^\perp \hookrightarrow H_d(M,E) \hookrightarrow H_0(M,E) = L_2(M,E)$$

By the Open Mapping Theorem, $P^{-1}: \operatorname{Im} P \to \ker P^{\perp}$ is a bounded operator, as is the inclusion $(\ker P)^{\perp} \to H_d(M,E)$. The last inclusion $H_d(M,E) \to L_2(M,E)$ is a compact operator by Rellich's Lemma 4.2.2. Thus by the Proposition 8.2.1, the map θ is a compact operator. Since $G = \theta \circ \pi$, where $\pi : L_2 \to \operatorname{Im} P$ is orthogonal projection onto the closed subspace $\operatorname{Im} P$, G is also a compact operator by 8.2.1. This proves (a) of (iii).

Since P is a differential operator, $P(C^\infty(M,E)) \subset C^\infty(M,E)$, and if $g \in C^\infty(M,E) \subset L_2(M,E)$, then its projection to the closed subspace $\operatorname{Im} P = (\ker P)^{\perp}$ is given by:

$$\pi(g) = g - \sum_{i=1}^{k} (g, f_i) f_i$$

where $\{f_i\}_{i=1}^k$ is an L_2-orthonormal basis for $\ker P$. By (i) above, all the f_i are smooth, thus the scalar combination $\sum_i (g, f_i) f_i$ is smooth, and hence $\pi(g)$ above is smooth. On the other hand, for a smooth $g = Pf$ in $\operatorname{Im} P$, it follows by elliptic regularity of Proposition 6.2.4 that f is also smooth. Thus the map θ above also maps smooth sections in $\operatorname{Im} P$ into smooth sections. Since $G = \theta \circ \pi$, it maps smooth sections to smooth sections. The fact that $GP = PG$ on smooth sections follows immediately from the definitions. This proves (b).

That G is a compact operator was seen in (a). That it is self-adjoint follows from the definition $G = \theta \circ \pi$, and θ is the inverse of the formally self-adjoint P, and $C^\infty(M,E)$ is dense in $L_2(M,E)$. The statement about its eigenvalues and the orthonormal decomposition of $L_2(M,E)$ into eigenspaces of G is the content of the Proposition 8.1.10. The eigenvalues are real since G is self-adjoint. This proves (c).

To see (d), note that $Ge_i = \mu_i e_i$, and $\mu_i \neq 0$ implies that e_i are orthogonal to $\ker P$, and hence so are Ge_i, so that:

$$\mu_i P e_i = PGe_i = P\theta e_i = PP^{-1} e_i = e_i$$

so that $Pe_i = \mu_i^{-1} e_i$ for all $\mu_i \neq 0$, and e_i become eigensections of P, corresponding to the real non-zero eigenvalues $\lambda_i = \mu_i^{-1}$. Since $(P - \lambda_i)e_i = 0$, and $P - \lambda_i$ is also elliptic of order d (it has the same leading symbol as P), it follows that $e_i \in C^\infty(M,E)$ for all i such that $\mu_i \neq 0$. For those i's which have $\mu_i = 0$, we have $e_i \in \ker P$, and we already know those are smooth by (b). Hence e_i are all smooth, and the rest of (d) follows from (c) above. □

Actually, we can refine (iii) (d) of the previous proposition. To be precise, we have the following proposition.

Proposition 8.4.9. Let M, E and P be as above in Proposition 8.4.8. Then let us arrange the absolute values of the eigenvalues λ_i of P as in (iii) (d) of the previous proposition in non-decreasing order as:

$$|\lambda_1| \leq |\lambda_2| \ldots. \leq |\lambda_k| \leq \ldots$$

Then there exists constants $C,\ \delta > 0$ and $N \in \mathbb{N}$ such that $|\lambda_n| \geq Cn^{\delta}$ for all $n \geq N$.

Proof: First we note that the eigenvalues of P^k will be λ_n^k, and obtaining the assertion for λ_n^k is sufficient to imply the same assertion for λ_n (with δ replaced by δ/k). So we may assume without loss of generality that P is of degree $d > n/2$ where $n = \dim M$.

Since we are assuming $d > n/2$, by (iv) of the Proposition 4.2.2 (viz. the Sobolev embedding theorem), we have for $f \in C^\infty(M,E)$ the inequality:

$$\|f\|_\infty = \sup_{x \in M} |f(x)| \leq C \|f\|_d \quad \text{for all } \ f \in C^\infty(M,E)$$

and combining this with the Garding-Friedrichs inequality Proposition 6.2.2 we have:

$$\|f\|_\infty \leq C(\|Pf\|_0 + \|f\|_0) \quad \text{for all} \quad f \in C^\infty(M,E) \tag{21}$$

We note that by elliptic regularity, all the eigensections ϕ_k of P are smooth sections. We assume they are orthonormal with respect to L_2-norm $\| \ \|_0$. Define:

$$F(a) := \operatorname{span}_{\mathbb{C}}\{\phi_k : P\phi_k = \lambda_k \phi_k \ \text{ and } \ |\lambda_k| \leq a\}$$

Let $m = \dim\, F(a)$. This dimension is finite by the fact that λ_k^{-1} have no cluster point except 0 from (iii) (c) of 8.4.8. We will make an estimate for m in terms of a, which will imply our assertion.

Note that for $f = \sum_{j=1}^m \alpha_j \phi_j \in F(a)$, we have $Pf = \sum_{j=1}^m \alpha_j \lambda_j \phi_j$, which shows that

$$\|Pf\|_0 \leq a \|f\|_0 \quad \text{for all } \ f \in F(a)$$

Plugging this into the inequality (21) above, we have for all choices of complex constants α_j, the inequality:

$$\left\| \sum_{j=1}^m \alpha_j \phi_j \right\|_\infty \leq C(1+a) \left\| \sum_{j=1}^m \alpha_j \phi_j \right\|_0 \tag{22}$$

In a local frame $\{e_i(x)\}_{i=1}^k$ orthonormal frame of E over $U \subset M$, where $k = \mathrm{rk}_{\mathbb{C}} E$, write:

$$\phi_j(x) = \sum_{i=1}^k \phi_j^i(x) e_i(x)$$

So that for any choice of constants α_j, we have for $x \in U$ that

$$\sum_{j=1}^m \alpha_j \phi_j(x) = \sum_{i=1}^k \left(\sum_j \alpha_j \phi_j^i(x) \right) e_i(x)$$

so that for any choice of constants $\alpha_j \in \mathbb{C}$, the inequality (22) implies:

$$\left| \sum_j \alpha_j \phi_j^i(x) \right| \leq \left\| \sum_{j=1}^m \alpha_j \phi_j \right\|_\infty \leq C(1+a) \left(\sum_j \mid \alpha_j \mid^2 \right)^{\frac{1}{2}} \quad \text{for each } \ i = 1, 2, ..k$$

For $x \in U$, choose $\alpha_j = \overline{\phi}_j^i(x)$. Then the last inequality reads:

$$\sum_j \mid \phi_j^i(x) \mid^2 \leq C(1+a) \left(\sum_j \mid \phi_j^i(x) \mid^2 \right)^{\frac{1}{2}}$$

that is:

$$\left(\sum_j \mid \phi_j^i(x) \mid^2 \right)^{\frac{1}{2}} \leq C(1+a)$$

Squaring both sides and summing over $i = 1, .., k$, we have:

$$\sum_{j=1}^m \|\phi_j(x)\|_x^2 = \sum_j \left(\sum_i \mid \phi_j^i(x) \mid^2 \right) = \sum_i \left(\sum_j \mid \phi_j^i(x) \mid^2 \right) \leq kC^2(1+a)^2 = C^2(1+a)^2$$

where C is a generic constant independent of a. This inequality is true for each $x \in M$, so we may integrate both sides over all of M to obtain

$$m = \int_M \sum_{j=1}^m \left(\|\phi_j(x)\|_x^2 \right) dV(x) \leq C^2(1+a)^2$$

which shows that $\frac{1}{C} m^{\frac{1}{2}} - 1 \leq a$. Since $\mid \lambda_j \mid \leq a$ for $j = 1, 2, .., m$, we can take $a = \max_{j=1}^m \mid \lambda_j \mid = \mid \lambda_m \mid$, so that we have:

$$\mid \lambda_m \mid \geq C m^{\frac{1}{2}} \quad \text{for } \ m \geq N$$

and the proposition follows. □

8.5. **The Fredholm Index.**

Definition 8.5.1 (Fredholm Index)**.** Let $T : \mathcal{H} \to \mathcal{H}$ be a Fredholm operator. The *Fredholm index of* T is defined by:

$$\mathrm{ind}\, T = \dim \ker T - \dim \mathrm{Coker}\, T$$

It makes sense, and is an integer, because of proposition 8.3.3. Similarly for $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, one again defines the index $\mathrm{ind}\, T$ by the same formula.

Example 8.5.2 (Index of examples above)**.** We can easily compute the indices of the various examples of Fredholm operators listed above. For an invertible operator $T : \mathcal{H}_1 \to \mathcal{H}_2$, the index is clearly 0. For a linear map $T : V \to W$ of finite dimensional vector spaces, the index is easily seen to be $\dim V - \dim W$ by elementary linear algebra. (Thus in the finite dimensional situation, the index depends only on the domain V and range W, and is not a very interesting invariant of T). For the unilateral right (resp. left) k-shift, the index is $(-k)$ (resp. k).

For an elliptic *self-adjoint* differential operator $P : C^\infty(M, E) \to C^\infty(M, E)$ of order d, M a *compact* Riemannian manifold, we have that $P : H_d(M, E) \to L_2(M, E)$ is Fredholm, by 8.4.5. Also by (ii) of 8.4.8, $\ker P = \operatorname{Coker} P$, and hence $\operatorname{ind} P = 0$.

Proposition 8.5.3. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ be Fredholm operators. Then $TS \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ is Fredholm, and

$$\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$$

In particular, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, the index is a *group homomorphism* from the group of units (=set of invertible elements) in the Calkin algebra $\mathcal{C}(\mathcal{H})$ to $\mathbb{Z}$.

Proof: In the sequel, we will denote the kernel of a linear operator T by $N(T)$. The fact that ST is Fredholm follows from the fact that post and precomposing compact operators with bounded operators again yields compact operators (see example 8.4.2).) Note that for any linear operator T, we have the following identity for a closed subspace W:

$$T^{-1}(W^\perp) = (T^*(W))^\perp \tag{23}$$

where the left side is the inverse image of $W^\perp$. From this (by taking $W^\perp = \{0\}$), one sees that $\operatorname{Coker} T^* = \operatorname{Im} T^{*\perp} = N(T)$, and $\operatorname{Coker} T = (\operatorname{Im} T)^\perp = N(T^*)$. Now one may do orthogonal decompositions of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ as follows:

$$\begin{aligned} \mathcal{H}_1 &= N(T) \oplus F \\ \mathcal{H}_2 &= N(T^*) \oplus G_1 = N(S) \oplus G_2 \\ \mathcal{H}_3 &= N(S^*) \oplus H \end{aligned}$$

where $T : F \to G_1$ and $S : G_2 \to H$ are isomorphisms.

The kernel of ST is given by (using the identity (23), and noting that $T_{|F} : F \to \operatorname{Im} T$ is an isomorphism) above):

$$\begin{aligned} N(ST) &= T^{-1}S^{-1}(0) = T^{-1}(N(S)) \\ &= N(T) + T_{|F}^{-1}(N(S) \cap \operatorname{Im} T) \\ &= N(T) + T_{|F}^{-1}\left(N(S) \ominus (N(S) \cap N(T^*)\right) \end{aligned}$$

so that

$$\dim\, N(ST) = \dim\, N(T) + \dim\, N(S) - \dim\,(N(S) \cap N(T^*))$$

Similarly,

$$\begin{aligned} \dim\, N((ST)^*) &= \dim\, N(T^*S^*) = \dim\, N(S^*) + \dim\, N(T^*) - \dim\,(N(T^*) \cap N(S^{**})) \\ &= \dim\, N(S^*) + \dim\, N(T^*) - \dim\,(N(T^*) \cap N(S)) \end{aligned}$$

Combining the two identities above, we get:

$$\operatorname{ind} ST = \dim\, N(ST) - \dim\, N((ST)^*) = \dim\, N(T) + \dim\, N(S) - \dim\, N(S^*) - \dim\, N(T^*) = \operatorname{ind} T + \operatorname{ind} S$$

proving the proposition. □

8.6. **Path components and Fredholm index.** We have the following crucial fact about Fredholm operators.

Proposition 8.6.1 (Invariance of index)**.** Let $t \mapsto T_t$ be a continuous map of an interval I to $\mathcal{B}(\mathcal{H})$, with T_t a Fredholm operator for each $t \in I$. Then:

$$\operatorname{ind} T_t = \operatorname{ind} T_s \quad \text{for all } \; t, s \in I$$

Thus the index remains a constant integer on each path component of the set of units (=invertible elements) in the Calkin Algebra.

Proof: We will show that the index is locally constant on I, and that will make $t \mapsto \operatorname{ind} T_t$ a continuous, and hence constant map. For a point $t \in I$, denote the kernel of T_t by K_t. Since T_t is Fredholm for each t, K_t is finite dimensional for all t. Let $V_t := K_t^{\perp}$, and $W_t := \operatorname{Im} T_t$. By Fredholmness of T_t, $W_t^{\perp}$ is also finite dimensional for each t.

Fix any $a \in I$. We claim that for a small enough $\epsilon > 0$, and for $|t-a| < \epsilon$, the index $\operatorname{ind} T_t = \operatorname{ind} T_a$. For simplicity of notation, denote K_a by K, V_a by V and W_a by W. Let $\pi : \mathcal{H} \to W$ denote the orthogonal projection, and $i : V \to \mathcal{H}$ denote the inclusion. Then $\pi \, T_a \, i : V \to W$ is an isomorphism, by definition. Since isomorphisms from V to W form an open set in $\mathcal{B}(V, W)$, it follows that there is an $\epsilon > 0$ such that $\pi \, T_t \, i : V \to W$ is an isomorphism for all t such that $|t - a| < \epsilon$.

Thus, for $|t - a| < \epsilon$, the index

$$\operatorname{ind}(\pi \, T_t \, i) = 0$$

By the proposition 8.5.3, and the facts that $\ker \pi = W^{\perp} = \operatorname{Coker} T_a$, $\operatorname{Coker} \pi = 0$, $\ker i = 0$, $\operatorname{Coker} i = V^{\perp} = \ker T_a$, it follows that

$$\operatorname{ind}(\pi \, T_t \, i) = \operatorname{ind} \pi + \operatorname{ind} T_t + \operatorname{ind} i = \dim \, W^{\perp} + \operatorname{ind} T_t + (-\dim \, V^{\perp}) = \operatorname{ind} T_t - \operatorname{ind} T_a$$

Thus $\operatorname{ind} T_t = \operatorname{ind} T_a$ for $|t - a| < \epsilon$, and the proposition is proved. □.

9. Elliptic Complexes on Compact Riemannian manifolds

9.1. **The de Rham complex.** Let M be a smooth connected oriented (i.e.the Jacobian of each coordinate change in the atlas being used has positive determinant) Riemannian manifold of dimension n, with Riemannian metric g. We recall that the volume n-form dV associated to this Riemannian metric, is given in a local coordinate system (ϕ, U) of an oriented atlas by:

$$dV_g(x) = \sqrt{\det g_{ij}(x)} \, dx_1 \wedge dx_2 \wedge .. \wedge dx_n$$

with coordinate functions x_i being the components of ϕ on the open set U. The expression above is independent of the coordinate system chosen, by the tranformation properties of the coordinate changes on the overlaps $U_i \cap U_j$, and the orientability of the atlas $\{(\phi_i, U_i)\}$. We will usually write dV instead of dV_g.

One also has the complex vector space of smooth complex-valued differential p-forms on M, which is denoted by $\bigwedge^p(M)$. Let ω be a differential p-form is given in a coordinate chart (ϕ, U) by the local expression:

$$\omega = \sum_{i_1 < i_2 < \ldots < i_p} \omega_{i_1 < i_2 < \ldots < i_p} dx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_p} = \sum_I \omega_I dx_I$$

where $I = (i_1 < i_2 < .. < i_p)$, $\; 1 \leq i_j \leq n$ denotes a multi-index of length p, and ω_I are all smooth functions on the open set U. Note that $\bigwedge^0(M)$ is just the vector space of smooth functions on M, and when M is oreinted, there is an isomorphism $f \mapsto f dV_g$ of $\bigwedge^0(M)$ with $\bigwedge^n(M)$ upon choosing a Riemannian volume element.

Then one can define the *exterior derivative operator*

$$d : \bigwedge^p(M) \to \bigwedge^{p+1}(M)$$

by $d\omega := \sum_I d\omega_I \wedge dx_I$, where $d\omega_I = \sum_j \frac{\partial \omega_I}{\partial x_j} dx_j$ is a 1-form. One easily checks that this definition of d is global, and does not depend on the choice of local coordinate charts. (In the case of $M = \mathbb{R}^3$, the exterior derivatives on $\bigwedge^0(\mathbb{R}^3)$, $\bigwedge^1(\mathbb{R}^3)$ and $\bigwedge^2(\mathbb{R}^3)$ lead to the familiar grad, curl and divergence operators.) It is well known that $d \circ d = 0$, and so we have a *cochain complex* of complex vector spaces:

$$\bigwedge^0(M) \xrightarrow{d} \bigwedge^1(M) \xrightarrow{d} \dots \bigwedge^p(M) \xrightarrow{d} \bigwedge^{p+1}(M) \dots \xrightarrow{d} \bigwedge^n(M)$$

which is called the *de Rham complex* of M. We also have the *skew-derivation formula* for the exterior derivative:

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^p \omega \wedge d\tau$$

where $\omega \in \bigwedge^p(M)$, $\tau \in \bigwedge^q(M)$.

The de Rham complex contains much of the algebraic topology of M, even though its definition is purely analytical. For example, we can define the i-th *de Rham cohomology of* M as the quotient:

$$H^i(M, \mathbb{C}) := \frac{\ker d : \bigwedge^i(M) \to \bigwedge^{i+1}(M)}{\operatorname{Im} d : \bigwedge^{i-1}(M) \to \bigwedge^i(M)}$$

It turns out by de Rham's theorem (to be stated below) that the dimension dim $H^i(M, \mathbb{C})$ is the i-th Betti number of M, and the alternating sum

$$\sum_{i=0}^{n} (-1)^i \dim H^i(M, \mathbb{C})$$

is the topological Euler characteristic $\chi(M)$ of M.

Now one brings in the Riemannian metric to introduce pointwise and global hermitian inner products on differential forms. The Riemannian metric g defines inner products for all real tangent vectors, and gives an identification of the real cotangent space with the tangent space by the identification $X \mapsto X^*$ where X^* is defined by the formula $X^*(Y) = g(X, Y)$ for all tangent vectors Y to M at x. Declaring the vector space isomorphism above to be an isometry puts a real positive definite inner product $\langle -, - \rangle$ on real cotangent vectors. More explicitly, in a coordinate chart (ϕ, U) with coordinates x_i we have:

$$\langle dx_i, dx_j \rangle = g^{ij}$$

where g^{ij} is the inverse of the $n \times n$ positive definite matrix $\left[g_{kl} := g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right) \right]$. Each of these g_{ij}'s is a smooth function of $x \in U$. Now we get inner products on all real p-covectors by the formula:

$$\left\langle dx_{i_1} \wedge dx_{i_2} \dots \wedge dx_{i_p}, dx_{j_1} \wedge dx_{j_2} \dots \wedge dx_{j_p} \right\rangle = \det[g^{i_l j_m}]$$

Thus we can talk of $\langle \omega(x), \tau(x) \rangle$ for two real p-forms $\omega, \tau \in \bigwedge^p(M, \mathbb{R})$. We do the canonical Hermitian extension of this real inner product on $\bigwedge^p(M, \mathbb{R})$ to a Hermitian inner product on its complexification $\bigwedge^p(M) = \bigwedge^p(M, \mathbb{R}) \otimes \mathbb{C}$. We continue to denote it by $\langle -, - \rangle$. By definition, for $\omega, \tau \in \bigwedge^p(M)$, the pointwise inner product $\langle \omega(x), \tau(x) \rangle$ is a smooth function of $x \in M$. We can then define the global inner product:

$$(\omega, \tau) := \int_M \langle \omega(x), \tau(x) \rangle \, dV(x)$$

of the smooth p-forms $\omega, \tau \in \bigwedge^p(M)$, and if M is compact, (ω, τ) will be finite for all $\omega, \tau \in \bigwedge^p(M)$.

The *Hodge star operator* is an operator:

$$* : \bigwedge^p(M) \to \bigwedge^{n-p}(M)$$

which is the unique operator obeying the identity:

$$\omega \wedge (*\tau) = \langle \omega, \tau \rangle \, dV = \tau \wedge *\omega$$

for ω, $\tau \in \bigwedge^p(M)$. That is, $\langle -, - \rangle$ being a non-degenerate pairing gives an identification of $\bigwedge^p(M)$ with its dual vector space $\bigwedge^{p*}(M)$, and $\wedge$ being a non-degenerate pairing of $\bigwedge^p(M)$ with $\bigwedge^{n-p}(M)$ provides an identification of $\bigwedge^{n-p}(M)$ with $\bigwedge^{p*}(M)$, so the Hodge $*$-operator is the resulting identification of $\bigwedge^p(M)$ with $\bigwedge^{n-p}(M)$. Using the fact that $\omega \wedge *\tau = (-1)^{p(n-p)}(*\tau) \wedge \omega$ and that $\langle \omega, \tau \rangle \, dV = \langle \tau, \omega \rangle \, dV$, it easily follows from the definition of $*$ above that

$$* \circ * = (-1)^{p(n-p)} \quad \text{on} \quad \bigwedge^p(M)$$

As expected, $*: \bigwedge^0(M) \to \bigwedge^n(M)$ is the isomorphism $f \mapsto f dV$ discussed earlier.

Using the Hodge $*$-operator, one can define the differential operator:

$$\begin{array}{rcl} \delta : \bigwedge^{p}(M) & \to & \bigwedge^{p-1}(M) \\ \omega & \mapsto & (-1)^{np+n-1} * d(*\omega) \end{array}$$

From this definition it follows that $*\delta\omega = (-1)^{np+n-1} * * d * \omega = (-1)^{np+n-1+(n-p+1)(p-1)} d * \omega = (-1)^p d(*\omega)$.

We note that if ω is a p-form and τ a $(p+1)$-form, then:

$$d(\omega \wedge *\tau) = d\omega \wedge *\tau + (-1)^p \omega \wedge d(*\tau) = d\omega \wedge *\tau - \omega \wedge (*\delta\tau)$$

Now $d(\omega \wedge *\tau)$ is an n-form on M, and if M is compact, or if one of τ, ω are of compact support, then by Stokes theorem we have (since $\partial M = \phi$) that:

$$\begin{aligned} (d\omega, \tau) &= \int_M d\omega \wedge *\tau = \int_M \omega \wedge *\delta\tau + \int_M d(\omega \wedge *\tau) = \int_M \omega \wedge *\delta\tau + \int_{\partial M} (\omega \wedge *\tau) \\ &= (\omega, \delta\tau) + 0 = (\omega, \delta\tau) \end{aligned} \tag{24}$$

That is, the operators d and δ are *formal adjoints* to each other on the spaces of smooth compactly supported forms, with respect to the global inner product (,) defined above.

9.2. **The Laplacian on differential forms.**

Definition 9.2.1. The *Laplace-Beltrami operator*, or *Laplacian* on $\bigwedge^p(M)$ is defined as:

$$\begin{array}{rcl} \Delta : \bigwedge^{p}(M) & \to & \bigwedge^{p}(M) \\ \omega & \mapsto & (d\delta + \delta d)\omega \end{array}$$

Since d and δ are both first-order differential operators, Δ is a second order differential operator.

One can also write down expressions for Δ in local coordinates, which are messy. For $\Delta : \bigwedge^0(M) \to \bigwedge^0(M)$, the expression is:

$$\Delta f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_i(\sqrt{g} g^{ij} \partial_j f) \quad \text{for } f \in \bigwedge^{0}(M)$$

where $\sqrt{g} := \sqrt{\det g_{ij}}$ and $\partial_j := \frac{\partial}{\partial x_j}$.

Remark 9.2.2 (Formal self-adjointness and positivity of Δ)**.** By (24) above, we also have for M compact, or one of $\omega, \tau \in \bigwedge^p(M)$ of compact support that:

$$(\Delta\omega, \tau) = ((d\delta + \delta d)\omega, \tau) = (\delta\omega, \delta\tau) + (d\omega, d\tau) = (\omega, \Delta\tau)$$

that is, Δ is *formally self-adjoint* with respect to the global inner product $(-, -)$ on $\bigwedge^p(M)$.

Further, by the above, if ω is of compact support, or M is compact,

$$(\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega)$$

Hence for M compact, $(\Delta\omega, \omega) \geq 0$ for all $\omega \in \bigwedge^i(M)$, and $\Delta\omega = 0$ for $\omega \in \bigwedge^i(M)$ if and only if $d\omega$ and $\delta\omega = 0$.

Instead of proving ellipticity of the Laplace operator separately, we will set up the general notion of an elliptic complex, and the Laplacian above will follow as a special case.

9.3. **Elliptic operators on compact manifolds.** In the sequel, TM will always denote the *complexified tangent bundle* $T_{\mathbb{C}}(M) := T_{\mathbb{R}}(M) \otimes_{\mathbb{R}} \mathbb{C}$. Likewise, for the cotangent bundle $T^*M := T^*_{\mathbb{C}}M = \hom(T_{\mathbb{R}}M, \mathbb{C})$.

Definition 9.3.1 (Algebra of differential operators on M)**.** Define the space $\mathcal{D}^0(M)$ of *linear differential operators of order* 0 to be $C^\infty(M)$.

Let $\chi(M)$ denote the space of vector fields on M. That is, $\chi(M)$ is the space $C^\infty(M, TM)$ of smooth sections of TM on M. Note that $\chi(M)$ has a natural *left-module* structure over the ring $C^\infty(M)$.

Define the space $\mathcal{D}^1(M)$ of *linear differential operators of order* 1 by

$$\mathcal{D}^1(M) := C^\infty(M) \oplus \ \chi(M)$$

This $\mathbb{C}$-vector space inherits the left C^∞-module structure from both its summands. In addtion, it also the structure of a *right* $C^\infty(M)$ module, defined by:

$$(Pg) = gP + [P, g] = gP + [\alpha X, g] = gP + \alpha X(g) \quad \text{for} \ \ P = \beta + \alpha X, \ \ X \in \chi(M), \ \alpha, \beta \in C^\infty(M)$$

This formula arises from the fact that the vector field X is a derivation on $C^\infty(M)$, or more simply because Pg is naturally defined by the formula:

$$Pg(f) = P(gf) \quad \text{for all} \ \ f \in C^\infty(M)$$

Note that the commutators satisfy:

(i): $[\mathcal{D}^1(M), \mathcal{D}^1(M)] \subset \mathcal{D}^1(M)$.

(ii): $[\mathcal{D}^0(M), \mathcal{D}^1(M)] = [\mathcal{D}^1(M), \mathcal{D}^0(M)] \subset \mathcal{D}^0(M)$

The $k-th$ *tensor power* of $\mathcal{D}^1(M)$ is defined as

$$\mathcal{T}^k := \mathcal{D}^1(M) \otimes_{C^\infty(M)} \mathcal{D}^1(M) \otimes ... \otimes_{C^\infty(M)} \mathcal{D}^1(M) \quad (k \text{ times})$$

uses the *right* $C^\infty(M)$-module structure of the i-th factor and the *left* $C^\infty(M)$-module structure of the $(i+1)$-th factor. Thus it has a natural left and right $C^\infty(M)$-module structure. Note that

$$\mathcal{T}^k = \otimes^k \mathcal{D}^1(M) = \left(\otimes^k \mathcal{D}^1(M)\right) \otimes C^\infty(M) \subset \left(\otimes^k \mathcal{D}^1(M)\right) \otimes \mathcal{D}^1(M) = \mathcal{T}^{k+1}$$

so that we can define:

$$\mathcal{T} := \cup_{k=0}^\infty \mathcal{T}^k = \cup_{k=0}^\infty (\otimes^k \mathcal{D}^1(M))$$

as an associative, non-commutative $C^\infty(M)$ algebra, fitered by $\mathcal{T}^k$. Let $\mathcal{I}$ be the left-ideal in $\mathcal{T}$ generated by all elements of the form

$$P_1 \otimes P_2 - P_2 \otimes P_1 - [P_1, P_2] \otimes 1, \qquad P_1, P_2 \in \mathcal{D}^1(M)$$

A simple calculation shows that for $g \in C^\infty(M), \ P_1, P_2 \in \mathcal{D}^1$, we have:

$$\begin{aligned}(P_1 \otimes P_2 - P_2 \otimes P_1 - [P_1, P_2] \otimes 1)g \;&=\; g(P_1 \otimes P_2 - P_2 \otimes P_1 - [P_1, P_2] \otimes 1)\\ &+\; ([P_1, g] \otimes P_2 - P_2 \otimes [P_1, g] - [[P_1, g], P_2] \otimes 1)\\ &+\; (P_1 \otimes [P_2, g] - [P_2, g] \otimes P_1 - [P_1, [P_2, g]] \otimes 1)\end{aligned}$$

(where one uses the Jacobi identity $([[P_1, P_2], g] + [[P_2, g], P_1] + [[g, P_1], P_2] = 0)$. Thus the left $C^\infty(M)$-ideal generated by the elements $P_1 \otimes P_2 - P_2 \otimes P_1 - [P_1, P_2] \otimes 1$ automatically becomes a right $C^\infty(M)$-ideal as well. Now we can go modulo this ideal $\mathcal{I}$.

Hence we define the *algebra of differential operators* on M to be the associative algebra:

$$\mathcal{D}^\infty(M) = \mathcal{T}/\mathcal{I}$$

The image of $D^d(M)$ of $\mathcal{T}^d$ is the left $C^\infty(M)$-module of *linear differential operators of order d.* Since $\mathcal{T}^d \subset \mathcal{T}^{d+1}$, we have $\mathcal{D}^d(M) \subset \mathcal{D}^{d+1}(M)$ for all d, and $\mathcal{D}^0(M) = C^\infty(M)$. $\mathcal{D}^\infty$ is a non-commutative, associative algebra over $\mathcal{C}^\infty(M)$, filtered by $\mathcal{D}^d(M)$. From the corresponding property of $\mathcal{T}^k$'s, it follows that:

$$\mathcal{D}^i(M).\mathcal{D}^j(M) \subset \mathcal{D}^{i+j}(M)$$

Finally, if E and F are two smooth complex vector bundles over M, the space of smooth sections of the bundle $\hom(E,F)$, namely $C^\infty(M,\hom(E,F))$, is a right (=same as left) $C^\infty(M)$-module in a natural way. Hence we may define the $C^\infty(M)$-modules:

$$\mathcal{D}^d(M;E,F) := C^\infty(M,\hom(E,F)) \otimes_{C^\infty(M)} \mathcal{D}^d(M), \quad \mathcal{D}^\infty(M;E,F) := C^\infty(M,\hom(E,F)) \otimes_{C^\infty(M)} \mathcal{D}^\infty(M)$$

Note that these left $C^\infty(M)$-modules are algebras if $E = F$.

Exercise 9.3.2. Verify (by using local coordinates) that $P : C^\infty(M,E) \to C^\infty(M,F)$ is a linear differential operator of order d in the sense of 6.1.1 iff it is an element of $\mathcal{D}^d(M;E,F)$.

We will denote $\mathcal{D}^d(M;E,F)$ simply as $\mathcal{D}^d(E,F)$, and sometimes even $\mathcal{D}^d$ when no confusion is likely, for notational convenience.

Lemma 9.3.3 (Leading symbols again)**.** Let $\pi : (T^*M) \to M$ denote the natural projection, where T^*M is the *real cotangent bundle* of M. Note that there is the scaling map $T^*M \to T^*M$ given by $\xi \mapsto t\xi$, which preserves each fibre T^*_xM. Define the *space of symbols of order* d by:

$$\mathrm{Sym}^d(E,F) := \left\{\sigma \in C^\infty(T^*M, \pi^* \hom(E,F)) : \sigma(t\xi) = t^d\sigma(\xi), \;\; \xi \in T^*(M)\right\}$$

(In other words, those smooth sections σ which are homogenous polynomials of degree d in the fibre variables). When $E = F = M \times \mathbb{C}$ the trivial line bundle, we denote $\mathrm{Sym}^d(E,F)$ simply by $\mathrm{Sym}^d(M)$. We have the following facts:

(i): The associated graded module to the filtered $C^\infty(M)$-module $\mathcal{D}^\infty(E,F)$ is the algebra $\mathrm{Sym}^\infty(E,F) := \oplus_{d=0}^\infty \mathrm{Sym}^d(E,F)$. The natural quotient map of $C^\infty(M)$-modules:

$$\begin{array}{rcl} \mathcal{D}^d(E,F) & \to & \mathrm{Sym}^d(E,F) \\ P & \mapsto & \sigma_L(P) \end{array}$$

is called the *leading symbol map.* When $E = F$, $\mathrm{Sym}^\infty(E,E)$ is a commutative algebra, and the map σ_L is an algebra homomorphism $\mathcal{D}^\infty(E,E) \to \mathrm{Sym}^\infty(E,E)$. In a local coordinate system, $\sigma_L(P)$ as defined above coincides with the leading symbol defined earlier in Definition 5.5.4.

(ii): If $\xi \in T^*_x(M)$ is any cotangent vector, and f is any smooth function satisfying $df(x) = \xi$, then the leading symbol can be computed from the formula:

$$\sigma_L(P)(\xi) = \lim_{t\to\infty} t^{-d}(e^{-itf}Pe^{itf})(x)$$

(iii): P is elliptic iff $\sigma_L(P)(\omega)$ is a bundle isomorphism at all points of $T^*M \setminus 0_M$, where 0_M denotes the zero section of T^*M.

Proof: First we note that

$$\mathcal{D}^d(E,F) = C^\infty(M;\hom(E,F)) \otimes_{C^\infty(M)} \mathcal{D}^d(M)$$

and similarly

$$\mathrm{Sym}^d(E,F) = C^\infty(M;\hom(E,F)) \otimes_{C^\infty(M)} \mathrm{Sym}^d(M)$$

hence we need prove all the assertions for the case of $E = F = M \times \mathbb{C}$, the trivial bundle, and then left-tensor everything with $C^\infty(M,\hom(E,F))$ to get it for general E and F. The second simplification one can make is to reduce it to $M = \mathbb{R}^n$. This is done by first covering M with charts U_i with each U_i diffeomorphic to $\mathbb{R}^n$. In fact, for any $U \subset M$ open, we can define the left and right $C^\infty(U)$ module $\mathcal{D}^d(U)$ by the definition above (applied to $M = U$). Indeed, for $V \supset U$ any two open subsets, there are the natural restriction maps $\chi(V) \to \chi(U)$ which preserves commutators of vector fields, and also the restriction map $C^\infty(V) \to C^\infty(U)$. Thus we have a natural restriction map $\mathcal{D}^1(V) \to \mathcal{D}^1(U)$ of first order differential operators. Thus restriction maps result:

$$\mathcal{T}(V) := \oplus_k(\otimes^k\mathcal{D}^1(V)) \to \mathcal{T}(U)$$

which are algebra homomorphisms. Clearly the ideal $\mathcal{I}_V$ generated by $P_1 \otimes P_2 - P_2 \otimes P_1 - [P_1,P_2] \otimes 1$ in $\mathcal{T}(V)$ maps to the corresponding ideal $I_U \subset \mathcal{T}(U)$, one has a natural restriction algebra homomorphism

$\mathcal{D}^\infty(V) \to \mathcal{D}^\infty(U)$ which maps $\mathcal{D}^d(V)$ to $\mathcal{D}^d(U)$. The fact that $\mathcal{D}^d$ is a sheaf of left and right modules over the sheaf $\mathcal{C}^\infty$ follows from the facts that (i) $\mathcal{D}^1$ is a sheaf, which implies that $\mathcal{T}$ is a sheaf, and (ii) $\mathcal{I}$ is also a sheaf of ideals inside $\mathcal{T}$ (verify!). Similarly, one forms the symbol sheaf Sym^d, whose sections over $U \subset M$ are precisely the sections σ in $C^\infty(T^*(U), \pi^*\mathbb{C})$ satisfying $\sigma(t\xi) = t^d\sigma(\xi)$. The symbol map also becomes a sheaf map with all of these definitions.

Because of the sheaf theoretic machinery above, all the assertions of the lemma need to be verified only locally, i.e. on $M = \mathbb{R}^n$. In this setting, $\mathcal{D}^1$ is the $C^\infty(\mathbb{R}^n)$ module of all operators of the kind $\sum_i a_i(x)\partial_i + b(x)$ where $a_i, b \in C^\infty$. Then clearly $\mathcal{D}^1/\mathcal{D}^0$ is the space χ of smooth vector fields on $\mathbb{R}^n$. It is trivial to check that for any smooth vector field $X(x) = \sum_{i=1}^n a_i(x)D_{i,x}$, (where $D_i = \frac{1}{\sqrt{-1}}\partial_i$), the smooth function σ on $T^*(\mathbb{R}^n)$ defined by:

$$\sigma(\xi_x) = \sqrt{-1}\,[\xi_x(X(x))] \quad \text{for } \xi_x \in T^*_x(\mathbb{R}^n)$$

satisfies $\sigma(t\xi) = t\sigma(\xi)$ by the linearity of the cotangent vector ξ_x. Since $\sigma(dx_{i,x}) = a_i(x)$ by this definition, it is natural to write

$$\sigma(\xi) = \sigma\left(\sum_{i=1}^n \xi_i\, dx_{i,x}\right) = \sum_i a_i(x)\xi_i$$

which gives precisely the leading symbol of $P = \sum_i a_i(x)D_{i,x}$. Conversely, given a $\sigma \in C^\infty(T^*(\mathbb{R}^n))$ satisfying $\sigma(t\xi) = t\sigma(\xi)$, it follows that σ is a linear functional on $T^*_x(\mathbb{R}^n)$, and one gets a C^∞ vector field in χ by setting

$$X(x) = \sum_{i=1}^n \sigma(dx_{i,x})D_{i,x}$$

It is checked immediately that these maps are inverses of each other. More generally, if $P = \sum_{|\alpha|\le d} a_\alpha D_x^\alpha$ is a differential operator in $\mathcal{D}^d$, then $\sigma_L(P)$ is $\sum_{|\alpha|=d} a_\alpha \xi^\alpha$, which being a homogeneous polynomial of degree d, satisfies $\sigma_L(P)(t\xi) = t^d\sigma_L(P)$. The space of smooth functions on $T^*(M)$ and obeying this scaling property are precisely those functions which are homogeneous polynomials of degree d in the variables $\xi_1, .., \xi_n$, and so Sym^d is exactly $\sigma_L(\mathcal{D}^d)$. Indeed, this definition of σ_L agrees with the earlier one in Definition 5.5.4. Now it is trivially checked that $\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q)$. Thus (i) is proved.

To see (ii), note that if $df(x) = \xi$, then $\partial_{j,x}f = \xi_j$, and hence for a C^∞ function g, we have:

$$D_{j,x}(e^{itf}g)(x) = te^{itf}\partial_{j,x}f(x)g(x) + e^{itf}D_{j,x}g(x) = te^{itf}\xi_j g(x) + e^{itf}(D_{j,x}g)(x)$$

More generally, using Leibnitz formula for differentiating a product:

$$\begin{aligned} D_{1,x}^{\alpha_1}...D_{n,x}^{\alpha_n}(e^{itf}g)(x) &= t^{|\alpha|}e^{itf}\xi_1^{\alpha_1}...\xi_n^{\alpha_n}g(x) \\ &+ \text{(terms involving strictly lower powers of } t) \end{aligned}$$

from which it follows that

$$\lim_{t\to\infty} t^{-d}(e^{-itf}Pe^{itf})(x) = \sigma_L(P)$$

and (ii) is proved.

(iii) is clear because saying that $\sigma_L(P)(\xi) \neq 0$ for all $|\ \xi\ |$ large enough is equivalent to saying that it is non-zero for all $\xi \neq 0$, by homogeneity of $\sigma_L(P)$. The lemma follows. □

For f a smooth function on M, and $P \in \mathcal{D}^d(M, E)$ a differential operator, we denote by $(ad\, f)P$ the differential operator $fP - Pf$. Using the fact that $[\mathcal{D}^0, \mathcal{D}^1] \subset \mathcal{D}^0$ and induction, it is easy to see that $(ad\, f)P \in \mathcal{D}^{d-1}(M, E)$, so that $(\mathrm{ad}\ f)^dP$ is a zero-th order operator.

Corollary 9.3.4. Let P and f as in (iii) of the Lemma 9.3.3 above. Then

(i):

$$\sigma_L(P) = \frac{(-i)^d}{d!}(ad\, f)^dP$$

(ii): Let P^* be the adjoint of P, defined with respect to some Hermitian inner products on E, F. Then

$$\sigma_L(P^*)(\xi) = (\sigma_L(P))^*(\bar{\xi})$$

Proof: Note that
$$\frac{d}{dt}(e^{-itf}Pe^{itf}) = (-i)e^{-itf}[(ad\,f)P]e^{itf}$$
so that we inductively have:
$$\left(\frac{d}{dt}\right)^d (e^{-itf}Pe^{itf}) = (-i)^d e^{-itf}[(ad\,f)^d P]e^{itf} = (-i)^d (ad\,f)^d P$$
since $(ad\,f)^d P$ is in $\mathcal{D}^0$, and commutes with e^{itf}. Applying L'Hospital's rule to the formula in (iii) of 9.3.3, we have (i)

To see (ii), if $df(x) = \xi$, we have by (iii) of 9.3.3 above that:
$$\sigma_L(P^*)(\xi) = \lim_{t\to\infty} t^{-d}(e^{-itf}P^*e^{itf}) = \lim_{t\to\infty} t^{-d}\left(e^{-it\bar{f}}Pe^{it\bar{f}}\right)^* = (\sigma_L(P))^*(\bar{\xi})$$
The corollary follows.

□

9.4. **Elliptic Complexes.**

Definition 9.4.1. Let $\{E^i\}_{i=0}^m$ be complex vector bundles with Hermitian metrics. Say that a sequence of differential operators:
$$... \to C^\infty(M, E^i) \xrightarrow{P_i} C^\infty(M, E^{i+1}) \to ....$$
is an *elliptic complex* if:

(i): $P_{i+1} \circ P_i = 0$ for all i.

(ii): The associated symbol sequence
$$.... \to \pi^* E^i \overset{\sigma_L(P_i)(\xi)}{\longrightarrow} \pi^* E^{i+1}$$
is exact for all $\xi \neq 0$ (i.e. all $\xi \in T^*M \setminus M$).

(iii): The order of each P_i is $d > 0$. (For most elliptic complexes of concern to us, $d = 1$).

Clearly, if we only have a two term sequence $C^\infty(M, E^0) \overset{P}{\to} C^\infty(M, E^1)$, then this two term complex is elliptic iff P is an elliptic operator of order $d > 0$.

Before looking at some examples of elliptic complexes, let us note the following:

Lemma 9.4.2. Let $\{C^\infty(M, E^*), P_*\}$ be a complex of differential operators (i.e. $P_{i+1} \circ P_i = 0$ for all i). Define the *Laplacian* of this complex by:
$$\Delta_P^i = P_i^* P_i + P_{i-1}P_{i-1}^* : C^\infty(M, E^i) \to C^\infty(M, E^i)$$
Then the complex above is elliptic iff Δ_P^i is an elliptic operator for each i.

Proof: Let us denote $\sigma_L(P_i) = p_i$. Let us assume that the complex is elliptic. Then, from (ii) of the Corollary 9.3.4, and (i) of 9.3.3 that σ_L is an algebra homomorphism, it follows that:
$$\sigma_L(\Delta_P^i)(\xi) = p_i^*(\xi)p_i(\xi) + p_{i-1}(\xi)p_{i-1}^*(\xi)$$
If for some $e \in \pi^* E^i$, $\sigma_L(\Delta_P^i)(\xi)e = 0$, and $\xi \neq 0$, it follows that with respect to the Hermitian inner product $\langle -, - \rangle$ on $\pi^* E^i$, we have:
$$\langle p_i(\xi)e, p_i(\xi)e\rangle + \left\langle p_{i-1}^*(\xi)e, p_{i-1}^*(\xi)e\right\rangle = 0$$
which implies that $p_i(\xi)e = 0$ and $p_{i-1}^*(\xi)e = 0$. Since the complex is elliptic, and $\xi \neq 0$, it follows that $e = p_{i-i}(\xi)v$ for $v \in \pi^*(E^{i-1})$. Since $p_{i-1}^*(\xi)e = 0$, it follows that $p_{i-1}^*(\xi)p_{i-1}(\xi)v = 0$. Thus $\left\langle v, p_{i-1}^*(\xi)p_{i-1}(\xi)v\right\rangle = 0$, which implies that $p_{i-1}(\xi)v = e = 0$. Thus $\sigma_L(\Delta_P^i)(\xi) : \pi^*E^i \to \pi^*E^i$ is a monomorphism, and hence an isomorphism. That is Δ_P^i is elliptic.

The converse is similar, and left as an exercise. □

Remark 9.4.3. Note that if $\mathcal{P}$ is an elliptic complex, and of finite length (i.e $E^i = 0$ for $i >> 0$), and dim $M > 0$, then we have

$$\sum_{i=0}^{\infty} \mathrm{rank}\, E^{2i} = \sum_{i=0}^{\infty} \mathrm{rank}\, E^{2i+1}$$

This is because we can choose a $\xi \neq 0$ in T_x^*M, and the fact that the symbol complex:

$$... \to E_x^i \xrightarrow{\sigma_L(\xi)} E_x^{i+1} \to ....$$

is exact means that the alternating sum:

$$\sum_{i=0}^{\infty} (-1)^i \dim\, E_x^i = \sum_{i=0}^{\infty} (-1)^i \mathrm{rank}\, E^i = 0$$

which implies our assertion.

Example 9.4.4 (The de-Rham Complex)**.** Set $E^i = \Lambda^i(T^*_{\mathbb{C}}M)$, the i-th exterior power of the (complexified) cotangent bundle of M. Then consider the de-Rham complex:

$$.... \to C^\infty(M, E^i) =: \Lambda^i(M, \mathbb{C}) \xrightarrow{d_i} \Lambda^{i+1}(M, \mathbb{C}) \to ....$$

If $\xi_x = \sum_i \xi_j dx_{j,x}$ is a real cotangent vector, in some local coordinate system, then since for $\omega = \sum_{|I|=i} \omega_I dx_I \in \Lambda^i(M, \mathbb{C})$ we have the representation of $d\omega$ in local coordinates:

$$d\omega = \sum_j dx_j \partial_j \wedge \omega = \sqrt{-1} \left(\sum_j dx_j D_{x_j} \right) \wedge \omega$$

it follows that $\sigma_L(d)(\xi) = \sqrt{-1} \left(\sum_j dx_j \xi_j \right) \wedge (-) = i\xi \wedge (-)$. One already knows that this is a complex of differential operators, i.e. $d_{i+1} \circ d_i = 0$, so to show that the complex is elliptic, it is enough to show that the operator $e(\xi) := \xi \wedge (-)$ is exact for $\xi \neq 0$. Since $\xi \neq 0$, we may complete it to a basis $\{e_i\}_{i=1}^p$ of $\pi^*(T^*M))_\xi = T_x^*(M)$ with $e_p = \xi$. Then, each $\alpha \in \Lambda^p(T^*_{\mathbb{C},x}(M))$ may be uniquely written as:

$$\alpha = \alpha_1 + \xi \wedge \alpha_2$$

where α_1, α_2 do not involve $\xi = e_p$. If $\xi \wedge \alpha = 0$, it follows that $\xi \wedge \alpha_1 = 0$, but since α_1 does not involve ξ, this implies $\alpha_1 = 0$. Thus $\alpha = \xi \wedge \alpha_2$. This proves that the de-Rham complex is elliptic.

From the lemma 9.4.2 above, it follows that all the Laplacians $\Delta^i = dd^* + d^*d$ of the de-Rham complex are elliptic operators.

Example 9.4.5 (Twisted Dolbeault Complex)**.** Let M be a compact *complex* manifold of $\dim_{\mathbb{C}} M = n$. Let E be a *holomorphic* vector bundle on M of $\mathrm{rk}_{\mathbb{C}} E = k$. We have the following well known decomposition (as complex vector bundles) for the complexification of the real tangent bundle $T_{\mathbb{R}}M$:

$$T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ is the holomorphic tangent bundle of M, and $T^{0,1}M$ is its complex conjugate bundle, and called the anti-holomorphic tangent bundle of M. In a local holomorphic coordinate chart $U \subset M$, we may write $v \in T^{1,0}M_{|U}$ as:

$$v = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial z_j}$$

and correspondingly $w \in T^{0,1}M_{|U}$ as:

$$w = \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial \bar{z}_j}$$

The decomposition of $T_{\mathbb{C}}M$ leads to a corresponding decomposition of $T^*_{\mathbb{C}}M = \hom_{\mathbb{R}}(T_{\mathbb{R}}M, \mathbb{C})$ as:

$$T^*_{\mathbb{C}}M = (T^{1,0}M)^* \oplus (T^{0,1}M)^*$$

Thus

$$\Lambda^i(T^*_{\mathbb{C}}M) = \oplus_{p+q=i}\left(\Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*\right) =: \oplus_{p+q=i}\Lambda^{p,q}(T^*_{\mathbb{C}}M)$$

Again, in a local holomorphic chart over U, and element $\omega \in \Lambda^{p,q}(T^*M)_{|U}$ has the representation:

$$\omega = \sum_{|I|=p,\ |J|=q} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

where we use the notation:

$$dz_I := dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \wedge .... \wedge d\overline{z}_{j_q}$$

We can tensor all this with the bundle E, and thus we have:

$$\Lambda^i(T^*M) \otimes_{\mathbb{C}} E = \oplus_{p+q=i}\Lambda^{p,q}(T^*_{\mathbb{C}}M) \otimes_{\mathbb{C}} E$$

Thus, for smooth sections of the above bundle, we have:

$$\Lambda^i(M,E) := C^\infty(M, \Lambda^i(T^*M) \otimes_{\mathbb{C}} E) = \oplus_{p+q=i}\Lambda^{p,q}(M,E)$$

where $\Lambda^{p,q}(M,E) := C^\infty(M, \Lambda^{p,q}(T^*_{\mathbb{C}}M) \otimes_{\mathbb{C}} E)$. Again, for U a coordinate chart, a section $\omega \in \Lambda^{p,q}(U,E)$ has the representation:

$$\omega = \sum_{|I|=p,\ |J|=q} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

where $\alpha_{IJ} \in C^\infty(U,E)$ are smooth sections of $E_{|U}$.

Now we can define the *Dolbeault operator*

$$\overline{\partial}^E : \Lambda^{p,q}(M,E) \to \Lambda^{p,q+1}(M,E)$$

by defining it on local representations as follows. On a coordinate chart U, write:

$$\omega_{|U} = \sum_{|I|=p,\ |J|=q} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

with $\alpha_{IJ} \in C^\infty(U,E) = \Lambda^{0,0}(U,E)$, and set

$$\overline{\partial}^E \omega_{|U} = \sum_{|I|=p,\ |J|=q} \overline{\partial}^E \alpha_{IJ} \wedge dz_I \wedge d\overline{z}_J$$

where

$$\overline{\partial}^E \alpha_{IJ} = \sum_{j=1}^n \frac{\partial \alpha_{IJ}}{\partial \overline{z}_j} d\overline{z}_j$$

The thing to verify is that all this is globally defined, and the reason it is globally defined is that $\overline{\partial}^E\alpha$ is globally defined as an element of $\Lambda^{0,1}(U,E)$ for $\alpha \in C^\infty(U,E) = \Lambda^{0,0}(U,E)$, and $U \subset M$ any open set. For, over a W satisfying $E_{|W}$ is holomorphically trivial, we can write α as $\alpha = \sum_{i=1}^k \alpha_i e_i$ where α_i are smooth functions on W, and $\{e_i\}_{i=1}^k$ is a *holomorphic frame* for $E_{|W}$. Then we set:

$$\overline{\partial}^E \alpha = \sum_{i=1}^k \overline{\partial}\alpha_i \, e_i$$

where, on a coordinate chart with coordinates $z_1, .., z_n$, we have

$$\overline{\partial}\alpha_i = \sum_{j=1}^n \frac{\partial \alpha_i}{\partial \overline{z}_j} d\overline{z}_j$$

the usual $\overline{\partial}$ operator on smooth complex valued functions. That this $\overline{\partial}$-operator on smooth functions is well-defined follows from the fact that coordinate changes on M are holomorphic.

If we change to another *holomorphic* frame $\{f_j\}_{j=1}^k$ for $E_{|V}$, where $V \subset M$ is another open set, we have the transition relation $e_i = \sum_j g_{ji} f_j$, where g_{ij} are *holomorphic functions* on $V \cap W$, and thus

$$\alpha = \sum_i \alpha_i e_i = \sum_j \left(\sum_i g_{ji}\alpha_i\right) f_j$$

and since g_{ji} are holomorphic, we have $\overline{\partial}(g_{ji}\alpha_i) = g_{ji}\overline{\partial}\alpha_i$, so that

$$\sum_i \overline{\partial}\alpha_i \, e_i = \sum_j \left(\sum_i g_{ji}\overline{\partial}\alpha_i \right) f_j = \sum_j \overline{\partial} \left(\sum_i g_{ji}\alpha_i \right) f_j$$

which shows that our definition of $\overline{\partial}^E \alpha$ makes global sense, independent of local holomorphic frames.

It is now easy to check, using local coordinates, that $\overline{\partial} \circ \overline{\partial} = 0$, so that we have the *twisted Dolbeault Complex*

$$.... \to \Lambda^{p,q}(M,E) = C^\infty(M, \Lambda^{p,q}T^*_{\mathbb{C}}(M) \otimes_{\mathbb{C}} E) \xrightarrow{\overline{\partial}^E} \Lambda^{p,q+1}(M,E) \to ....$$

of differential operators. That is, we are taking the complex vector bundle $E^q := \Lambda^{p,q}T^*_{\mathbb{C}}M \otimes E$, with a fixed p. We can easily equip these smooth complex vector bundles with some Hermitian metrics, arising from a Hermitian metric on the bundles $T_{\mathbb{C}}M$ and E.

To check that this is an elliptic complex, one needs to calculate the symbol of $\overline{\partial}^E$. We note that the the complex vector bundle $(T^{0,1}M)^*$ can be identified with the real cotangent bundle T^*M, by forgetting its complex structure, and with this notation $\overline{\partial} = i\sum_j d\overline{z}_j \frac{\partial}{\partial \overline{z}_j} \wedge (-)$, so that its symbol is given by:

$$\sigma_L(\overline{\partial})(\xi) = \frac{i}{2}\sigma_L \left(\sum_j d\overline{z}_j (D_{x_j} + iD_{y_j}) \wedge (-) \right) = \frac{i}{2} \sum_j d\overline{z}_j (\xi^1_j + i\xi^2_j) = \frac{i}{2}\xi \wedge (-)$$

where $\xi = \sum_j \xi_j d\overline{z}_j = \sum_j (\xi^1_j + i\xi^2_j) d\overline{z}_j \in T^{0,1}(M)^*$. The reason that $\sigma_L(\overline{\partial})(\xi)$ is exact for $\xi \neq 0$ is the same as that for the de Rham complex above, so we omit the argument.

9.5. **The Hodge Theorem for Elliptic Complexes.**

Definition 9.5.1. Let $\mathcal{P}$ denote an elliptic complex:

$$..... \to C^\infty(M, E^i) \xrightarrow{P_i} C^\infty(M, E^{i+1}) \to ....$$

on a compact Riemannian manifold M. We define the i-th *cohomology* of this complex to be the $\mathbb{C}$-vextor space:

$$H^i(M, \mathcal{P}) := \frac{\ker P_i}{\operatorname{Im} P_{i-1}}$$

For example, in the case of the de-Rham complex of Example 9.4.4 above, this gives the de Rham cohomology of M (with complex coefficients). In the case of the twisted Dolbeault complex of Example 9.4.5 above, it gives the (p,q)-*Dolbeault cohomology with coefficients in* E, and is denoted by $H^{p,q}(M,E)$, which algebraic geometers write as $H^q(M, \Omega^p(E))$ for reasons we needn't explore here.

Theorem 9.5.2 (Hodge Theorem for Elliptic Complexes)**.** Let $\mathcal{P}$ be an elliptic complex on a compact Riemannian manifold M. Let $\Delta^i_P : C^\infty(M, E^i) \to C^\infty(M, E^i)$ be the Laplacian introduced in the Lemma 9.4.2. Then:

(i): $H^i(M, \mathcal{P}) \simeq \ker \Delta^i_P$, and this cohomology is a finite dimensional space.

(ii): (*Kodaira-Hodge decomposition*) The L_2-space of sections $L_2(M, E^i) = H_0(M, E^i)$ admits the L_2-orthogonal direct sum decomposition:

$$L_2(M, E^i) = \ker \Delta^i_P \oplus P^*_i(H_d(M, E^{i+1})) \oplus P_{i-1}(H_d(M, E^{i-1}))$$

where each space on the right is a closed Hilbert subspace.

Proof: Let us first prove (ii), and then (i) will follow as a consequence. Since $\mathcal{P}$ is an elliptic complex, by the Lemma 9.4.2, the operators

$$\Delta^i_P : C^\infty(M, E^i) \to C^\infty(M, E^i)$$

are elliptic operators of the same order $2d \geq 2$, where $d = \operatorname{ord} P_i \geq 1$. Also by its definition, it is self-adjoint. By (i) of Proposition 8.4.8, the kernel

$$\mathcal{H}^i_P := \ker \Delta^i_P$$

is a finite dimensional subspace inside $C^\infty(M, E^i)$, and therefore a finite-dimensional closed subspace of $H_d(M, E^i)$ for all d. Let $\pi : L_2(M, E^i) = H_0(M, E^i) \to \mathcal{H}^i_P$ be the L_2-orthogonal projection. Then for all $f \in L_2(M, E^i)$, we have $f - \pi(f) \in \mathcal{H}^{i,\perp}_P$. Then let

$$G^i_P : L_2(M, E^i) \to L_2(M, E^i)$$

denote the Green operator from Proposition 8.4.8. By (a) of (iii) in that proposition, we have:

$$\Delta^i_P G^i_P (f - \pi(f)) = f - \pi(f)$$

Again, by (a) of (iii) of the Proposition 8.4.8, we have $G^i_P(\pi(f)) = 0$, since $\pi(f) \in \ker \Delta^i_P$. Thus we have:

$$f = \pi(f) + \Delta^i_P G^i_P(f) = \pi(f) + P^*_i(P_i G^i_P f) + P_{i-1}(P^*_{i-1} G^i_P f) \quad \text{for all} \quad f \in L_2(M, E^i)$$

By the construction of the Green operator in 8.4.8, $G^i_P f \in H_{2d}(M, E^i)$, which implies that $P_i G^i_P f \in H_d(M, E^{i+1})$. Similarly, $P^*_{i-1} G^i_P f \in H_d(M, E^{i-1})$. The computation above therefore shows that:

$$L_2(M, E^i) = \mathcal{H}^i_P + P_{i-1}(H_d(M, E^{i-1}) + P^*_i(H_d(M, E^{i+1})$$

We denote the last two spaces above by $\operatorname{Im} P_{i-1}$ and $\operatorname{Im} P^*_i$ respectively.

To check that the decomposition is orthogonal, we easily check that $\mathcal{H}^i_P = \ker P_i \cap \ker P^*_{i-1}$ from the definition of Δ^i_P. Hence for $\alpha \in \mathcal{H}^i_P$, we have:

$$(\alpha, P^*_i \beta) = (P_i \alpha, \beta) = 0$$

for all $\beta \in H_d(M, E^{i+1})$. Hence $\mathcal{H}^i_P$ is orthogonal to $\operatorname{Im} P^*_i$. Similarly, it is orthogonal to $\operatorname{Im} P_{i-1}$. Finally, if we have $\alpha = P_{i-1}\beta$ and $\gamma = P^*_i \delta$, then:

$$(\alpha, \gamma) = (P_{i-1}\beta, P^*_i \delta) = (P_i P_{i-1} \beta, \delta) = 0$$

since $P_i P_{i-1} = 0$. This shows that $\operatorname{Im} P^*_i$ and $\operatorname{Im} P_{i-1}$ are also mutually orthogonal. We need to check that both these images are closed. Note that if $\alpha \in L_2(M, E^i)$ and $\alpha \in \mathcal{H}^i_P + P_{i-1}(H_d(M, E^{i-1}))$, then $P_i \alpha = 0$. Conversely, if $P_i : L_2(M, E^i) \to H_{-d}(M, E^{i+1})$ annihilates α, we write:

$$\alpha = \alpha_1 + P_{i-1}\beta + P^*_i \gamma$$

by the decomposition above, where $\gamma \in H_d(M, E^{i+1})$. Now note that $P_i \alpha = 0$ implies that the element $P_i P^*_i \gamma \in H_{-d}(M, E^{i+1})$ is zero. This implies that under the natural pairing $\langle -, - \rangle$ of $H_d(M, E^{i+1})$ and $H_{-d}(M, E^{i+1})$ (see (iii) of Proposition 4.2.2), we have:

$$\langle \gamma, P_i P^*_i \gamma \rangle = (P^*_i \gamma, P^*_i \gamma)_0 = 0$$

which implies that $P^*_i \gamma = 0$, and $\alpha \in \mathcal{H}^i_P + \operatorname{Im} P_{i-1}$. Thus $\operatorname{Im} P^*_i$ is precisely the orthogonal complement of the subspace $\ker P_i$ in $L_2(M, E^i)$, and since the orthogonal complement of *any* subspace is closed, we have $\operatorname{Im} P^*_i$ is closed. Similarly, $\operatorname{Im} P_{i-1}$ is the orthogonal complement of $\ker P^*_{i-1}$ in $L_2(M, E^i)$ and also closed. This proves (ii).

In fact, since both G^i_P and Δ^i_P map smooth forms to smooth forms, as do P_{i-1} and P^*_i, and $\mathcal{H}^i_P \subset C^\infty(M, E^i)$, we can restrict the decomposition above to obtain:

$$C^\infty(M, E^i) = \mathcal{H}^i_P \oplus P_{i-1}(C^\infty(M, E^{i-1})) \oplus P^*_i(C^\infty(M, E^{i+1}))$$

which is L_2-orthogonal, but of course left hand space and the two right hand spaces are no longer closed in $L_2(M, E^i)$. Again it is readily checked that

$$\ker \{P_i : C^\infty(M, E^i) \to C^\infty(M, E^i)\} = \mathcal{H}^i_P \oplus P_{i-1}(C^\infty(M, E^{i-1}))$$

which implies that

$$H^i(M, \mathcal{P}) = \frac{\ker \{P_i : C^\infty(M, E^i) \to C^\infty(M, E^i)\}}{P_{i-1}(C^\infty(M, E^{i-1}))} = \mathcal{H}^i_P$$

and, indeed the natural composite map:

$$\mathcal{H}^i_P \to \ker\{P_i : C^\infty(M,E^i) \to C^\infty(M,E^i)\} \to H^i(M,\mathcal{P})$$

is the required isomorphism. This proves (i), and the theorem follows. □

Corollary 9.5.3 (Hodge-deRham Theorem)**.** By the Theorem 9.5.2 above applied to the elliptic de Rham complex of Example 9.4.4 above, we have that the i-th de Rham cohomology of a compact manifold M satisfies:

$$H^i_{dR}(M,\mathbb{C}) \simeq \mathcal{H}^i$$

where $\mathcal{H}^i = \ker\{\Delta^i : \Lambda^i(M,\mathbb{C}) \to \Lambda^{i+1}(M,\mathbb{C})\}$ is the space of *harmonic i-forms* on M. In particular, by the above theorem, this cohomology is finite dimensional. This is, incidentally, provable by using the de Rham theorem which is highly non-trivial, and the fact that a compact smooth manifold is a finite CW-complex, which again uses non-trivial Morse Theory. That is, the finite dimensionality of the de Rham cohomology of a compact smooth manifold, whichever way one chooses to prove it, is a very deep result.

Corollary 9.5.4 (Hodge-Dolbeault Theorem)**.** By the Theorem 9.5.2 applied to the twisted Dolbeault complex of Example 9.4.5, it follows that the twisted Dolbeault cohomology $H^{p,q}(M,E)$ of a compact complex manifold M and holomorphic coefficient bundle E satisfies:

$$H^{p,q}(M,E) \simeq \mathcal{H}^{p,q}(M,E)$$

where the space on the right is the kernel of the *Hodge-Dolbeault Laplacian* $\square := \overline{\partial}^E\overline{\partial}^{E*} + \overline{\partial}^{E*}\overline{\partial}^E$ inside $\Lambda^{p,q}(M,E)$. Again the theorem implies that this Dolbeault cohomology is finite dimensional. In the case when E is the trivial line bundle $M\times\mathbb{C}$, and $p=0$, the Dolbeault cohomology $H^{p,q}(M,E)$ is simply denoted $H^{0,q}_{\overline{\partial}}(M)$ or simply $H^{0,q}(M)$. Then, by the above, the alternating sum:

$$\sum_{q=0}^{n}(-1)^q \dim\, H^{0,q}(M)$$

is finite. Again, that this is finite for the situation above has to be proved as above, and is a very deep fact.

9.6. **Index of an elliptic complex.** We observed in (a) of (iii) in Proposition 8.4.8 that for a formally self-adjoint elliptic differential operator $P : C^\infty(M,E) \to C^\infty(M,E)$ of order $d>0$, the (finite dimensional) cokernel $\operatorname{Coker} P = (\operatorname{Im} P)^\perp = \ker P$, so that the index of the Fredholm operator $P : H_d(M,E) \to H_0(M,E)$ is $\operatorname{ind} P = \dim\ker P - \dim\operatorname{Coker} P = 0$. Thus we won't get any interesting index by considering the indices of the elliptic (of order 2d) Laplacians Δ^i_P of an elliptic complex $\mathcal{P}$. On the other hand, a profound idea due to Dirac (who introduced it to explain electron spin) suggests that we find a "square root" of the Laplacian to get an interesting index.

What one does instead is construct an operator of order d as follows.

Definition 9.6.1 (The Dirac operator of an elliptic complex)**.** Let M be a compact oreinted Riemannian manifold, and let $\mathcal{P}$ be the elliptic complex:

$$\ldots \to C^\infty(M,E^i) \xrightarrow{P_i} C^\infty(M,E^{i+1}) \to \ldots..$$

where P_i is a differential operator of order $d>0$ for each i. Let us assume that this complex is of finite length, i.e. $E^i = 0$ for i large enough. Define $E^+ = \oplus_{i=0}^\infty E^{2i}$ and $E^- = \oplus_{i=0}^\infty E^{2i+1}$. Note that by the Remark 9.4.3, the smooth complex vector bundles E^+ and E^- have the same rank. Then we define the following operators of order d:

$$\begin{aligned} D^+ &:= P_+ + P_-^* : C^\infty(M,E^+) \to C^\infty(M,E^-) \\ D^- &:= P_- + P_+^* : C^\infty(M,E^-) \to C^\infty(M,E^+) \end{aligned}$$

where $P^+ := \oplus_i P_{2i}$, $P_- = \oplus_i P_{2i+1}$. These operators are called the *Dirac operators* of the elliptic complex $\mathcal{P}$.

Proposition 9.6.2. In the setting of the Definition 9.6.1 above, we have:

(i): D^+ and D^- are formal adjoints of each other, and are differential operators of order d.

(ii): The composite $D^-D^+ = \oplus_{i\geq 0}\Delta_P^{2i}$ and similarly the composite $D^+D^- = \oplus_{i\geq 0}\Delta_P^{2i+1}$. Thus the two term complex:

$$D^+ : C^\infty(M, E^+) \to C^\infty(M, E^-)$$

is an elliptic complex, with associated Laplacian being

$$\Delta_P^+ := D^-D^+ = \oplus_{i\geq 0}\Delta_P^{2i} : C^\infty(M, E^+) \to C^\infty(M, E^+)$$

This Laplacian Δ_P^+ is elliptic and (formally) self-adjoint. Similarly one can construct the other elliptic formally self-adjoint Laplacian $\Delta_P^- := D^+ : D^-$ acting on $C^\infty(M, E^-)$.

(iii): The operators

$$D^\pm : H_d(M, E^\pm) \to L_2(M, E^\mp)$$

are elliptic, and hence Fredholm. Their Fredholm index is given by:

$$\text{ind}\, D^+ = \sum_{i=0}^{\infty}(-1)^i(\dim\ker\Delta_P^i) = \sum_{i=0}^{\infty}(-1)^i \dim\, H^i(M, \mathcal{P}) = -\text{ind}\, D^-$$

Proof: The assertion (i) is clear from the definitions.

(ii) is also clear from the definitions. The assertion that this two term complex is elliptic follows from the fact that the associated Laplacian is precisely $\Delta_P^+ = \oplus_{i\geq 0}\Delta_P^{2i}$, which is elliptic, and formally self adjoint, by the Lemma 9.4.2.

We have seen that a two term complex is elliptic iff the operator in this complex is elliptic, so D^+ (and hence its formal adjoint D^-) is an elliptic operator. One easily checks that $D^+f = 0$ iff $\Delta_P^+ f = D^-D^+f = 0$ for $f \in H_d(M, E^+)$, by using the fact that

$$(D^+f, g) = (f, D^-g), \ \text{ for all } \ f \in H_d(M, E^+), \ \ g \in L_2(M, E^-)$$

which follows from the duality of $H_d(M, E^+)$ and $H_{-d}(M, E^+)$ of (iii) in Proposition 4.2.2 and that the above formula holds for f, g smooth (i.e. D^+ and D^- are formal adjoints of each other). Likewise for the adjoint D^-, we have $f \in \ker D^-$ iff $f \in \ker \Delta_P^-$. Hence the index of D^+ and D^- satisfy:

$$\text{ind}\, D^+ = \dim\,\ker D^+ - \dim\,\ker D^- = \sum_{i\geq 0}(\dim\,\ker\Delta_P^{2i} - \dim\,\ker\Delta_P^{2i+1}) = \sum_{i\geq 0}(-1)^i \dim\,\ker\Delta_P^i = -\text{ind}\, D^-$$

The fact that $\dim\,\ker\Delta_P^i = \dim H^i(M, \mathcal{P})$ follows from (i) of the Hodge Theorem 9.5.2. The proposition follows. □

Note that for the De Rham complex of Example 9.4.4, the Dirac operator is $d + d^* = d + \delta$, and its index is the Euler characteristic of M. For the twisted Dolbeault complex of Example 9.4.5, the associated Dirac operator is $\overline{\partial}^E + \overline{\partial}^{E*}$, and its index is the quantity $\sum_q(-1)^q H^{p,q}(M, E)$.

Remark 9.6.3. Aside from the fact that the Dirac operator construction leads to an interesting index, it also shows that no generality is lost by considering two-term elliptic complexes instead of a general elliptic complex of finite length. We will henceforth restrict ourselves to this setting for analytical considerations, though finite length elliptic complexes will always be in the background because they arise from natural geometric considerations, e.g. the de Rham complex, the twisted Dolbeault complex, and the signature and spin complexes that will arise later.

10. Heat kernels

10.1. **Heat Operators on Compact Manifolds.** We now confine ourselves to the setting of the Proposition 9.6.2. That is, we have two Hermitian smooth vector bundles $E^{\pm}$ on M, and elliptic operators of order $d > 0$ D^+ and D^- fulfilling all the conclusions of 9.6.2. In particular, by the Propositions 8.4.8 and 8.4.9, we know that the spectra of Δ_P^+ and Δ_P^- are discrete, with absolute values of eigenvalues $\mid \lambda_n \mid \geq Cn^{\delta}$ for some $\delta > 0$. Actually, one can say more:

Proposition 10.1.1. In the setting of Proposition 9.6.2, the spectrum of Δ_P^+ and Δ_P^- satisfies $\lambda_n \geq 0$. Thus if we arrange the eigenvalues of Δ_P^+ in non-decreasing order:

$$\lambda_0 \leq \lambda_1 \leq .... \leq \lambda_n \leq ....$$

we have constants $C, \delta > 0$ so that $\lambda_n \geq Cn^{\delta}$ for all n. Likewise for Δ_P^-.

Proof: Let e_n be a basis of L_2-orthonormal smooth eigensections in $L_2(M, E^+)$, for the elliptic self-adjoint operator Δ_P^+, via (d) of the Proposition 8.4.8. Then

$$\lambda_n = (\Delta_P^+ e_n, e_n) = (D^- D^+ e_n, e_n) = (D^+ e_n, D^+ e_n) \geq 0$$

where the right equality follows from the fact that $D^+ e_n$ is also smooth, and D^+ and D^- are formal adjoints of each other by (i) of 9.6.2. The last assertion follows from the Proposition 8.4.9. □.

Proposition 10.1.2. Let $t \in (0, \infty)$. Define the operator $e^{-t\Delta_P^+}$ by defining its action on the eigensections e_n of the last proposition by $e^{-t\Delta_P^+} e_n = e^{-t\lambda_n} e_n$ (i.e. by "functional calculus"). Then this extends to a bounded self-adjoint operator:

$$e^{-t\Delta_P^+} : L_2(M, E^+) \to L_2(M, E^+)$$

called the *heat operator* of Δ_P^+. For all $t \in (0, \infty)$, this operator is infinitely smoothing, gets defined on $H_d(M, E^+)$ for all d, and when viewed as an operator $H_d \to L_2$, is compact, for all d. The analogous statement holds for $e^{-t\Delta_P^-}$.

Proof: Write an element $f \in L_2(M, E^+)$ as:

$$f = \sum_{n=0}^{\infty} a_n e_n$$

where $\sum_n \mid a_n \mid^2 = \|f\|^2 < \infty$. Since we have $\lambda_n \geq Cn^{\delta}$ by the last Proposition 10.1.1, we have $e^{-t\lambda_n} \leq e^{-tCn^{\delta}}$ for all n. Since $t > 0$, it follows that there is a constant $A(t)$ such that $e^{-t\lambda_n} \leq A(t)$ for all n. Thus:

$$\sum_n \mid e^{-t\lambda_n} a_n \mid^2 \leq A(t)^2 \sum_n \mid a_n \mid^2$$

and the heat operator $e^{-t\Delta_P^+}$ is a bounded operator on $L_2(M, E^+)$, with operator norm $\leq A(t)$. It is self-adjoint since Δ_P^+ is formally self adjoint, and smooth functions are dense in $L_2(M, E^+)$.

To see that it is infinitely smoothing, note that by the Corollary 6.2.3 (Garding inequality) applied to the elliptic operator $Q := (\Delta_P^+)^k$ (which is of order $2kd$), we have that the Sobolev $2kd$-norm is given by:

$$\|e_n\|_{2kd}^2 = \|Qe_n\|_0^2 + \|e_n\|_0^2 = (\lambda_n^k + 1)$$

Again, since $t > 0$ and $\lambda_n \geq Cn^{\delta}$, it easily follows that

$$\sum_{n=0}^{\infty} e^{-2t\lambda_n} (\lambda_n^k + 1) \leq B_k(t) < \infty$$

Thus, for the partial sum $g_N := \sum_{n=0}^{N} e^{-t\lambda_n} a_n e_n$, we will have

$$\begin{aligned} \|g_N\|_{2kd} &\leq \sum_{n=0}^{N} e^{-t\lambda_n} \mid a_n \mid \|e_n\|_{2kd} = \sum_{n=0}^{N} e^{-t\lambda_n} (\lambda_n^k + 1)^{1/2} \mid a_n \mid \\ &\leq \left(\sum_{n=0}^{N} e^{-2t\lambda_n} (\lambda_n^k + 1) \right)^{1/2} \left(\sum_{n=0}^{N} \mid a_n \mid^2 \right)^{1/2} \leq B_k(t)^{1/2} \|f\|_0 \end{aligned}$$

From which it follows that $e^{-t\Delta_P^+} f \in H_{2kd}(M, E^+)$ for all k, which implies that it is smooth by the Sobolev Lemma (iv) of Proposition 4.2.2.

Since $e^{-t\Delta_P^+}$ is infinitely smoothing, it is in $\Psi^{d-1}(M)$ for all d, and a bounded operator $H_d(M, E^+)$ into $H_1(M, E^+)$ for all d. By Rellich's Lemma (vi) of 4.2.2, since the inclusion $H_1 \subset H_0 = L_2(M, E^+)$ is compact,

$$e^{-t\Delta_P^+} : H_d(M, E^+) \to L_2(M, E^+)$$

is a compact operator for all d. □

Proposition 10.1.3 (Some facts about the Heat Operator)**.** In the setting of the previous proposition, we have the following:

(i): The for $f \in L_2(M,E^+)$, we let $f_{\mathcal{H}^+}$ denote the orthogonal projection $\pi(f)$ to the finite dimensional Δ^+-harmonic space $\ker \Delta^+ = \oplus_{i\geq 0} \ker \Delta_P^{2i}$ by the Kodaira-Hodge decomposition of (ii) in 9.5.2. Then

$$\lim_{t\to 0} e^{-t\Delta^+} f = f; \quad \lim_{t\to\infty} e^{-t\Delta^+} f = f_{\mathcal{H}^+} \quad \text{for all } \ f \in L_2(M,E^+)$$

where the convergence, of course, is in the L_2-norm.

(ii): If $f \in C^\infty(M,E^+)$, then $e^{-t\Delta^+}f$ converges to $f_{\mathcal{H}^+}$ as $t\to\infty$ and to f as $t \to 0$ in the norm $\|-\|_{k,\infty}$ for all k.

(iii): For $t > 0$, there is a *smooth integral kernel*:

$$k_t^+ \in C^\infty(M\times M, \hom_{\mathbb{C}}(\pi_2^* E^+, \pi_1^* E^+))$$

(where π_1, π_2 are the first and second projections of $M\times M$ to M) satisfying:

$$(e^{-t\Delta^+} f)(x) = \int_M k_t^+(x,y) f(y) dV(y) \quad \text{for all } \ f \in C^\infty(M,E^+)$$

(iv): For $t \in (0,\infty)$, the sum $\sum_{i=0}^\infty e^{-t\lambda_i}$, called the *trace of the heat operator* for obvious reasons, and denoted by $\operatorname{tr} e^{-t\Delta^+}$ is given by the integral:

$$\operatorname{tr} e^{-t\Delta^+} = \int_M \operatorname{tr} k_t^+(x,x) dV(x)$$

Analogous facts obtain for $e^{-t\Delta^-}$.

Proof: Let e_n be L_2-orthogonal eigensections for Δ^+ corresponding to the eigenvalues λ_n. Assume that in our non-decreasing arrangement of eigenvalues $\lambda_1 = \lambda_2 = ...\lambda_p = 0$, so that $\{e_n\}_{n=1}^p$ is an orthonormal basis for $\mathcal{H}^+$. Also $\lambda_{p+1} > 0$. Now expand $f \in L_2(M,E^+)$ as $f = f_{\mathcal{H}^+} + \sum_{n\geq p+1} a_n e_n$. Then $e^{-t\Delta^+} f = f_{\mathcal{H}^+} + \sum_{n\geq p+1} e^{-t\lambda_n} a_n e_n$, and:

$$\left\| e^{-t\Delta^+} f - f_{\mathcal{H}^+} \right\|^2 = \sum_{n\geq p+1} e^{-2t\lambda_n} \mid a_n \mid^2 \leq e^{-2t\lambda_{p+1}} \sum_{n\geq p+1} \mid a_n \mid^2 \leq e^{-t\lambda_{p+1}} \|f\|^2$$

which clearly shows, since $\lambda_{p+1} > 0$, that $\lim_{t\to\infty} e^{-t\Delta^+} f = f_{\mathcal{H}^+}$ and the second assertion of (i) follows.

For the first assertion, note that:

$$\left\| e^{-t\Delta^+} f - f \right\|^2 = \sum_{n\geq p+1} (e^{-t\lambda_n} - 1)^2 \mid a_n \mid^2$$

Now, given any $\epsilon > 0$, choose an $N > p+1$ such that $\sum_{n\geq N+1} \mid a_n \mid^2 \leq \epsilon$. Also since $\lambda_n \geq \lambda_{p+1} > 0$ for $n \geq p+1$ by the Proposition 10.1.1, we can choose $\eta > 0$ so that $(e^{-t\lambda_n} - 1)^2 \leq \epsilon$ for $t \in (0,\eta)$ and all the finitely many n satisfying $p+1 \leq n \leq N$. Then we estimate:

$$\begin{aligned} \sum_{n\geq p+1} (e^{-t\lambda_n} - 1)^2 \mid a_n \mid^2 \quad &\leq \sum_{p+1\leq n\leq N} (e^{-t\lambda_n} - 1)^2 \mid a_n \mid^2 + C \sum_{n\geq N+1} \mid a_n \mid^2 \\ &\leq \epsilon \|f\|^2 + C\epsilon \quad \text{for} \quad 0 < t < \eta \end{aligned}$$

which proves that $\lim_{t\to 0} e^{-t\Delta^+} f = f$ in L_2 and the first assertion of (i) follows.

Now we prove (ii). In view the Sobolev Embedding Theorem (iv) of Proposition 4.2.2, and the Corollary 6.2.3 (Garding-Friedrichs inequality applied to the elliptic operator Δ^{+k} of order $2kd$), it is enough to show that that for $f \in C^\infty(M,E^+)$ (contained in $H_{2kd}(M,E^+)$ for all $k \geq 1$):

(a): $(\Delta^+)^k e^{-t\Delta^+} f$ converges to $(\Delta^+)^k f$ in $L_2(M,E^+)$ as $t \to 0$ (resp. converges to $(\Delta^+)^k f_{\mathcal{H}^+}$, which incidentally is zero for $k \geq 1$, as $t \to \infty$) and, (b): $e^{-t\Delta^+}f$ converges to f in $L_2(M,E^+)$ as $t\to 0$ (resp. converges to $f_{\mathcal{H}^+}$ in $L_2(M,E^+)$ as $t\to\infty$).

The statement (b) follows from (i) above. For the statement (a), note that $(\Delta^+)^k e^{-t\Delta^+}f = e^{-t\Delta^+}(\Delta^+)^k f$, and ${\Delta^+}^k(f_{\mathcal{H}^+}) = ({\Delta^+}^k f)_{\mathcal{H}^+}$, and thus (a) follows by applying (i) to the section ${\Delta^+}^k f \in L_2(M,E^+)$. This proves (ii).

Now we give the construction for k_t^+. For the smooth eigensection e_n of E^+ corresponding to the eigenvalue λ_n, denote by e_n^* the section of $E^{+*} = \hom(E^+,\mathbb{C})$ defined by $e_n^*(x)(w) = \langle w, e_n(x)\rangle_x$ for $w\in E_x^+$ (remember Hermitian metrics are linear in the first slot, and conjugate linear in the second!). Then $e_n^*(y)\otimes e_n(x)$ becomes a smooth section of $\hom(\pi_2^*E^+,\pi_1^*E^+)$, its value at a an element $v\in E_y^+ = (\pi_2^*E^+)_{(x,y)}$ being the element $e_n^*(y)(v)e_n(x) = \langle v, e_n(y)\rangle\, e_n(x) \in E_x^+ = (\pi_1^*E^+)_{(x,y)}$.

Define the formal sum:

$$k_t(x,y) = \sum_{i=0}^{\infty} e^{-t\lambda_n}(e_n^*(y)\otimes e_n(x))$$

Note that $L_2(M\times M, \hom(\pi_2^*E^+,\pi_1^*E^+))$ has a canonical L_2-inner product arising out of the natural tensor product Hermitian metric on the bundle $\hom(\pi_2^*E^+,\pi_1^*E^+)$. The corresponding global inner product on $M\times M$ (with respect to the volume element of the product Riemannian metric) has the orthonormal basis $\{e_m^*(y)\otimes e_n(x)\}$. So the series above certainly converges in L_2-norm, by the estimate $\lambda_n \geq Cn^\delta$ of Proposition 10.1.1. To show that this kernel is a smooth section on $M\times M$, we apply the elliptic operators ${\Delta_y^+}^k \times {\Delta_x^+}^j$ for arbitrary j and k, and note that the differentiated series will have coefficients $\lambda_n^{k+j}e^{-t\lambda_n}$, to which again $\lambda_n \geq Cn^\delta$ may be applied, to show that this differentiated series is again L_2 over $M\times M$. Now appeal to the Sobolev Lemma and Garding-Friedrichs as always.

To see it is the required kernel, we compute its effect on each e_m:

$$\int_{y\in M} k_t(x,y)e_m(y)dV(y) = \sum_{n=0}^{\infty} e^{-t\lambda_n}\int_{y\in M} e_n^*(y)\otimes e_n(x)(e_m(y))dV(y)$$

$$= \sum_{n=0}^{\infty} e^{-t\lambda_n} e_n(x)\int_{y\in M}\langle e_n(y),(e_m(y)\rangle\, dV(y) = \sum_{n=0}^{\infty} e^{-t\lambda_n} e_n(x)(e_n,e_m) = e^{-t\lambda_m}e_m(x)$$

since $(e_n,e_m) = \delta_{nm}$ by L_2-orthornormality of e_n's. This shows that the integral operator defined by k_t has the same effect on each e_m as the heat operator $e^{-t\Delta^+}$, and the two operators are therefore the same on $L_2(M,E^+)$. This proves (iii).

To see (iv), we first define what we mean by $\operatorname{tr} k_t^+(x,x)$. k_t is a smooth section of the bundle $\hom_{\mathbb{C}}(\pi_2^*E^+,\pi_1^*E^+)$. The maps π_1 and π_2 agree on the diagonal, and indeed if one identifies the diagonal Δ_M inside $M\times M$ with M via the map $(x,x)\mapsto x$, the bundles $\pi_2^*E^+$ and $\pi_1^*E^+$ both get identified with the bundle E^+. Thus restricting the smooth section k_t^+ to the diagonal gives the smooth section, denoted by $k_t^+(x,x)$, of the bundle $\hom_{\mathbb{C}}(E^+,E^+)$. On this bundle there is the natural trace map:

$$\begin{aligned} \operatorname{tr} : \hom_{\mathbb{C}}(E^+,E^+) &\to \mathbb{C} \\ T_x &\mapsto \sum_i \langle T_x(f_i), f_i\rangle_x \end{aligned}$$

where f_i is any $\langle -,-\rangle_x$ orthonormal basis of E_x^+, (viz. it is the invariant trace of $T_x : E_x^+ \to E_x^+$).

Now we simply calculate, for $x\in M$, and f_i some $\langle -,-\rangle_x$-orthonormal basis of E_x^+:

$$\begin{aligned} \operatorname{tr} k_t^+(x,x) &= \sum_i \sum_{n=0}^{\infty} e^{-t\lambda_n}\langle (e_n^*(x)\otimes e_n(x))(f_i), f_i\rangle_x \\ &= \sum_{n=0}^{\infty} e^{-t\lambda_n}\sum_i \langle\langle f_i, e_n(x)\rangle\, e_n(x), f_i\rangle_x \\ &= \sum_{n=0}^{\infty} e^{-t\lambda_n}\sum_i |\, \langle e_n(x), f_i\rangle_x\, |^2 = \sum_{n=0}^{\infty} e^{-t\lambda_n}\, \|e_n(x)\|_x^2 \end{aligned}$$

which implies that:

$$\int_M \operatorname{tr} k_t^+(x,x)dV(x) = \sum_{n=0}^{\infty} e^{-t\lambda_n} \int_M \|e_n(x)\|_x^2 \, dV(x) = \sum_{n=0}^{\infty} e^{-t\lambda_n}(e_n, e_n) = \sum_{n=0}^{\infty} e^{-t\lambda_n}$$

since e_n's form an orthonormal basis with respect to the global inner product $(-,-)$. This proves (iv), and the proposition follows. □

Remark 10.1.4. The proof of assertion (ii) of the foregoing proposition reflects a classical fact about the heat operator $e^{-t\Delta^+}$, which is that it starts with an arbitrarily irregular f (a distribution, i.e. in some $H_s(M, E^+)$) at $t = 0$, and makes it smooth at any positive time $t > 0$. Indeed as $t \to \infty$, it converts the irregular f into its smooth harmonic part $f_{\mathcal{H}^+}$. Thus it time-evolves the irregular initial data f into a smooth section for any $t > 0$, and into its smooth harmonic part as $t \to \infty$.

Remark 10.1.5. We explicitly constructed the kernel for the heat operator $e^{-t\Delta^+}$. However, it is a fact that an operator (defined on distributional sections $\mathcal{D}'(M, E)$) on a compact Riemannian manifold is infinitely smoothing iff it is given by an integral operator with a smooth integral kernel. For convenience's sake, let us consider an operator K which maps $H_s(M, \mathbb{C}) \to L_2(M, \mathbb{C})$ as a bounded operator, for all s, and whose image is contained in $C^\infty(M, \mathbb{C})$. We know that on $\mathbb{R}^n$, the Dirac distribution δ_x is a compactly supported distribution lying in $H_{-k}(\mathbb{R}^n)$ for all $k > n/2$ (see the Corollary 3.2.2). If M is of dimension n, since the support of δ_x is x, it becomes an element of $H_{-k}(M)$ for all $n > k/2$ (by using a partition of unity definition of $H_s(M)$). Thus $K(\delta_x)$ is a smooth function. Define:

$$k(y, x) := K(\delta_x)(y)$$

One now has to verify that this is the required integral kernel. For the converse, one has to verify that integral operators with smooth integral kernels on a *compact* manifold are infinitely smoothing, by differentiating under the integral sign using compactness of M, or using the Sobolev Embedding Theorem coupled with clever uses of integral inequalities.

As we have remarked earlier, integral operators with smooth integral kernels *do not* give rise to infinitely smoothing operators on non-compact manifolds. For example, the Fourier transform on $\mathbb{R}$ is an integral operator with smooth kernel $e^{-i\xi.x}$, but converts a smooth function like $(1+x^2)^{-1}$ into a non-smooth function.

Proposition 10.1.6 (Facts about the heat kernel)**.**

(i): The section $k_t^+(x, y)$ defined in (iii) of the Proposition 10.1.3 satisfies the pointwise adjointness formula:

$$\langle k_t^+(x,y)v, w\rangle_x = \langle v, k_t^+(y,x)w\rangle_y \quad \text{for } v \in (\pi_2^* E^+)_{(x,y)} = E_y^+, \ w \in (\pi_1^* E^+)_{(x,y)} = E_x^+$$

(ii): $k_t^+(x, y)$ satisfies the *heat equations*

$$\left(\frac{\partial}{\partial t} + \Delta_x^+\right) k_t^+(x,y) = 0 = \left(\frac{\partial}{\partial t} + (\Delta^+)_y^\vee\right) k_t^+(x,y) \ \text{ for } \ t \in (0,\infty), \ (x,y) \in M \times M$$

where

$$(\Delta^+)^\vee : C^\infty(M, E^{+*}) \to C^\infty(M, E^{+*})$$

is the pointwise dual of Δ^+ with respect to the Hermitian metric $\langle -, - \rangle$ on E^+.

(iii): If $f \in L_2(M, E^+)$ is a square integrable section, we have seen in the Proposition 10.1.2 that $e^{-t\Delta^+} f$ is smooth in x. It is also smooth in t for $t \in (0, \infty)$, and if we define $F(x, t) := e^{-t\Delta^+} f$, the F satisfies:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta^+\right) F(x,t) &= 0 \quad \text{for } \ t \in (0,\infty), \ \ x \in M \\ F(x,0) &:= \lim_{t \to 0} F(x,t) = f \end{aligned}$$

There are completely analogous statements for k_t^- and Δ^-.

Proof: To see (i), we note that for $v \in E_y^+$, $w \in E_x^+$:

$$\langle (e_n^*(y) \otimes e_n(x))(v), w \rangle_x = \left\langle \langle v, e_n(y) \rangle_y e_n(x), w \right\rangle_x = \langle v, e_n(y) \rangle_y \langle e_n(x), w \rangle_x$$

and interchanging the roles of x and y, v and w, we have:

$$\langle (e_n^*(x) \otimes e_n(y)(w), v \rangle_y = \langle w, e_n(x) \rangle_x \langle e_n(y), v \rangle_y$$

Since the right hand sides of the two equations above are complex, conjugates of each other, we have:

$$\langle (e_n^*(y) \otimes e_n(x))(v), w \rangle_x = \langle v, e_n^*(x) \otimes e_n(y)(w) \rangle_y$$

from which (i) follows by multiplying by $e^{-t\lambda_n}$ and summing over n.

To see (ii), note that since the series for $k_t(x, y)$ is absolutely and uniformly convergent in both variables, as are the differentiated series with respect to $\partial_t = \frac{\partial}{\partial t}$ and Δ_x^+ and Δ_y^+ (from the eigenvalue estimate $\lambda_n \geq Cn^\delta$), we can apply these operators term by term. Hence:

$$\begin{aligned} \partial_t(\sum_n e^{-t\lambda_n} e_n^*(y) \otimes e_n(x) &= \sum_n (-\lambda_n e^{-t\lambda_n} e_n^*(y) \otimes e_n(x) \\ -\sum_n e^{-t\lambda_n} e_n^*(y) \otimes \Delta_x^+ e_n(x) &= -\Delta_x^+(k_t(x, y)) \end{aligned}$$

Also, from the equation $\Delta_y^+ e_n(y) = \lambda_n e_n(y)$, one finds that for the pointwise adjoint operator $\Delta_y^{+\vee}$ defined by the adjoint formula:

$$\left\langle \Delta^{+\vee}\psi, f \right\rangle = \left\langle \psi, \Delta^+ f \right\rangle, \quad f \in C^\infty(M, E^+), \quad \psi \in C^\infty(M, E^{+*})$$

one easily finds that $(\Delta^+)^\vee e_n^* = \lambda_n e_n^*$, and the second formula of (ii) follows as well.

To see (iii) note that if $f \in L_2(M, E^+)$, we may write $f = \sum_n a_n e_n$, with $\sum_n \mid a_n \mid^2 = \|f\|^2 < \infty$. Furthermore, $F(x, t) = \sum_n e^{-t\lambda_n} a_n e_n$ is a series which lies in $H_s(M, E^+)$ for all s, and converges in $\|-\|_s$ for each s (meaning the Sobolev s-norm of the tails $\sum_{n \geq N} e^{-t\lambda_n} a_n e_n$ converges to 0 for all s, by the facts that $\lambda_n \geq Cn^\delta$, and $\|e_n\|_{2kd}^2 = \lambda_n^k + 1$). Hence the series on the right converges in $\|-\|_{\infty,k}$ for all k, by the Sobolev Embedding Theorem (iv) of 4.2.2. Hence if one applies ∂_t, or Δ^+ term-by-term to this series, the resulting series converge to $\partial_t(F(x, t))$ and $\Delta^+ F(x, t)$ respectively. However, upon term by term differentiation we have:

$$\partial_t(e^{-t\Delta^+} f) = -\sum_n \lambda_n e^{-t\lambda_n} a_n e_n = -\sum_n e^{-t\lambda_n} a_n \Delta^+ e_n = \Delta^+(e^{-t\Delta^+} f)$$

since Δ^+ and $e^{-t\Delta^+}$ commute. This proves that $\partial_t F(x, t) + \Delta^+ F(x, t) = 0$. The fact that $\lim_{t\to 0} F(x, t) = f$ follows from (i) of Proposition 10.1.3. The proposition follows.

An analogue of (iii) can be proved for $f \in H_s(M, E^+)$ and any s, (i.e. for all distributional sections $f \in H_{-\infty}(M, E^+) = \mathcal{D}'(M, E^+)$, but we omit the proof. It is completely analogous, because f can still be expanded in a Fourier series $\sum_n a_n e_n$. One needs to note that e_n need no longer be orthonormal in $\|-\|_{2kd}$, but we still have $(e_n, e_m)_{2kd} = (\lambda_n^k + 1)\delta_{nm}$ for all $k \neq 0$ (by proving the analogue of Corollary 6.2.3 for $k \leq 0$, which in turn stems from the duality of H_{2kd} and H_{-2kd} from (iii) of Proposition 4.2.2). □

10.2. **An integral formula for the index of D^+.** The following proposition is the key to the entire heat-equation approach for the index theorem.

Theorem 10.2.1 (McKean-Singer)**.** Let M be a compact Riemannian manifold, and $\mathcal{P}$ an elliptic complex on M. Let:

$$D^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$$

be the corresponding Dirac operators, as in Definition 9.6.1, and let

$$k_t^\pm(x, y) \in C^\infty(M \times M, \hom_{\mathbb{C}}(\pi_2^* E^\pm, \pi_1^* E^\pm))$$

denote the heat kernels of the heat evolution operators $e^{-t\Delta^\pm}$ respectively, as in (iii) of the Proposition 10.1.3. Then:

$$\operatorname{ind} D^+ = \int_M (\operatorname{tr} k_t^+(x, x) - \operatorname{tr} k_t^-(x, x)) dV(x) = -\operatorname{ind} D^-$$

In particular, the quantity on the right is an integer independent of t.

Proof: Let $\lambda_n \geq 0$ and $\mu_m \geq 0$ be the eigenvalues of the two Laplacians $\Delta^+ = D^- D^+$ and $\Delta^- = D^+ D^-$ respectively. Let the eigensections of Δ^+ be denoted e_n, which are orthonormal in $L_2(M, E^+)$ with respect to its L_2- inner product, which we denote by $(-,-)_+$ (with $(-,-)_-$ denoting the L_2-inner product on $L_2(M, E^-)$).

We now note that if e_n is an eigensection of Δ^+ with eigenvalue λ_n, we have:

$$\Delta^- D^+ e_n = (D^+ D^-) D^+ e_n = D^+(D^- D^+) e_n = D^+(\Delta^+ e_n) = \lambda_n D^+ e_n$$

so that $D^+ e_n$, if non-zero, is an eigensection of Δ^- corresponding to the same eigenvalue λ_n.

Furthermore, for all n, m, by the fact that λ_n and λ_m are both real, and the adjointness of D^+ and D^-, we have:

$$\lambda_n (e_n, e_m)_+ = (D^- D^+ e_n, e_m)_+ = (D^+ e_n, D^+ e_m)_- = (e_n, D^- D^+ e_m)_+ = \lambda_m (e_n, e_m)_+$$

which implies in view of the foregoing that:

(i): If $\lambda_n \neq 0$ (i.e. $\lambda_n > 0$), the section $D^+ e_n$ is a non-zero eigensection of Δ^- corresponding to the same eigenvalue λ_n as e_n.

(ii): For $n \neq m$ we have $D^+ e_m$ orthogonal to $D^+ e_m$.

Similar facts obtain for the eigensections $f_n \in C^\infty(M, E^-)$ of the other Laplacian Δ^-. From this it follows that for $\lambda_n \neq 0$ and $\mu_m \neq 0$, D^+ maps the finite-dimensional λ_n-eigenspace of Δ^+ isomorphically into a subspace of the λ_n-eigenspace of Δ^-, and D^- similarly maps the finite-dimensional μ_m-eigenspace of Δ^- isomorphically into a subspace of the μ_m-eigenspace of Δ^+. It follows that the *non-zero eigenvalues* $\lambda_n > 0$ of Δ^+ are in bijective correspondence with the non-zero eigenvalues $\mu_m > 0$ of Δ^-, and also occur with exactly the same multiplicity. Thus:

$$\operatorname{tr} e^{-t\Delta^+} - \operatorname{tr} e^{-t\Delta^-} = \sum_n e^{-t\lambda_n} - \sum_m e^{-t\mu_m} = \sum_{\lambda_n = 0} 1 - \sum_{\mu_m = 0} 1 = \dim \ker \Delta^+ - \dim \ker \Delta^-$$

But the left hand side of this equation is precisely:

$$\int_M (\operatorname{tr} k_t^+(x,x) - \operatorname{tr} k_t^-(x,x)) dV(x)$$

by (iv) of Proposition 10.1.3, and the right hand side of the equation is $\operatorname{ind} D^+ = -\operatorname{ind} D^-$ by the proof of (iii) in Proposition 9.6.2.

The last assertion is clear in view of the fact that $\operatorname{ind} D^+$ is independent of t. The theorem follows. $\square$.

Now, in the sequel, the main aim is to identify the integrand

$$\operatorname{str} k_t(x,x) := \operatorname{tr} k_t^+(x,x) - \operatorname{tr} k_t^-(x,x)$$

called the *supertrace* of the heat evolution operator. This is impossible in full generality. However, one can do what is called an *asymptotic expansion* in powers of $t^{1/d}$ (where d is the order of the differential operator D^+) for small times t, and using the fact that the left hand side is independent of t, compute just the coefficient of t^0 (the constant term) in this symptotic expansion. That such an asymptotic expansion exists in general is proved in Gilkey. However, since we shall be interested only in four specific elliptic complexes (the de-Rham, Twisted-Dolbeault, Signature and Spin complexes), for each of which the corresponding Laplacian $\Delta^+ = D^- D^+$ is a *generalised Laplacian*, i.e. a second order operator whose leading symbol is the same as that of the classical Laplace-Beltrami operator, we will concentrate only on such elliptic complexes.

11. Fundamental solutions

To motivate whatever follows, we need to construct the heat kernel for the Laplacian $\Delta = -\sum_j \partial_j^2$ on $\mathbb{R}^n$. Our assertion of the existence of a heat kernel in Proposition 10.1.3 doesn't quite apply, since $\mathbb{R}^n$ is non-compact, and does not have a discrete spectrum. But fortunately, one can explicitly write down the heat kernel (or Gauss kernel as it is sometimes known) in the case of $\mathbb{R}^n$.

11.1. **The Euclidean heat kernel.**

Proposition 11.1.1 (Euclidean heat kernel)**.** For $x, y \in \mathbb{R}^n$, define the function:

$$k_t(x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

(i): $k_t(x,y)$ is symmetric in x and y, and is a fundamental solution to the heat equation, viz.,

$$(\partial_t + \Delta_x)k_t(x,y) = 0 = (\partial_t + \Delta_y)k_t(x,y)$$

(ii): For $f \in L_2(\mathbb{R}^n)$ the function:

$$F(x,t) = e^{-t\Delta} f := \int_{\mathbb{R}^n} k_t(x,y)f(y)dy$$

is a smooth function of both t and x, and satisfies:

$$(\partial_t + \Delta)F(x,t) = 0$$

(iii): Let $y \in \mathbb{R}^n$, and let δ_y denote the Dirac distribution at y. Then there is a smooth function $w(-,t) \in \mathcal{S}(\mathbb{R}^n)$ such that:

$$(a) \quad (\partial_t + \Delta)w(x,t) = 0 \quad \text{for all} \quad x \in \mathbb{R}^n, \quad t > 0$$

and

$$(b) \quad \lim_{t\to 0}(f, w(-,t)) = f(y) \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^n)$$

This $w(-,t)$ is called a *fundamental solution of the heat equation on* $\mathbb{R}^n$ *with pole at* y, and is uniquely determined by the conditions (a) and (b).

(**Caution**: in this proposition, dy denotes Euclidean volume element $dy = dy_1 ... dy_n$, and is related to the earlier volume element $dV(y)$ of §1 by $dV(y) = (2\pi)^{-n/2}dy$.)

Proof: Direct differentiation yields that:

$$\begin{aligned}
\partial_t(t^{-n/2}e^{-|x-y|^2/4t}) &= \left(t^{-n/2}\frac{|\ x-y\ |^2}{4t^2} + (-n/2)t^{-n/2-1}\right)e^{-|x-y|^2/4t} \\
&= \left(\frac{-n}{2} + \frac{|\ x-y\ |^2}{4t}\right)t^{-n/2-1}e^{-|x-y|^2/4t} \\
\Delta_x(t^{-n/2}e^{-|x-y|^2/4t}) &= -t^{-n/2}\sum_i \partial_{x,i}\left(-\frac{(x_i-y_i)}{2t}e^{-|x-y|^2/4t}\right) = t^{-n/2}\sum_i\left(\frac{1}{2t} - \frac{(x_i-y_i)^2}{4t^2}\right)e^{-|x-y|^2/4t} \\
&= t^{-n/2-1}\left(\frac{n}{2} - \frac{|\ x-y\ |^2}{4t}\right)e^{-|x-y|^2/4t}
\end{aligned}$$

from which the statement (i) follows. For the second, note that if we denote the function:

$$\rho_t(x) = (2t)^{-n/2}(e^{-|x|^2/4t})$$

then $F(x,t) = \rho_t * f$, where the convolution is the same as the one introduced in §1 (i.e. in the space variables, with respect to the volume $dV(y) = (2\pi)^{-n/2}dy_1...dy_n$). Since ρ_t is in the Schwartz class (since $t > 0$), and $f \in L_2(\mathbb{R}^n) = H_0(\mathbb{R}^n)$ implies f is a tempered distribution, it follows that the convolution $F(x,t)$ is smooth in the space variable x by the Lemma 1.4.7. and also that $\Delta_x F(x,t) = (\Delta\rho_t) * f$.

In the time variable, one uses that $\partial_t\rho_t(x) = (-n/2+ \mid x \mid^2 /4t)t^{-1}\rho_t(x)$ and the Dominated convergence theorem to take ∂_t under the integral sign and get

$$\partial_t(F(x,t)) = (\partial_t\rho_t) * f$$

Hence we have:

$$(\partial_t + \Delta_x)(F(x,t)) = (\partial_t + \Delta)(\rho_t) * f = \int_{\mathbb{R}^n}(\partial_t + \Delta_y)k_t(x,y)f(y)dy = 0$$

by applying (i). This proves (ii).

To see (iii), note that the Dirac distribution δ_y is a tempered distribution (as we remarked in Example 1.3.5, it is a compactly supported distribution), and so taking the convolution with the Schwartz class function ρ_t for $t > 0$

$$w(x,t) := \rho_t * \delta_y$$

gives a smooth function (see Proposition 1.4.7). To see that it is in the Schwartz class, note that its Fourier transform is $\widehat{\rho_t}(\xi)e^{-i\xi.y}$, which is in $\mathcal{S}(\mathbb{R}^n)$ since $\widehat{\rho_t}$ is in $\mathcal{S}(\mathbb{R}^n)$. If one writes down the formula for the convolution of distributions, we find:

$$w(x,t) = \delta_y(\rho_t^x) = \rho_t^x(y) = (2t)^{-n/2}e^{-|x-y|^2/4t} = (2\pi)^{n/2}k_t(x,y)$$

which is clearly a smooth function of $x \in \mathbb{R}^n$ for $t > 0$. That it satisfies the heat equation is an immediate consequence of (i) that $(\partial_t + \Delta_x)k_t(x,y) = 0$ for $t > 0$. This shows (a) of (iii).

To see (b), note that for $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$(f, w(-,t)) = \int_{\mathbb{R}^n} f(x)w(x,t)dV(x) = \int_{\mathbb{R}^n} \rho_t(x-y)f(x)dV(x) = \int_{\mathbb{R}^n} \rho_t(y-x)f(x)dV(x) = (\rho_t * f)(y)$$

and the proposition follows by noting that for $\phi(x) = \rho_{\frac{1}{2}}(x)$:

$$\int_{\mathbb{R}^n} \phi(x)dV(x) = \int_{\mathbb{R}^n} \rho_{\frac{1}{2}}(x)dV(x) = \int_{\mathbb{R}^n} e^{-|x|^2/2}dV(x) = 1$$

and that by setting $\epsilon^2 = 2t$, we have

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon) = (2t)^{-n/2}\rho_{\frac{1}{2}}(x/\epsilon) = (2t)^{-n/2}e^{-|x|^2/2\epsilon^2} = (2t)^{-n/2})e^{-|x|^2/4t} = \rho_t(x)$$

But by the Lemma 1.2.3, we have ϕ_ϵ are approximate identities, and $\phi_\epsilon * f = \rho_t * f \to f$ uniformly on $\mathbb{R}^n$, for all $f \in \mathcal{S}(\mathbb{R}^n)$. Thus $(f, w(-,t)) \to (\rho_t * f)(y)$ has limit $f(y)$ as $t \to 0$. This proves (b). To see that (a) and (b) uniquely determine $w(-,t)$, let us assume $u(-,t) \in \mathcal{S}(\mathbb{R}^n)$ also satisfies (a) and (b). Denoting $w_t := w(-,t)$, $u_t := u(-,t)$ for notational convenience, note that the time derivative of the L_2-norm of $w_t - u_t$ is given by (because both w_t and u_t are rapidly decreasing, Stokes Formula is applicable):

$$\partial_t(w_t - u_t, w_t - u_t) = -2(\Delta(w_t - u_t), w_t - u_t) = -2(d(w_t - u_t), d(w_t - u_t)) \leq 0$$

and so $\|w_t - u_t\|^2$ is a non-increasing function of t. Also, by (b), for every f we have:

$$\lim_{t\to 0}(f, w_t - u_t) = f(y) - f(y) = 0$$

which means $\lim_{t\to 0}\|w_t - u_t\| \to 0$, and by the fact that $\|w_t - u_t\|$ is non-increasing in t, it follows that $w_t \equiv u_t$ for all $t > 0$. This proves the proposition.

□

11.2. Fundamental solutions of the Heat equation for the Dirac Laplacians.

Now let M be a compact Riemannian manifold, and let $D^\pm$ be the Dirac operators introduced in Definition 9.6.1.

Definition 11.2.1 (Fundamental solutions)**.** Let $x \in M$ and let $v \in E_x^+$. We say that a smooth section $w(-,t) \in C^\infty(M, E^+)$ is a *fundamental solution with pole* (x, v) if:

(i): The section $w(-,t)$ satisfies the heat equation for Δ^+, viz.

$$(\partial_t + \Delta^+)w(x,t) = 0 \quad \text{for all } x \in M, \quad t > 0$$

(ii): $\lim_{t\to 0}(s, w(-,t)) = \langle s(x), v\rangle_x$ for all $s \in C^\infty(M, E^+)$

The second condition means that w_t approaches the "Dirac distributional-section" (at the point x) which is given by $\delta_x v$ as $t \to 0$.

One can obviously make a similar definition for E^- and Δ^-.

Proposition 11.2.2 (Existence and uniqueness of fundamental solutions)**.** Let M, $E^\pm$ be as above. Then given $v \in E_x^+$, there exists a fundamental solution with pole (x, v) to the heat equation for Δ^+, and this solution is unique. Likewise for E^- and Δ^-.

Proof: We merely apply (iii) of the Proposition 10.1.3 to $f = \delta_x v$, this "Dirac distributional section" $\delta_x v$. We also note that this $f \in H_{-k}(M, E^+)$ for all $k > n/2$, and by the Remark 10.1.5, we will have that

$$w(z,t) = \int_M k_t^+(z,y)\delta_x(y) v dV(y) = k_t^+(z,x)v$$

is a smooth section of E^+ for $t > 0$. Indeed, the right hand side is clearly smooth in z since k_t^+ is smooth in z (x is held fixed here) for $t > 0$. Furthermore, for $t > 0$, we have

$$(\partial_t + \Delta^+)(w(z,t)) = (\partial_t + \Delta_z^+)(k_t^+(z,x)v) = 0$$

by using (ii) of Proposition 10.1.6.

For the convergence as $t \to 0$, we have:

$$\begin{aligned} (s, w(-,t)) &= \int_M \langle s(z), w(z,t)\rangle_z \, dV(z) = \int_M \langle s(z), k_t^+(z,x)v\rangle_z \, dV(z) \\ &= \int_M \langle k_t^+(x,z)s(z), v\rangle_x \, dV(z) = \left\langle \int_M k_t^+(x,z)s(z)dV(z), v \right\rangle_x \\ &= \left\langle (e^{-t\Delta^+}s)(x), v \right\rangle_x \end{aligned}$$

where we have used the adjointness-symmetry property (i) of Proposition 10.1.6 to arrive at the second line. Now, by (ii) of Proposition 10.1.3, we have by the smoothness of s that $\lim_{t\to\infty} e^{-t\Delta^+}s \to s$ in the $\|-\|_{\infty,0}$ (i.e the convergence is uniform over M), which means that the limit at x satisfies:

$$\lim_{t\to 0}(e^{-t\Delta^+}s)(x) = s(x)$$

and hence $\lim_{t\to 0}(s, w(-,t)) = \langle s(x), v\rangle_x$, and our assertion follows. Likewise for E^- and Δ^-.

To see uniqueness, just verbatim repeat the argument for uniqueness given in (iii) of the Proposition 11.1.1, only noting that for $w_t - u_t$, we have:

$$-(\Delta^+(w_t - u_t), w_t - u_t) = -(D^+(w_t - u_t), D^+(w_t - u_t)) \leq 0$$

by the formal-adjointness of D^+ and D^- proved in (i) of Proposition 9.6.2. $\square$

Exercise 11.2.3. For $M = S^1$, and $E^+ = \Lambda^0 T^*(M)$, $E^- = \Lambda^1 T^*(M)$, $D^+ = d$, $D^- = \delta$ (i.e. the two term deRham elliptic complex for S^1, whose associated Dirac complex is itself), explicitly write down the heat kernels k_t^+ and k_t^- and carry out the verifications of all the preceding propositions in this subsection and the previous one by hand.

12. Asymptotic expansions of the heat kernel

This approach is due to Minakshisundaram and Pleijel. First, assuming that one has an asymptotic expansion, one computes the coefficients in this expansion by substituting in the heat equation and equating coefficients term-by-term. Then one appeals to elliptic estimates to prove that the formal procedure above makes sense.

12.1. Asymptotic expansions.

Definition 12.1.1. Let f be any function on $(0, \infty)$. A formal series $\sum_{k=0}^\infty a_k t^{n_k}$ (where $n_k \in \mathbb{Z}$) is said to be an *asymptotic expansion for f near* 0 if:

(i): $n_k < n_{k+1}$ for all k (so that $n_k \to \infty$ as $k \to \infty$), and,

(ii): For each $l \geq 0$, there exists a $C_l \geq 0$ such that

$$\left| f(t) - \sum_{k=0}^{l} a_k t^{n_k} \right| \leq C_l t^{n_{l+1}}$$

This will be denoted by $f(t) \sim \sum_{k=0}^\infty a_k t^{n_k}$ (Compare with asymptotic expansions of symbols introduced in Definition 5.3.2).

For example, for the function $k_t(x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$ introduced above, regarded as a function of t, we would have that:

$$(4\pi)^{-n/2} \sum_{k=0}^{\infty} \left(\frac{| x-y |^{2k}}{4^k \, k!} \right) t^{-n/2-k}$$

is an asymptotic expansion near 0. Note that the expansion starts with $t^{-n/2}$.

For a motivation, knowing the heat kernel on $\mathbb{R}$ for the Laplacian $\Delta = -\partial_x^2$, let us try to find an asymptotic expansion for the heat kernel of the operator $L = \Delta + b(x)\partial_x + c(x)$ where b and c are smooth functions. It is in fact enough to find the fundamental solution: $u(x,t)$ satisfying $(\partial_t + L)u(x,t) = 0$, and $\lim_{t\to 0} u(x,t) = \delta_x$. Then one gets the heat kernel by $k_t(x,y) = u(x-y,t)$ (verify!). To this end we have:

Proposition 12.1.2. Let $L = \Delta + b(x)\partial_x + c(x)$ as above, where b and c are smooth functions on $\mathbb{R}$. Then there is an *asymptotic fundamental solution* to the corresponding heat equation $(\partial_t + L)u(x,t) = 0$. That is, there is a formal series:

$$(4\pi t)^{-1/2} e^{-x^2/4t} (u_0(x) + tu_1(x) + ... + t^k u_k(x) + ...)$$

where $u_j(x)$ are smooth functions of x, with $u_0(0) = 1$, such that for the partial sum

$$S_k(x,t) := (4\pi t)^{-1/2} e^{-x^2/4t} \left(\sum_{j=0}^{k} t^j u_j(x) \right)$$

we have:

$$(\partial_t + L)S_k(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t} t^k r_k(x)$$

where $r_k(x)$ is a smooth function of x. Furthermore, $u_j(0)$ are algebraic expressions (i.e. polynomials) in the jets (derivatives of all orders) of b and c at 0.

Proof: The idea is to determine the $u_j(x)$ by a recursive formula. So in the PDE $(\partial_t + L)u(x,t) = 0$, let us substitute the series

$$(4\pi t)^{-1/2} e^{-x^2/4t} \left[u_0(x) + tu_1(x) + t^2 u_2(x) + ... \right]$$

for $u(x,t)$. The coefficient of t^k in the expression within square-brackets is u_k.

Note that the formal series on differentiating with respect to x is

$$\partial_x u(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t} \left[-\frac{x}{2t}(u_0 + tu_1 + ...) + (u_0' + tu_1' + ...) \right]$$

where u_i' denotes $\partial_x u_i$. Note that the coefficient of t^k in the expression within square-brackets is:

$$u_k' - \frac{x}{2} u_{k+1}$$

Differentiating again with respect to x, we have:

$$\partial_x^2 u(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t} \left[\left(\frac{x^2}{4t^2} - \frac{1}{2t} \right)(u_0 + tu_1 + ...) - \frac{x}{t}(u_0' + tu_1' + ...) + (u_0'' + tu_1'' + ...) \right]$$

The coefficient of t^k in the expression within square-brackets is:

$$\frac{x^2}{4} u_{k+2} - \frac{1}{2} u_{k+1} - x u_{k+1}' + u_k''$$

Taking the t-derivative, we have:

$$\partial_t u(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t} \left[(u_1 + 2tu_2 + 3t^2 u_3 + ...) + \left(\frac{x^2}{4t^2} - \frac{1}{2t} \right)(u_0 + tu_1 + ...) \right]$$

The coefficient of t^k in the expression within square brackets is:

$$(k+1)u_{k+1} + \frac{x^2}{4} u_{k+2} - \frac{1}{2} u_{k+1} = (k + \frac{1}{2})u_{k+1} + \frac{x^2}{4} u_{k+2}$$

Now substitute this into the heat equation for L to get:

$$(\partial_t + L)u(x,t) = (\partial_t - \partial_x^2 + b(x)\partial_x + c(x))u(x,t) = (4\pi t)^{-1/2}e^{-x^2/4t}\left[\sum_{k=-2}^{\infty}\alpha_k t^k\right]$$

where:

$$\begin{aligned}\alpha_k &= (k+\frac{1}{2})u_{k+1} + \frac{x^2}{4}u_{k+2} - \frac{x^2}{4}u_{k+2} + \frac{1}{2}u_{k+1} + xu'_{k+1} - u''_k + b(x)u'_k - b(x)\frac{x}{2}u_{k+1} + c(x)u_k \\ &= xu'_{k+1} + \left(k+1-\frac{xb(x)}{2}\right)u_{k+1} + Lu_k\end{aligned}$$

Setting $\alpha_k = 0$ gives a recursive differential equation for u_{k+1} in terms of u_k. That is, the equation:

$$xu'_{k+1} + \left(k+1-\frac{xb(x)}{2}\right)u_{k+1} + Lu_k = 0 \tag{25}$$

Since $u_{-1} = 0$ by definition, we have on substituting $k = -1$ in the equation (25) above the following differential equation for u_0:

$$u'_0 - \frac{b(x)}{2}u_0 = 0$$

which implies that $u_0 = Ae^{-\frac{1}{2}\int_0^x b(y)dy}$ for some constant A, and setting the requirement that $u_0(0) = 1$ implies that

$$u_0 = e^{-\frac{1}{2}\int_0^x b(y)dy}$$

More generally, consider the integrating factor:

$$R_k(x) = x^{k+1}e^{-\frac{1}{2}\int_0^x b(y)dy}$$

we get $\log R_k(x) = (k+1)\log x - \frac{1}{2}\int_0^x b(y)dy$ so that:

$$\frac{1}{R_k(x)}\frac{d}{dx}(R_k(x)u_{k+1}) = \left(\frac{k+1}{x} - \frac{b(x)}{2}\right)u_{k+1} + u'_{k+1} = \frac{1}{x}\left[\left((k+1) - \frac{xb(x)}{2}\right)u_{k+1} + xu'_{k+1}\right] = -\frac{1}{x}Lu_k$$

by (25), so that

$$u_{k+1}(x) = \frac{-1}{R_k(x)}\left(\int_0^x \frac{R_k(y)}{y}Lu_k(y)dy\right)$$

gives the explicit recursive formula for u_{k+1} in terms of u_k.

Now if we take the partial sum:

$$S_k(x) = (4\pi t)^{-\frac{1}{2}}e^{-x^2/4t}\left(u_0 + tu_1 + t^2u_2 + ... + t^k u_k\right)$$

with u_k defined as above, then write:

$$(\partial_t + L)S_k = (4\pi t)^{-\frac{1}{2}}e^{-x^2/4t}\left[\beta_{-2}t^{-2} + ...\beta_k t^k\right]$$

since α_j contains no contribution from the term $t^{j+2}u_{j+2}$, we have that the coefficient $\beta_j = \alpha_j$ for all $j \le k-1$. Also $\beta_k = Lu_k$, since the rest of the expression for α_k involves u_{k+1}.

So we finally have:

$$(\partial_t + L)S_k = (4\pi t)^{-1/2}t^k e^{-x^2/4t}(Lu_k)$$

which implies the differential equation asserted for S_k.

We need to show that the u_k's defined above are smooth. We do this by induction. The function $u_0 = \exp(-\frac{1}{2}\int b(y)dy)$ is clearly smooth by definition. Also the integrating factor R_k is given by:

$$R_k(x) = x^{k+1}u_0(x)$$

from the above proof. Hence if we inductively assume that u_k is smooth, we will have:

$$-\frac{R_k(y)}{y}Lu_k(y) = y^k\gamma_k(y)$$

where $\gamma_k(y) = -u_0(y)Lu_k(y)$ is smooth in y. Hence the integral:

$$-\int_0^x \frac{R_k(y)}{y} Lu_k(y)dy = \int_0^x y^k \gamma_k(y)dy = x^{k+1}\rho_k(x)$$

where $\rho_k(x)$ is a smooth function in x (using integration by parts, for example). Thus the formula for $u_{k+1}(x)$ in the proof above reads

$$u_{k+1}(x) \quad = \quad \frac{1}{R_k(x)}\left(-\int_0^x \frac{R_k(y)}{y} Lu_k(y)dy\right) = \frac{1}{R_k(x)}(x^{k+1}\rho_k(x)) = \frac{\rho_k(x)}{u_0(x)}$$

which is clearly smooth in x since u_0 is a nowhere vanishing smooth function. Note that adding a constant of integration to the indefinite integral $\int_0^x \frac{R_k(y)}{y} Lu_k(y)dy$ will destroy this property, because we need this integral to yield the factor x^{k+1}. Hence, by induction, all the u_k are smooth.

The final assertion is that $u_k(0)$ are polynomial expressions in the various jets (higher derivatives) of b and c at zero. Indeed, we claim that $u_k^{(r)}(0)$ are *all* polynomials in the various jets of b and c at 0. We do this by double induction on k and r. For $k = 0$, by definition $u_0(0) = 1$, and $u_0'(x) = \frac{b(x)}{2}u_0(x)$ implies by Leibnitz rule that:

$$u_0^{(r+1)}(0) = \frac{1}{2}\sum_{0 \le j \le r} \frac{r!}{(r-j)!\, j!} b^{(r-j)}(0)u_0^{(j)}(0)$$

so that induction on r shows that our claim is true for $k = 0$. Assume inductively that it is true for u_k, i.e. $u_k^{(r)}(0)$ is a polynomial in the various jets of b and c at 0 for all r. Since $L = -\partial_x^2 + b(x)\partial_x + c(x)$, it follows by the induction hypothesis that $(Lu_k)^{(r)}(0)$ is also a polynomial in the various jets of b and c at 0 for all r. From the equation (25) it follows that:

$$u_{k+1}(0) = \frac{1}{k+1}(Lu_k)(0)$$

so that the claim is true for $u_{k+1}(0)$. Differentiating the equation (25) $(r+1)$ times with respect to x yields:

$$xu_{k+1}^{(r+2)} + (r+1)u_{k+1}^{(r+1)} + \left(k+1-\frac{xb(x)}{2}\right)u_{k+1}^{(r+1)} + \sum_{0\le j\le r} A_j(x)u_{k+1}^{(j)} + (Lu_k)^{(r+1)} = 0$$

where $A_j(x)$ is a polynomial in $x, b(x), ..., b^{(j)}(x)$. Setting $x = 0$ in this equation shows that:

$$(r+k+2)u_{k+1}^{(r+1)}(0) = -\sum_{0\le j\le r} A_j(0)u_{k+1}^{(j)}(0) - (Lu_k)^{(r+1)}(0)$$

Since $A_j(0)$ is a polynomial in the various jets of b at 0, and by the last para $(Lu_k)^{(r+1)}(0)$ is a polynomial in the jets of b and c at 0, the last equation above implies by induction on r that $u_{k+1}^{(r+1)}$ is a polynomial in the jets of b and c at 0. The proposition follows. □

Let us prove another technical lemma which will be used later on.

Lemma 12.1.3. Let $g = \sum_{i,j} g_{ij}dx_i \otimes dx_j$ be a Riemannian metric on a suitably small ball U around the origin in $\mathbb{R}^n$, and let g^{ij} be the corresponding metric on 1- forms, i.e. $g^{ij} = g_{ij}^{-1}$ is the matrix inverse of g. This metric defines a Riemannian distance on this ball, which we denote by δ. Define a smooth function on U:

$$f(x,t) = (4\pi t)^{-n/2}\exp\left(\frac{-\delta(0,x)^2}{4t}\right)$$

Then

$$\partial_t f - \sum_{i,j} g^{ij}\partial_i\partial_j f = \left(\frac{1}{t}a_1 + a_2\right) f$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and a_1, a_2 are smooth functions of x, with $a_1(0) = 0$.

Proof: Let us denote the scalar Laplacian (on functions) on U by Δ. We claim that

$$\Delta = -\sum_{i,j} g^{ij}\partial_i\partial_j + L$$

where L is a 1st-order operator. This is because we saw in the Example 9.4.4 that $\sigma_L(d) = i\xi\wedge(-)$, and also by (ii) of the Corollary 9.3.4 that $\sigma_L(d^*) = (-i\xi)\angle(-)$, the adjoint of $\sigma_L(d)$. Thus $\sigma_L(d^*d) = \mid \xi \mid^2 = \sum_{i,j} g^{ij}\xi^i\xi^j$. Thus

$$\Delta = \sum_{i,j} g^{ij} D_{x,i} D_{x,j} + L = -\sum_{ij} g^{ij}\partial_i\partial_j + L$$

where $L = \sum_i \alpha_i(x)\partial_i + \beta(x)$ is a first-order operator.

Now note that for the first-order operator $L = \sum_i \alpha_i(x)\partial_i + \beta(x)$ as above, we have

$$Lf = (4\pi t)^{-n/2}\sum_i \alpha_i(x)\left[-\frac{1}{4t}\partial_i(\delta(0,x)^2)\exp\left(\frac{-\delta(0,x)^2}{4t}\right)\right] + \beta(x)f = \left(\frac{1}{t}c_1(x) + c_2(x)\right) f$$

where c_i are smooth, and also

$$c_1(0) = \frac{-1}{4}\sum_i \alpha_i(0)\partial_i(\delta(0,x)^2)(0) = \frac{-1}{4}\sum_i \alpha_i(0)(\sum_j g_{ij}x_j)(0) = 0$$

since

$$\delta(0,x) = \|x\| + o(\|x\|^2) = (\sum_{i,j} g_{ij}(0)x_ix_j)^{1/2} + o(\|x\|^2)$$

where $\|x\|$ denotes the norm of x in the tangent space $T_0(\mathbb{R}^n)$ with respect to $g_{ij}(0)$.

Thus it is enough to prove that:

$$\partial_t f + \Delta f = \left(\frac{1}{t}a_1 + a_2\right) f$$

where a_i are smooth, and $a_1(0) = 0$. Now it is convenient to use geodesic polar coordinates on U, i.e. polar coordinates on $T_0(\mathbb{R}^n) = \mathbb{R}^n$ transferred to U by the exponential map. We may shrink U to guarantee that the exponential map is a diffeomorphism of a neighbourhood of 0 in $T_0(\mathbb{R}^n)$ onto U. In these polar coordinates $\delta(0,x) = r$, and

$$f(x,t) = (4\pi t)^{-n/2}\exp\left(\frac{-r^2}{4t}\right)$$

which is a radial function. It is also known that if $x = \exp_0(x_1,..,x_n)$ is a vector in $T_0(\mathbb{R}^n)$, then $r^2 = \delta(0,x)^2 = \|x\|^2 = \sum_{i,j} g_{ij}(0)x_ix_j$. Thus for the function $f(r,t)$, which does not depend on any of the other polar coordinates $v_2,..,v_n$ on the unit sphere, we have:

$$\partial_i f = \partial_r f\frac{\partial r}{\partial x_i} = \frac{1}{r}\sum_k g_{ik}(0)x_k\partial_r f$$

Differentiating again, multiplying with $-g^{ij}$ and summing over i,j yields:

$$\Delta f = -\partial_r^2 f - \frac{(n-1)}{r}\partial_r f$$

Now:

$$\partial_r f = \frac{-r}{2t} f$$

and so

$$\partial_r^2 f = \frac{-1}{2t} f + \frac{r^2}{4t^2} f$$

Thus

$$\Delta f = \frac{n}{2t} f - \frac{r^2}{4t^2} f$$

Finally,

$$\partial_t f = -\frac{n}{2t} f + \frac{r^2}{4t^2} f$$

Hence $\partial_t f + \Delta f = 0$, which is of the required form. □

12.2. **Generalised Laplacians.** We will now look at elliptic operators of a special kind, because these will be of primary interest in whatever follows.

In this section, E is to be thought of as either E^+ or E^-, the complex Hermitian vector bundles arising in the Dirac complex. Also, the operator Δ^E which will be cropping up in this section will be the operators Δ^+ or Δ^- in our future considerations.

Definition 12.2.1. Let E be a complex vector bundle on a compact Riemannian manifold M, with Hermitian metric $\langle -, - \rangle$. Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a differential operator of order 2. We say that P is a *generalised Laplacian* if:

(i): P is a formally self-adjoint, viz., $(Pf,g) = (f,Pg)$ for all $f,g \in C^\infty(M,E)$, where (f_1,f_2) is the usual global Hermitian inner product on $C^\infty(M,E)$ defined by:

$$(f_1,f_2) = \int_M \langle f_1(x), f_2(x) \rangle_x \, dV(x) \quad f_i \in C^\infty(M,E)$$

(ii): The leading symbol of P satisfies:

$$\sigma_L(P)(\xi) = \mid \xi \mid^2 I_{E_x}, \quad \xi \in T^*M_x$$

In future we will suppress I_{E_x} from the notation, with the understanding that the scalar $\mid \xi \mid^2$ means that scalar times the identity endomorphism of $(\pi^* E)_\xi = E_x$.

Remark 12.2.2. Using (i) of the Corollary 9.3.4, we have for a second operator that:

$$\sigma_L(P)(\xi) = \frac{-1}{2}(ad\, f)^2 P = \frac{-1}{2}[f,[f,P]]$$

for f such that $df(x) = \xi$. Thus P is a generalised Laplacian iff P is formally self-adjoint of order 2 and:

$$[f,[f,P]] = -2 \mid \xi \mid^2 = -2 \mid df \mid^2$$

for each $f \in C^\infty(M)$.

One can easily construct a generalised Laplacian P as above by using a connection ∇^E on the bundle E and the Levi-Civita connection ∇ on the Riemannian manifold M, as we see below.

First we recall that there is a trace map defined by:

$$\begin{array}{rcl} \mathrm{tr} : C^\infty(M, T^*M \otimes T^*M) & \to & C^\infty(M) \\ f & \mapsto & f(g) \end{array}$$

where $g \in C^\infty(M, TM \otimes TM)$ is the Riemannian metric (on the *cotangent bundle*) given by $g = \sum_{i,j} g^{ij} \partial_i \otimes \partial_j$ in local coordinates. By tensoring with I_E, we get a map as below, which we also denote by tr,

$$\mathrm{tr} : C^\infty(M, T^*M \otimes T^*M \otimes E) \to C^\infty(M,E)$$

If $s \in C^\infty(M, T^*M \otimes T^*M \otimes E)$, then in a local coordinate system x_i at a point, we may write

$$s = \sum_{i,j} s(\partial_i, \partial_j) dx_i \otimes dx_j$$

where $s(\partial_i, \partial_j)$ is a local smooth section of E. Then

$$\mathrm{tr} s = \sum_{i,j} s(\partial_i,\partial_j) dx_i \otimes dx_j \left(\sum_{k,l} g^{k,l} \partial_k \otimes \partial_l \right) = \sum_{i,j} s(\partial_i,\partial_j) g^{ij} \tag{26}$$

Now let

$$\nabla^E : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$$

be a connection on E. By taking the natural "tensor product connection" $I_{T^*M} \otimes \nabla^E + \nabla \otimes I_E$ (where ∇ is the Levi-Civita connection on T^*M, we also get a connection:

$$\nabla^{T^*M \otimes E} : C^\infty(M, T^*M \otimes E) \to C^\infty(M, T^*M \otimes T^*M \otimes E)$$

Definition 12.2.3 (The operator Δ^E)**.** Define the second order differential operator:

$$\Delta^E = -\mathrm{tr}(\nabla^{T^*M \otimes E} \circ \nabla^E) : C^\infty(M, E) \to C^\infty(M, E)$$

Lemma 12.2.4. The operator Δ^E defined above is a generalised Laplacian.

Proof: First, if $s \in C^\infty(M, E)$, we have, in local coordinates x_i at a point:

$$\nabla^E s = \sum_i dx_i \otimes \nabla_i^E s$$

where we set $\nabla_i^E s := \nabla_{\partial_i}^E s$ to simplify notation.

Then we compute $\nabla^{T^*M \otimes E}$ of both sides:

$$\begin{aligned} \nabla^{T^*M \otimes E} \nabla^E s &= \sum_i \nabla^{T^*M \otimes E}(dx_i \otimes \nabla_i^E s) = \sum_i \nabla(dx_i) \otimes \nabla_i^E s + \sum_i dx_i \otimes \nabla^E \nabla_i^E s \\ &= \sum_{i,j} \left(dx_j \otimes \nabla_j(dx_i) \otimes \nabla_i^E s + dx_i \otimes dx_j \otimes \nabla_j^E \nabla_i^E s\right) \end{aligned}$$

where ∇ denotes the Levi-Civita connection. Now note that for a tangent vector Y:

$$\nabla_j(dx_i)(Y) = \partial_j(dx_i(Y)) - dx_i(\nabla_j Y) = \partial_j Y_i - dx_i(\nabla_j Y)$$

Thus, for tangent vectors $X = \sum_i X_i \partial_i$ and $Y = \sum_j Y_j \partial_j$, we have:

$$(\nabla^{T^*M \otimes E} \nabla^E s)(X, Y) = \sum_{i,j} X_j(\partial_j Y_i - dx_i(\nabla_j Y))\nabla_i^E s + \sum_{i,j} X_i Y_j \nabla_j^E \nabla_i^E s \tag{27}$$

Now again note that the second term of the equation (27) above is:

$$-\sum_{i,j} X_j dx_i(\nabla_j Y)\nabla_i^E s = -\sum_j X_j \sum_i (\nabla_j Y)_i \nabla_i^E s = -\sum_j X_j \nabla_{\nabla_j Y}^E s = -\nabla_{\nabla_X Y}^E s$$

Also:

$$\nabla_X^E \nabla_Y^E s = \sum_j X_j \nabla_j^E (\sum_i Y_i \nabla_i^E s) = \sum_j X_j Y_i \nabla_j^E \nabla_i^E s + X_j(\partial_j Y_i)\nabla_i^E s$$

which are precisely the first and third terms of (27). In conclusion:

$$(\nabla^{T^*M \otimes E} \nabla^E s)(X, Y) = \nabla_X^E \nabla_Y^E s - \nabla_{\nabla_X Y}^E s$$

Thus, by applying the equation (26) above, we find that:

$$\Delta^E s = -\sum_{i,j} g^{ij} \left(\nabla_i^E \nabla_j^E - \sum_k \Gamma_{ij}^k \nabla_k^E \right) s$$

where the Christoffel symbols Γ_{ij}^k are defined by

$$\nabla_i \partial_j = \sum_k \Gamma_{ij}^k \partial_k$$

Since the leading symbols of ∇_i is just ∂_i, it follows that Δ^E has the same leading symbol as the operator given in local coordinates by:

$$-\sum_{ij} g^{ij} \partial_i \partial_j = \sum_{i,j} g^{ij} D_{x,i} D_{x,j}$$

But his leading symbol precisely $\sum_{i,j} g^{ij}\xi^i\xi^j = \mid \xi \mid^2$, the symbol of the Laplacian. So Δ^E is a generalised Laplacian. We remark here that the leading symbol depends only on the Riemannian metric g on M, and does not depend on the connection ∇^E on E. □

Remark 12.2.5. We have already remarked in the proof of the Lemma 12.1.3 that for the usual Laplace-Beltrami operator on functions, (i.e. the Laplacian on $C^\infty(M)$ of the deRham complex) that $\sigma_L(\Delta) = -\sum_{i,j} g^{ij}(x)\partial_i\partial_j$. Thus, from the above proposition it follows that *no matter what connection one puts on* E, we have:

$$\Delta^E = \Delta + L$$

where L is a first order differential operator. L, of course, will depend on E. We will study it in greater detail later, and see the connection with the Bochner and Weitzenbock formulas.

12.3. **Fundamental solutions of the Heat Equation for generalised Laplacians.** By the Proposition 11.2.2, we have the existence and uniqueness of a fundamental solution $w(x,t)$ to the Dirac Laplacians $\Delta^\pm$. *For the elliptic complexes we consider in the sequel, all of these Dirac Laplacians $\Delta^\pm$ will be generalised Laplacians.* (Indeed, they will all arise as Δ^E as in Definition 12.2.3, and Lemma 12.2.4 will imply that they are generalised Laplacians). The proof of the existence and uniqueness of the fundamental solution $w(x,t)$ used the eigensections and eigenvalues of $\Delta^\pm$, which gives little information about the behaviour of the fundamental solution, because one cannot explicitly compute these eigenvalues and eigensections.

The objective of this section is to gain more information by *actual construction of the fundamental solution* of Proposition 11.2.2, by starting out with a Gaussian type fundamental solution as in $\mathbb{R}^n$, and applying an iterated approximation process using asymptotic solutions for generalised Laplacians. Because this iterative procedure is explicit, it will in theory "solve" the problem of computing the fundamental solution.

Since by definition a generalised Laplacian Δ^E has the same leading symbol as the Laplace-Beltrami operator Δ, it follows that

$$P = \Delta + L$$

where

$$L = \sum_{i=1}^n b_i(x)\partial_i + c(x)$$

is a first order operator. We have a handle on the fundamental solution for Δ by Lemma 12.1.3, we can try to mimic the argument of Proposition 12.1.2 to obtain a fundamental solution for the heat equation:

$$(\partial_t + P)u = 0$$

by asymptotic methods.

Proposition 12.3.1. Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a generalised Laplacian on the compact Riemannian manifold M. Let 0 be some preassigned point on M. Then, on a suitably small neighbourhood U of 0, there is an asymptotic solution:

$$u(x,t) \sim (4\pi t)^{-n/2} \exp\left(-\frac{\delta(0,x)^2}{4t}\right)(u_0(x) + tu_1(x) + ...t^k u_k(x) + ..)$$

where $\delta(0,x)$ is the Riemannian distance between 0 and x in the metric g, and $u_k(x)$ are smooth sections of E on U. That is to say,

$$(\partial_t + P)S_k(x,t) = (4\pi t)^{-n/2} \exp\left(-\frac{\delta(0,x)^2}{4t}\right) t^k r_k(x) \quad \text{for } x \in U, \quad t \in (0,\infty)$$

where $S_k(x,t)$ is the partial sum $\sum_{j=0}^k t^j u_j(x)$, and $r_k(x)$ is a smooth function on U. $u_0(0)$ can be given any preassigned non-zero (vector) value $v \in E_0$, and for all k, each component of $u_k(0)$ is a polynomial in the $v_i(0)$ and the various jets of g^{ij}, b_i and c at 0 ($b_i(x)$ and $c(x)$ being the coefficients occurring in the first order operator $L := P - \Delta$ as in the last paragraph).

Proof: First of all note that we may use a coordinate chart U around 0 on which E is trivial, and which is diffeomorphic to $\mathbb{R}^n$. So, we take $M = \mathbb{R}^n$, and E the trivial bundle. By coordinatewise application, we can also assume that E is the trivial line bundle. Since P is a generalised Laplacian, we can take:

$$P = \Delta + L = -\sum_{i,j} g^{ij}(x)\partial_i\partial_j + L, \quad \text{where} \quad L = \sum_i b_i(x)\partial_i + c(x)$$

and $b_i(x)$ and $c(x)$ are smooth. 0 is the origin in $\mathbb{R}^n$. We will use the geodesic normal coordinates $(x_1, .., x_n)$ introduced in the proof of Lemma 12.1.3.

Now if $f, v \in C^\infty(U)$ are two smooth functions, we have by Leibnitz's formula:

$$\begin{aligned} P(fv) &= -\sum_{i,j} g^{ij}(x)\partial_i\partial_j(fv) + \sum_i b_i(x)\partial_i(fv) + c(x)(fv) \\ &= -\sum_{i,j} g^{ij}(x)(v\partial_i\partial_j f + 2(\partial_i v)(\partial_j f) + f\partial_i\partial_j v) + \sum_i b_i(x)(f\partial_i v + v\partial_i f) + c(x)fv \\ &= fPv - v\sum_{i,j} g^{ij}(x)\partial_i\partial_j f - 2\sum_{i,j} g^{ij}(x)(\partial_i v)(\partial_j f) + v\sum_i b_i(x)\partial_i f \end{aligned} \tag{28}$$

Thus we have,

$$\frac{1}{f}(\partial_t(fv) + P(fv)) = (\partial_t v + Pv) + \frac{v}{f}(\partial_t f - \sum_{i,j} g^{ij}(x)\partial_i\partial_j f) - \frac{2}{f}\sum_{i,j} g^{ij}(x)(\partial_i v)(\partial_j f) + \frac{v}{f}\sum_i b_i(x)\partial_i f \tag{29}$$

Now set $f = f(x,t) = (4\pi t)^{-n/2}\exp(\frac{-\delta(0,x)^2}{4t})$ in the above formula. By the Lemma 12.1.3, we have (upon shrinking U if necessary) that:

$$\partial_t f - \sum_{i,j} g^{ij}\partial_i\partial_j f = \left(\frac{1}{t}a_1 + a_2\right) f$$

where a_i are smooth, and $a_1(0) = 0$. Also, since we are using geodesic normal coordinates, we have $r^2 = \|x\|^2 = \sum_{i,j} g_{i,j}(0)x_i x_j = \sum_i x_i^2$. Now f being a radial function (i.e. only a function of r), we have:

$$\frac{1}{f}\partial_i f = \frac{1}{f}\partial_r f\,\partial_i r = -\frac{2r}{4t}\frac{x_i}{r} = -\frac{x_i}{2t}$$

Substituting these two facts into the equation (29), we have:

$$\frac{1}{f}\left(\partial_t(fv) + P(fv)\right) = (\partial_t v + Pv) + v\left(\frac{1}{t}a_1 + a_2\right) - 2g^{ij}\partial_i v\left(\frac{-x_j}{2t}\right) + v\sum_j b_j(x)\left(-\frac{x_j}{2t}\right)$$

which implies that:

$$(\partial_t(fv) + P(fv)) = f\left[\partial_t v + Pv) + u\left(\frac{1}{t}a_1 + a_2\right) + \sum_j \frac{x_j}{t}\left(\sum_i g^{ij}\partial_i v - \frac{1}{2}vb_j\right)\right] \tag{30}$$

Since we are using geodesic normal coordinates, the radial vector field ∂_r has length one, and is in the same direction as $x = \sum_j x_j e_j$, where $e_j(x) = \sum g^{ij}(x)\partial_i$ is an orthonormal frame at $x \in U$. Thus

$$\partial_r = \frac{1}{\|x\|}\sum_j x_j e_j = \frac{1}{r}\sum_{i,j} g^{ij}(x)x_j\partial_i$$

Substituting this into the equation (30) above, we get:

$$(\partial_t(fv) + P(fv)) = f\left[(\partial_t v + Pv) + u\left(\frac{1}{t}a_1 + a_2\right) - \left(\sum_j \frac{x_j b_j}{2t}v + \frac{r}{t}\partial_r v\right)\right] \tag{31}$$

Now let $v = \sum_{k=0}^\infty t^k u_k(x)$, so that

$$u(x,t) = fv = (4\pi t)^{-n/2}\exp(\frac{-\delta(0,x)^2}{4t})(\sum_k t^k u_k(x))$$

Since we want to satisfy $(\partial_t + P)u = (\partial_t(fv) + P(fv)) = 0$, we compute the coefficient of t^k in the box-brackets on the right hand side of equation (31) and set it equal to zero. The coefficient of t^k on the right hand side of equation (31) is

$$\begin{aligned}(k+1)u_{k+1} \quad &+ \quad Pu_k + a_1 u_{k+1} + a_2 u_k - \frac{1}{2}\sum_j b_j x_j u_{k+1} + r\partial_r u_{k+1} \\ &= \quad r\partial_r u_{k+1} + \left(k + 1 + a_1 - \frac{1}{2}\sum_j x_j b_j\right) u_{k+1} + a_2 u_k + Pu_k\end{aligned}$$

which leads to the recursive differential equation:

$$r\partial_r u_{k+1} + \left(k + 1 + a_1 - \frac{1}{2}\sum_j x_j b_j\right) u_{k+1} + (P + a_2)u_k = 0 \tag{32}$$

From the Lemma 12.1.3, we have $a_1(0) = 0$, so we may write $a_1(x) = \sum_j x_j A_j(x)$ for some smooth functions A_j by the first order Taylor expansion. We write $x = ry$ where $\|y\| = 1$, i.e. y is on the unit sphere S^{n-1}. Then let $B(r, y) := -2\sum_j (y_j A_j(r, y) + y_j b_j(r, y))$. If we also write $\Lambda := P + a_2$, our equation above becomes:

$$r\partial_r u_{k+1} + \left(k + 1 - \frac{1}{2} rB(r, y)\right) u_{k+1} + \Lambda u_k = 0$$

This equation is an ODE in the variable r, with $y \in S^{n-1}$ being treated as a smooth parameter, and identical to the earlier single-variable equation (25), with r playing the role of x, $B(r, y)$ playing the role of the earlier $b(x)$, and Λ playing the role of the earlier L. Thus it is solved along any ray $y = y_0 \in S^{n-1}$ by exactly the same procedure as in the Proposition 12.1.2. The resulting u_j are smooth in x, because of the inductive formula

$$u_{k+1}(r, y) = -\frac{1}{R_k(x)} \int_0^r \frac{R_k(x)\Lambda u_k(x)}{\|x\|} d(\|x\|); \quad R_k(x) = \|x\|^{k+1} \exp\left(-\int_0^r B(x) d(\|x\|)\right)$$

using the same argument as in Proposition 12.1.2, after noting the fact that B and the coefficients of the differential operator Λ are smooth in x (since $b_j(x)$, $c(x)$, $a_1(x)$, $a_2(x)$ are all smooth in x).

To see the last assertion about $u_k(0)$, we have as before that $u_k(0)$ will be algebraic in the various jets of B and the coefficients of Λ at 0. That is, they will be algebraic in the jets of g^{ij}, b_j, c, a_1 and a_2 at 0. We just need to show that the jets of a_1 and a_2 at 0 are algebraic in the jets of g^{ij} at 0. If we go back to the proof of Lemma 12.1.3, we find that a_1 and a_2 defined there are precisely c_1 and c_2, where c_1 and c_2 are defined by:

$$c_1(x) = -\frac{1}{4}\sum_i \alpha_i(x)\partial_i r^2 = -\frac{1}{2}\sum_i x_i \alpha_i(x), \quad c_2(x) = \beta(x)$$

where

$$\Delta = -\sum_{i,j} g^{ij}\partial_i\partial_j + \sum_i \alpha_i(x)\partial_i + \beta(x)$$

Now from the calculation of the Laplacian for the metric $g = g_{ij}$, one knows that $\alpha_i(x)$ is an algebraic expression in the first derivatives of g, and $\beta(x)$ is an algebraic expression in the second derivatives of g. Hence the jets of c_1 and c_2 at 0 are algebraic expressions in the jets of g at 0, from the equations for c_1 and c_2 above. The proposition follows. □

Proposition 12.3.2 (Duhamel's Principle)**.** Let M be a compact Riemannian manifold, and let

$$\Delta^+ : C^\infty(M, E^+) \to C^\infty(M, E^+)$$

be the Dirac Laplacian corresponding to an elliptic complex on M. Let us assume that Δ^+ is of order 2. Let σ_t be a smoothly varying section in $C^\infty(M, E^+)$, (i.e. $\sigma_{(-)} \in C^\infty((0, \infty) \times M, p^*E^+)$ where $p : (0, \infty) \times M \to M$ is the second projection). Then there exists a unique smooth solution ρ_t which is also smooth in t, satisfying:

(i): $\rho_0 = 0$, and

(ii): ρ_t satisfies the *inhomogenous time-dependent* heat equation:

$$(\partial_t + \Delta^+)\rho_t = \sigma_t$$

for all $t \in (0, \infty)$.

Likewise for E^- and Δ^-.

Proof: If $\sigma_t = \sigma$ were independent of t, our ρ_t would be $e^{-t\Delta^+}\sigma$. In general, we add up the contributions $e^{-(t-s)\Delta^+}\sigma_s$. That is, define:

$$\rho_t = \int_0^t e^{-(t-s)\Delta^+}\sigma_s ds$$

Note that the integral makes sense, since the integrand is smooth in s, and on differentiating both sides with respect to t (and using the dominated convergence theorem), we have:

$$\begin{aligned}\partial_t \rho_t &= e^{-(t-t)\Delta^+}\sigma_t + \int_0^t \partial_t(e^{-(t-s)\Delta^+}\sigma_s)ds \\ &= \sigma_t - \int_0^t \Delta^+(e^{-(t-s)\Delta^+}\sigma_s)ds \\ &= \sigma_t - \Delta^+\rho_t\end{aligned}$$

The uniqueness follows from the fact that for another solution u_t satisfying both (i) and (ii), we have:

$$\partial_t(\rho_t - u_t) = -\Delta^+(\rho_t - u_t)$$

so that

$$\partial_t(\rho_t - u_t, \rho_t - u_t) = -2(\Delta^+(\rho_t - u_t), \rho_t - u_t) = -2(D^+(\rho_t - u_t), D^+(\rho_t - u_t)) \leq 0$$

which shows that the L_2-norm $\|\rho_t - u_t\|^2$ is a non-increasing function of t. But since it is zero at $t = 0$ by (ii), it follows that it is identically zero. $\square$.

Corollary 12.3.3. For the ρ_t found above, we have the Sobolev norm estimates:

$$\|\rho_t\|_{2k} \leq t \sup_{0 \leq s \leq t} \|\sigma_s\|_{2k}$$

for all $k = 0, 1, 2...,$.

Proof: We first note that for any $f \in C^\infty(M, E^+)$ and for all $\mu \geq 0$ we have $e^{-\mu\lambda_i} \leq 1$ for all the eigenvalues $\lambda_i \geq 0$ of Δ^+, and consequently the inequality of L_2-norms:

$$\left\| e^{-\mu\Delta^+} f \right\| \leq \|f\| \quad \text{for all } \ \mu \geq 0$$

Now, by the Corollary 6.2.3 (Garding's Inequality) it follows that:

$$\begin{aligned}\left\| e^{-\mu\Delta^+} f \right\|_{2k}^2 &= \left\| \Delta^{+\,k} e^{-\mu\Delta^+} f \right\|^2 + \left\| e^{-\mu\Delta^+} f \right\|^2 = \left\| e^{-\mu\Delta^+} (\Delta^{+\,k} f) \right\|^2 + \left\| e^{-\mu\Delta^+} f \right\|^2 \\ &\leq \left\| \Delta^{+,k} f \right\|^2 + \|f\|^2 = \|f\|_{2k}^2 \quad \text{for all } \ \mu \geq 0\end{aligned}$$

Hence

$$\|\rho_t\|_{2k} \leq \int_0^t \left\| e^{-(t-s)\Delta^+}\sigma_s \right\|_{2k} ds \leq \int_0^t \|\sigma_s\|_{2k} ds \leq t \sup_{0 \leq s \leq t} \|\sigma_s\|_{2k}$$

and the corollary follows. $\square$

Now we can prove that "asymptotic fundamental solutions" converge to real fundamental solutions. More precisely, we have:

Proposition 12.3.4. In the setting of the previous proposition, let w_t be the unique fundamental solution to the heat equation for the Dirac operator Δ^+, with pole at (x, v) (whose existence was proved in the Proposition 11.2.2). Let u_t be a smooth section, varying smoothly in t (see last proposition for definition) which satisfies:

(i): For all $s \in C^\infty(M, E^+)$, we have

$$\lim_{t\to 0}(s, u_t) = \langle s(x), v\rangle_x$$

(That is u_t converges to the Dirac distributional section $\delta_x v$ as $t \to 0$), and

(ii):

$$(\partial_t + \Delta^+)u_t = t^N r_t(x)$$

where r_t is a smooth section of E^+, smoothly varying for $t \in (0, \infty)$ and *continuous* in $t \in [0, \infty)$ and *uniformly bounded in the Sobolev* $2k$*-norm* $\|-\|_{2k}$ for $t \in [0, T]$ and some $T > 0$ and some $k \geq 0$. (This means $\|r_t\|_{2k} \leq C$ for all $t \in [0, T]$, where C is a positive constant.)

Then we have:

$$\|w_t - u_t\|_{\infty,l} \leq C_l t^{N+1} \quad \text{for all} \quad l < 2k - n/2 \ \text{ and all } \quad t \in (0, T]$$

where $C_l > 0$ is some constant.

Proof: By the Duhamel Principle Proposition 12.3.2, there exists a smoothly varying smooth section ρ_t of E^+ satisfying:

$$(\partial_t + \Delta^+)\rho_t = t^N r_t$$

and also satisfying $\rho_0 = 0$. Then the smoothly varying section $w_t := u_t - \rho_t$ satisfies:

$$(\partial_t + \Delta^+)w_t = (\partial_t + \Delta^+)u_t - (\partial_t + \Delta^+)\rho_t = t^N r_t - t^N r_t = 0$$

Also, for any smooth section $s \in C^\infty(M, E^+)$, we have:

$$\lim_{t\to 0}(s, w_t) = \lim_{t\to 0}(s, u_t) = \langle s(x), v\rangle_x$$

since $\rho_0 = 0$.

Thus w_t is *the unique* fundamental solution of the heat equation with pole at (x, v). It follows that $u_t = \rho_t + w_t$ and by the Corollary 12.3.3

$$\|w_t - u_t\|_{2k} = \|\rho_t\|_{2k} \leq t \sup_{0\leq s\leq t} \left\|s^N r_s\right\|_{2k} = t^{N+1} \sup_{0\leq s\leq T} \|r_s\|_{2k} \quad \text{for } \ t \in (0, T]$$

By the hypothesis on the Sobolev $2k$-norm $\|r_s\|_{2k}$ for $s \in [0, T]$ it follows that for $t \in [0, T]$ we have:

$$\sup_{0\leq s\leq T} \|r_s\|_{2k} \leq C \quad \text{for all } \ s \in [0, T]$$

Thus it follows that:

$$\|w_t - u_t\|_{2k} \leq Ct^{N+1} \quad \text{for all } \ t \in (0, T]$$

Now one uses Sobolev's Embedding Theorem (iv) of Proposition 4.2.2 which asserts that

$$\|-\|_{\infty,l} \leq C \|-\|_{2k} \quad \text{for all } \ l < 2k - n/2$$

to get the assertion for $\|-\|_{\infty,l}$ with $l < 2k - n/2$. □

Theorem 12.3.5 (Asymptotic fundamental solution for the heat equation of a generalised Dirac Laplacian)**.** Let

$$\Delta^+ = D^- D^+ : C^\infty(M, E^+) \to C^\infty(M, E^+)$$

be the Dirac Laplacian of the Dirac complex defined by an elliptic complex $\mathcal{P}$ on the compact Riemannian manifold M of dimension n. *Assume that Δ^+ is a generalised Laplacian in the sense of Definition 12.2.1.* Let $v \in E_a^+$ be some vector, and let w_t be the fundamental solution to the heat equation for Δ^+ with pole at (a, v), which exists and is unique by the Proposition 11.2.2. Then there exists an *asymptotic fundamental solution* $u(x,t) = u_t(x)$ with pole at (a, v) which is given by a formal series:

$$u(x,t) = (4\pi t)^{-n/2} \exp\left(-\frac{\delta(x,a)^2}{4t}\right)\left(u_0(x) + tu_1(x) + t^2 u_2(x) + ... + t^k u_k(x) + ...\right) \quad x \in M, \quad t \in (0, \infty)$$

where $\delta(x, a)$ denotes the Riemannian distance between x and a, and $u_i(x)$ are smooth functions of x. The value $u_0(a) = v$, and in a suitable local coordinate neighbourhood of a the and local framing of E^+, for every k, each component of the vector $u_k(a)$ is a polynomial in the p-jets at the point a of g^{ij} and the coefficients $b_i,\ c$ occurring in the first-order operator:

$$\Delta^+ + \sum_{ij} g^{ij} \partial_i \partial_j = \sum_l b_l(y) \partial_l + c(y)$$

This asymptotic solution satisfies:

(i): For each smooth section $s \in C^\infty(M, E^+)$,

$$\lim_{t \to 0} (s, u_t) = \langle s(a), v \rangle_a$$

(ii): Given any positive integer $N > 0$, for the partial sum

$$S_m(x,t) := (4\pi t)^{-n/2} \exp\left(-\frac{\delta(x,a)^2}{4t}\right)\left(\sum_{k=0}^{m} t^k u_k(x)\right)$$

we have:

$$(\partial_t + \Delta^+) S_m(x,t) = t^N r_{m,t}(x) \quad \text{for all } \ m \geq N + n/2$$

where $r_{m,t}(x) = r_m(x,t)$ is a smoothly varying section in $C^\infty(M, E^+)$ and continuous for $t \in [0, \infty)$. Indeed, $r_m(x, 0) \equiv 0$. If we fix some $T > 0$, then its Sobolev $2k$-norm on M satisfies:

$$\|r_{m,t}\|_{2k} \leq C_{k,m} \quad \text{for all } \ 2k \leq m - N - n/2 \quad \text{and all } \ t \in [0, T].$$

Finally,

(iii): For the $T > 0$ in (ii) above, we have the norm estimate:

$$\|w_t - S_m(-,t)\|_{l,\infty} \leq C_l t^{N+1}$$

for each $0 \leq l \leq m - N - n$ and all $t \in (0, T]$.

Likewise for Δ^- and E^-.

Proof: Let U be a neighbourhood of $a \in M$ such that U is diffeomorphic to a neighbourhood of 0 in $\mathbb{R}^n = T_a(M)$ via the exponential map $\exp_a : T_a(M) \to M$ of the Riemannian manifold M. Since Δ^+ is a generalised Laplacian by hypothesis, we may further guarantee that U is small enough for the Proposition 12.3.1 to apply to $P = \Delta^+$.

By restricting to a ball around a contained in U, we may assume without loss of generality that $U = B(a, 3\epsilon)$. Then, by that Proposition we have a formal series:

$$\widetilde{u}(x,t) \sim (4\pi t)^{-n/2} \exp\left(-\delta(a,x)^2/4t\right)\left(\widetilde{u}_0 + t\widetilde{u}_1 + ... + t^k \widetilde{u}_k + ...\right)$$

where $\widetilde{u}_i(x)$ are smooth functions defined on U. Furthermore, $\widetilde{u}_0(a) = v$, and in a suitable framing of E^+ on U, each component of each $\widetilde{u}_k(a)$ is a polynomial in the p-jets of g^{ij}, b_i, c at a. Also on U we have, by the proof of Propositions 12.1.2 and 12.3.1 that:

$$(\partial_t + \Delta^+)\widetilde{S}_m(x,t) = (4\pi t)^{-n/2} t^m \exp\left(-\delta(x,a)^2/4t\right) \Lambda \widetilde{u}_m(x)$$

where $\Lambda = \Delta^+ + a_2$ is also a generalised Laplacian defined on U. If $m \geq N + n/2$, the function

$$\widetilde{r}_m(x,t) := t^{m-N-n/2} \exp\left(-\delta(x,a)^2/4t\right) \Lambda \widetilde{u}_m(x) \tag{33}$$

is a smoothly varying section of $E^+_{|U}$, continuous and uniformly bounded in the norm $\|-\|_x$ of E^+_x, for all $x \in U$ and all $t \in [0,T]$. That is,

$$\sup_{x\in U,\, t\in[0,T]} \|\widetilde{r}_m(x,t)\|_x < \infty$$

Note that the equation (33) above implies that $\widetilde{r}_m(x,0) \equiv 0$ for $m > N + n/2$. Since

$$\partial_i(t^p \exp(-\delta(x,a)^2/4t)) = \left(p - \frac{x_i}{2}\right) t^{p-1} \exp(-\delta(x,a)^2/4t)$$

on U, we see that for $m \geq N + n/2 + 2k$, the L_2-norm:

$$\|\partial^\alpha \widetilde{r}_m\|^2_{0,U} := \int_{x\in U} \|\partial^\alpha \widetilde{r}_m(x,t)\|^2_x \, dV(x)$$

will be finite and uniformly bounded for all $|\alpha| \leq 2k \leq m - N - n/2$. Thus the Sobolev $2k$-norm of $\widetilde{r}_m(-,t)$ on U satisfies:

$$\|\widetilde{r}_m(-,t)\|_{2k,U} \leq C_{k,m} \quad \text{for all } \ 2k \leq m - N - n/2, \ \text{ and all } \ \ t \in [0,T]$$

Thus we have:

$$(\partial_t + \Delta^+)\widetilde{S}_m(x,t) = t^N \widetilde{r}_m(x,t) \quad \text{for all} \quad x \in U, \quad \text{and} \ \ m > N + n/2 \tag{34}$$

with $\widetilde{r}_m(x,t)$ a smoothly varying section of $E^+_{|U}$ for $t \in (0,\infty)$, continuous in $t \in [0,\infty)$, and uniformly bounded in Sobolev norm $\|-\|_{2k,U}$ by a positive constant $C_{k,m}$ for all $t \in [0,\infty)$ and all $2k \leq m - N - n/2$.

The first step is to globalise $\widetilde{u}(x,t)$ for all $x \in M$. We do this via a cut-off function. Let

$$\psi : \mathbb{R} \to [0,\infty)$$

be a smooth function such that $\psi(s) \equiv 1$ for $|s| \leq \epsilon$ and $\psi(s) \equiv 0$ for $|s| \geq 2\epsilon$.

To simplify notation, denote $r := r(x) := \delta(x,a)$. Then define:

$$u(x,t) = \psi(r(x))\widetilde{u}(x,t)$$

so that

$$u(x,t) \sim (4\pi t)^{-n/2} \exp\left(-\frac{\delta(x,a)^2}{4t}\right) \left(u_0(x) + t u_1(x) + ... + t^k u_k(x) + ...\right)$$

where $u_k(x) := \psi(r)\widetilde{u}_k(x)$. Since $\psi(r(x))$ is identically 1 on $B(0,\epsilon)$, the function $\psi(r(x))$ is a smooth function of x, and hence the u_k's defined above are smooth functions of x on all of M. Furthermore:

$$\begin{aligned} u_k(x) &= \widetilde{u}_k(x) \quad \text{for} \ \ \delta(x,a) \leq \epsilon \\ &= 0 \qquad\ \ \text{for} \ \ \delta(x,a) \geq 2\epsilon \end{aligned}$$

Hence the statement about $u_k(a)$ follows from the corresponding statements about $\widetilde{u}_k(a)$.

For notational convenience, denote:

$$f(x,t) := (4\pi t)^{-n/2} \exp\left(-\frac{\delta(x,a)^2}{4t}\right)$$

Since ψ is supported in U, we have for a smooth section $s \in C^\infty(M, E^+)$:

$$\begin{aligned}
\lim_{t\to 0}(s, u_t) &= \lim_{t\to 0}\int_M \langle s(x), u(x,t)\rangle_x \, dV(x) = \lim_{t\to 0}\int_U \langle s(x), \psi(r(x))\widetilde{u}(x,t)\rangle_x \, dV(x) \\
&= \lim_{t\to 0}\int_U \langle \psi(r(x))s(x), \widetilde{u}(x,t)\rangle_x \, dV(x) \\
&= \lim_{t\to 0}\int_U \langle \psi(r(x))s(x), f(x,t)\widetilde{u}_0(x)\rangle_x \, dV(x) = \langle \psi(a)s(a), v\rangle_a \\
&= \langle s(a), v\rangle_a
\end{aligned}$$

because $\psi(a) = 1$ and $\widetilde{u}_0(a) = v$ by the Proposition 12.3.1, and $f(x,t)$ is an approximate identity at $x = a$ for compactly supported smooth sections in U, and $\psi(r(x))s(x)$ is a smooth section compactly supported in U. This proves (i) of the Theorem.

Now we prove (ii). We have by definition that $S_m(x,t) = \psi(r(x))\widetilde{S}_m(x,t)$. Hence

$$\partial_t S_m(x,t) = \psi(r)\partial_t \widetilde{S}_m(x,t) \quad \text{for all} \;\; x \in M \tag{35}$$

where the right hand side is interpreted to be identically zero for $x \notin U$ (i.e. $\delta(x,a) \geq 3\epsilon$).

On the other hand,

$$\Delta^+(\psi(r)\widetilde{S}_m(x,t)) = \psi(r)\Delta^+\widetilde{S}_m(x,t) + \mu(r)L\widetilde{S}_m(x,t) \tag{36}$$

where

$$\mu(r) := a(r)\psi'(r) + b(r)\psi''(r)$$

and L $(= \sum_i \alpha_i(x)\partial_i + \beta(x)$ in U, and $\equiv 0$ oustside some $V \supset B(0, 2\epsilon))$ is some first order linear differential operator in the space variables on M. We already understand the first term, from the foregoing discussion, and we need to estimate the second term. Since

$$\widetilde{S}_m(x,t) = f(x,t)\sum_{k=0}^m t^k \widetilde{u}_k(x)$$

we compute for $x \in U$:

$$\begin{aligned}
L\widetilde{S}_m(x,t) &= \left[\sum_{i=1}^n \alpha_i(x)\partial_i(x) + \beta(x)\right] f \sum_{k=0}^m t^k \widetilde{u}_k(x) \\
&= (Lf)\sum_{k=0}^m t^k \widetilde{u}_k(x) + f\sum_{k=0}^m t^k (L - \beta(x))\widetilde{u}_k(x) \\
&= f\left(\frac{1}{t}c_1(x) + c_2(x)\right)\sum_{k=0}^m t^k \widetilde{u}_k(x) + f\sum_{k=0}^m t^k w_k(x) \\
&= t^{-1} f(x,t) P_m(x,t) \quad \text{for} \quad x \in U
\end{aligned}$$

where $P_m(x,t)$ is a polynomial of degree m in t whose coefficients are smooth sections of $E^+_{|U}$. Note that we have used the first paragraph of the Lemma 12.1.3 to substitute $Lf = (\frac{1}{t}c_1 + c_2)f$.

Since $\psi'(r)$ and $\psi''(r)$ identically vanish for $0 \leq r \leq \epsilon$ and $r \geq 2\epsilon$, it follows that $\mu(r) \equiv 0$ for $0 \leq r \leq \epsilon$ and $r \geq 2\epsilon$.

Consider the section:

$$h_m(x,t) := t^{-N-1}\mu(r)f(x,t)P_m(x,t)$$

Since $f(x,t) \leq (4\pi t)^{-n/2}e^{-\epsilon^2/4t}$ for $r \geq \epsilon$ and $t \in [0,\infty)$, and $\mu(r)$ vanishes identically for $r \leq \epsilon$ and $r \geq 2\epsilon$, it follows that the section above is a smooth section of E^+ with compact support in the annulus $\epsilon \leq r \leq 2\epsilon$, for every $N \geq 0$ and all $t \in [0,\infty)$. At $t = 0$, it is the identically zero function. Hence we may write:

$$\mu(r)L\widetilde{S}_m(x,t) = t^N(t^{-N-1}\mu(r)f(x,t)P_m(x,t)) = t^N h_m(x,t) \quad \text{for all} \quad x \in M, \text{ and all } \; m \geq 0 \tag{37}$$

where $h_m(x,t) = t^{-N-1}\mu(r)f(x,t)P_m(x,t)$, P_m being a polynomial of degree m in t whose coefficients are smooth sections in the variable $x \in M$, $h_m(x,t) \equiv 0$ for $r(x) = \delta(x,a) \leq \epsilon$ and $r(x) \geq 2\epsilon$ and all $t \in [0,\infty)$, with $h_m(x,0) \equiv 0$ on M.

To get a hold on the Sobolev $2k$-norm of $h_m(x,t)$, note that P_m is a polynomial of degree m in t, whose coefficients are smooth sections. Also each spatial derivative of $f(x,t)$ will yield $(t^{-1}a_1 + a_2)f$, and any spatial derivative of $\mu(r)$ will again yield a smooth function compactly supported in the annulus $\epsilon \leq r \leq 2\epsilon$. Hence, for the L_2-norm:

$$\int_M \|\partial_x^\alpha h_m(x,t)\|_x^2 \, dV(x) \leq Ct^{-N-1-|\alpha|} \int_{\epsilon \leq r \leq 2\epsilon} f(x,t)^2 dV(x) \leq Ct^{-N-1-|\alpha|} e^{-\epsilon^2/2t} \quad \text{for } t \in [0,T]$$

Thus we have:

$$\sup_{t \in [0,T]} \|h_m(-,t)\|_{2k} < \infty \quad \text{for all } k \tag{38}$$

Now we can combine all the equations (34), (35), (36) and (37) to compute:

$$\begin{aligned}(\partial_t + \Delta^+)S_m(x,t) &= (\partial_t + \Delta^+)\psi(r)\widetilde{S}_m(x,t) = \psi(r)(\partial_t + \Delta^+)\widetilde{S}_m(x,t) + \mu(r)L\widetilde{S}_m(x,t) \\ &= \psi(r)(t^N \widetilde{r}_m(x,t)) + t^N h_m(x,t) \\ &= t^N r_m(x,t)\end{aligned} \tag{39}$$

for all $x \in M$, all $m \geq N + n/2$ and all $t \in (0,\infty)$, and $r_m(x,t) := \psi(r)\widetilde{r}_m(x,t) + h_m(x,t)$. Also,

(a): From the equations (34) and (37) it follows that $r_m(x,0) = \psi(r(x))\widetilde{r}_m(x,0) + h_m(x,0) \equiv 0$.

(b): From the statement following equation (34), the fact that

$$\|\psi(r)\widetilde{r}_m(-,t)\|_{2k} \leq C\, \|\widetilde{r}_m(-,t)\|_{2k,U}$$

and from the inequality (38), it follows that:

$$\sup_{t \in [0,T]} \|r_m(-,t)\|_{2k} \leq C_k \quad \text{for all} \quad 2k \leq m - N - n/2$$

This establishes (ii) of the Theorem. The final assertion (iii) now follows from the Corollary 12.3.3. □

Example 12.3.6 (The Circle)**.** For the circle, one can explicitly write down the heat kernel, and the fundamental solution by tinkering with the fundamental solution for $\mathbb{R}$.

Let $S^1(R)$ denote the circle of radius R around the origin in $\mathbb{R}^2$, and let $\theta \in (-\pi,\pi)$ denote the usual angle coordinate in the open set $S^1(R)\setminus\{-R\}$. The Riemannian metric is $R^2 d\theta^2$, and the corresponding Riemannian volume of $S^1(R)$ is $2\pi R$. We consider the Dirac complex of the de-Rham complex of the circle, viz. with $E^+ = \Lambda^0(T^*_{\mathbb{C}}S^1(R))$, $E^- = \Lambda^1(T^*_{\mathbb{C}}S^1(R))$ and $D^+ = d$, $D^- = \delta$, and $\Delta^+ = \delta d = -R^{-2}\partial_\theta^2$ the scalar Laplacian on functions. Since $g^{ij} = g^{11} = R^{-2}$, the scalar Laplacian on functions is $\Delta^+ = -\sum_{ij} g^{ij}\partial_i\partial_j = -\frac{1}{R^2}\partial_\theta^2$.

Denote

$$f(\theta,t) := (4\pi t)^{-1/2}\exp(-R^2\theta^2/4t)$$

For $x \in S^1(R)$ and $t > 0$, define:

$$u(x,t) = \sum_{n \in \mathbb{Z}} f(\theta + 2n\pi, t)$$

where $x = Re^{2\pi i\theta}$. Note that by definition above, which "logarithm" of x we take is immaterial for the definition of u. We first need to check that the series above converges for each $x \in S^1(R)$, and each $t > 0$. But this is clear, since for $t > 0$, the factor of $\exp(-n^2R^2\pi^2/t)$ will occur in the n-th term, and the series will converge very rapidly and indeed uniformly and absolutely. Likewise with the t-derivative and all θ-derivatives of the series. So it is permissible to differentiate term-by-term and integrate term by term etc.

Since $(\partial_t - R^{-2}\partial_\theta^2)f(\theta,t) = 0$, it follows that $u(x,t)$ satisfies the heat equation. Note that

$$\lim_{t\to 0} t^{-\frac{1}{2}}\exp(-R^2n^2\pi^2/t) = 0 \quad \text{for all} \quad n \neq 0,\ t > 0$$

Also, for any smooth function $s \in C^\infty(S^1(R))$, we can lift s to a smooth function $\widetilde{s}$ which is compactly supported in say $U = (-2\pi, +2\pi)$, with $s(Re^{i\theta}) = \widetilde{s}(\theta)$ for $\theta \in (-\pi, \pi)$. Then it is easy to check that:

$$\lim_{t\to 0} \int_\pi^\infty \widetilde{s}(\theta) f(\theta, t) R\, d\theta = 0, \quad \lim_{t\to 0} \int_{-\infty}^{-\pi} \widetilde{s}(\theta) f(\theta, t) R\, d\theta = 0$$

because $f(\theta, t) \leq (4\pi t)^{-1/2} \exp(-R^2\pi^2/4t) \leq C \exp(-\alpha/t)$ for $\theta \geq \pi$ and $\theta \leq -\pi$, where α and C are some positive constants.

From the two observations above, we have:

$$\begin{aligned}
\lim_{t\to 0}(s, u_t) &= \lim_{t\to 0} \int_{-\pi}^{\pi} s(Re^{i\theta})(4\pi t)^{-1/2} \exp(-R^2\theta^2/4t) R\, d\theta \\
&= \lim_{t\to 0} \int_{-\pi}^{\pi} \widetilde{s}(\theta)(4\pi t)^{-1/2} \exp(-R^2\theta^2/4t) R\, d\theta \\
&= \lim_{t\to 0} \int_{-\infty}^{\infty} \widetilde{s}(x/R)(4\pi t)^{-1/2} \exp(-x^2/4t) d\theta = \widetilde{s}(0) = s(Re^{i0})
\end{aligned}$$

Hence, by the uniqueness statement of the Proposition 11.2.2, applied to that $u(x,t)$ is the fundamental solution to the heat equation with pole at $(Re^{i0}, 1)$. By suitably translating the space variable of $u(x,t)$, one can write down the fundamental solution with pole at any other point $(x, 1)$.

Now we can determine all the coefficients $u_i(a)$ in the asymptotic expansion of the fundamental solution, where $a = Re^{i0} = R \in S^1(R)$. Since

$$u(x,t) = \sum_{n\in\mathbb{Z}} f(\theta + 2n\pi, t)$$

and for $n \neq 0$, the term $f(\theta + 2n\pi, t)$ contains the factor $(4\pi t)^{-1/2} \exp(-R^2 n^2/4t) \leq Ce^{-\alpha/t}$ for some $C, \alpha > 0$, we find that:

$$\lim_{t\to 0} t^{-k} f(\theta + 2n\pi, t) = 0 \quad \text{for all} \ \ k \geq 0 \quad \text{and} \ \ n \neq 0$$

In other words,

$$u(x,t) \sim (4\pi t)^{-1/2} \exp(-R^2\theta^2/4t) \quad \text{as} \ \ t \to 0$$

Now $R\theta = \delta(a, x)$, the Riemannian distance between $a = Re^{i0}$ and $x = Re^{i\theta}$ in $S^1(R)$. So we find, on comparing the expression for $u(x,t)$ in the Proposition 12.3.5 that

$$u_0(a) = 1, \ \ u_i(a) = 0 \quad \text{for all} \ \ i \geq 1$$

This fact has a lot of interesting consequences. Note that the eigenvalues of Δ^+ are precisely $\lambda_n = n^2/R^2$, and the corresponding (normalised) eigenfunctions are $e_n(\theta) = (2\pi R)^{-1/2} e^{i\,n\theta}$, where $n \in \mathbb{Z}$. From the the construction of the fundamental solution of Δ^+ (from the heat kernel in (iii) of Proposition 10.1.3 and the fundamental solution in Proposition 11.2.1, we have:

$$u(x,t) = k_t^+(x, a) = \sum_{n\in\mathbb{Z}} e^{-t\lambda_n} e_n^*(a) \otimes e_n(x) = (2\pi R)^{-1} \sum_{n\in\mathbb{Z}} e^{-tn^2/R^2} e^{i\,n\theta} \quad \text{where} \ \ x = Re^{i\theta}, \ \ a = Re^{i0}$$

Since our asymptotic expansion for $u(x,t)$ just consists of the first term and no others, it follows that the partial sum:

$$S_m(x,t) = (4\pi t)^{-1/2} \exp(-R^2\theta^2/4t) \quad \text{for all} \quad m \geq 0$$

Then (iii) of the Theorem 12.3.5 (for $l = 0$, say) now tells us that

$$\left\| (2\pi R)^{-1} \sum_{n\in\mathbb{Z}} e^{-tn^2/R^2} e^{i\,n\theta} - (4\pi t)^{-1/2} \exp(-R^2\theta^2/4t) \right\|_{0,\infty} \leq Ct^{N+1} \quad \text{for} \ \ t \in (0, T] \ \ m \geq N+1$$

which implies that

$$(2\pi R)^{-1} \sum_{n\in Z} e^{-tn^2/R^2} e^{i\,n\theta} \sim (4\pi t)^{-1/2} \exp(-R^2\theta^2/4t) \ \ \text{as} \ \ t \to 0 \quad \text{for each} \ \ \theta \in (-\pi, \pi)$$

Setting $\theta = 0$ in the above formula, one obtains *Jacobi's asymptotic formula*

$$\sum_{n \in Z} e^{-tn^2/R^2} \sim (2\pi R)(4\pi t)^{-1/2} = R\sqrt{\pi/t} \quad \text{as} \quad t \to 0 \tag{40}$$

So here is a beautiful college-level mathematical formula that uses the asymptotic expansion of the heat kernel on a compact Riemannian manifold for its proof!

Also note that the left hand side of (40) is precisely the trace of the heat operator $e^{-t\Delta^+}$, so the Jacobi formula above says that:

$$\lim_{t \to 0} (4\pi t)^{1/2} \left(\operatorname{tr} e^{-t\Delta^+} \right) = 2\pi R = \operatorname{Vol}\left(S^1(R)\right)$$

Thus the $t \to 0$ asymptotics of the trace of the heat operator encodes the Riemannian volume of $S^1(R)$. Indeed, this is a general fact, as we see below.

Proposition 12.3.7 (You can hear the volume of a manifold)**.** For the scalar Laplacian $\Delta : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$, we have:

$$\lim_{t \to 0} (4\pi t)^{n/2} \left(\operatorname{tr} e^{-t\Delta} \right) = \operatorname{Vol}(M)$$

Proof: We first remark that for the scalar Laplacian Δ on any compact Riemannian manifold, the eigenvalues $\lambda_n \geq 0$, because

$$\lambda_n = (\Delta e_n, e_n) = (\delta d e_n, e_n) = (de_n, de_n)$$

where $\{e_n\}$ is an orthonormal basis of smooth eigenfunctions, with e_n belonging to the eigenvalue λ_n. Because the operator Δ is elliptic and formally self-adjoint, the Proposition 8.4.9 shows that $\lambda_n \geq Cn^\delta$, and the existence of the heat kernel:

$$k_t(x, y) \in C^\infty(M \times M, \mathbb{C})$$

defined by $k_t(x, y) = \sum_n e^{-t\lambda_n} e_n^*(y) e_n(x)$ goes through exactly as in (iii) of Proposition 10.1.3. The fundamental solution $w(x, t)$ of the heat equation with the pole $(a, 1)$, with $a \in M$ is as before given by

$$w^a(x, t) = k_t(x, a)$$

Then, since the asymptotic expansion and Duhamel Principle carry over to generalised Laplacians on any bundle E (in this case the trivial bundle $M \times \mathbb{C}$), we have the conclusions of Theorem 12.3.5 in this setting as well, though it was stated for Dirac Laplacians.

We also have:

$$\operatorname{tr}(e^{-t\Delta}) = \sum_n e^{-t\lambda_n} = \int_M e^{-t\lambda_n} \int_M e_n^*(a) e_n(a) dV(a) = \int_M k_t(a, a) dV(a)$$

On the other hand, we have by (iii) of Theorem 12.3.5 that:

$$\|k_t(-, a) - S_m^a(-, t)\|_{0,\infty} = \|w_t^a(x) - S_m^a(x, t)\|_{0,\infty} \leq Ct^{N+1} \;\text{ for }\; m \geq N + n, \quad t \in (0, T]$$

where $S_m^a(x, t)$ is the partial sum of the asymptotic solution $u^a(x, t)$ with pole $(a, 1)$. On setting $x = a$, this implies that:

$$\left| k_t(a, a) - (4\pi t)^{-n/2} \sum_{k=0}^{m} u_k(a) t^k \right| \leq Ct^{N+1} \quad \text{for all} \;\; m \geq N + n, \quad t \in (0, T] \tag{41}$$

Note that $s(a) = \lim_{t \to 0}(s, u^a(-, t))$ for all $s \in C^\infty(M)$. Letting $f^a(x, t) = (4\pi t)^{-n/2} \exp(-\delta(x, a)^2/4t)$ we also have $s(a) = \lim_{t \to 0}(s, f^a(-, t))$ for all $s \in C^\infty(M)$. Since $u^a(x, t) = f^a(x, t)(u_0(x) + O(t))$, we have

$$s(a) = \lim_{t \to 0}(s, u^a(x, t)) = \lim_{t \to 0}(s, f^a(-, t) u_0(-)) = \lim_{t \to 0}(\overline{u}_0 s, f^a(-, t)) = \overline{u}_0(a) s(a)$$

which implies that $u_0(a) = 1$. (In fact, we remarked in the proof of Theorem 12.3.5 that $u_0(a) = \widetilde{u}_0(a) = v$ from the Proposition 12.3.1, if u^a is the asymptotic fundamental solution with pole (a, v)). Thus, from the equation (41) above, it follows that:

$$\left|(4\pi t)^{n/2}k_t(a,a) - \sum_{k=0}^{m} t^k u_k(a)\right| \leq Ct^{N+1+n/2} \quad \text{for all} \quad m \geq N+n, \;\; t \in (0,T]$$

which implies that

$$\lim_{t\to 0}(4\pi t)^{n/2}\mathrm{tr}\, e^{-t\Delta} = \lim_{t\to 0}(4\pi t)^{n/2}\int_M k_t(a,a)dV(a) = \lim_{t\to 0}\int_M u_0(a)dV(a) = \mathrm{Vol}(M)$$

and the proposition follows. □

13. Clifford Algebras and Spin Structures

13.1. **Clifford Algebras.**

Definition 13.1.1. Let V be an inner product space, with a symmetric bilinear form $\langle -, - \rangle$. The *Clifford algebra* on V, denoted $Cl(V)$ is an associative unital $\mathbb{R}$-algebra together with an $\mathbb{R}$-linear map:

$$\phi : V \to Cl(V)$$

satisfying:

(i): $\phi(v)^2 = -\langle v, v\rangle .1$ for all $v \in V$.

(ii): If $\psi : V \to A$ is any $\mathbb{R}$-linear map into an associative unital algebra A satisfying $\psi(v)^2 = -\langle v, v\rangle 1_A$ for all $v \in V$, then there exists a *unique* $\mathbb{R}$-algebra homomorphism $\widetilde{\psi}$ which makes the diagram:

$$\begin{array}{ccc} V & \stackrel{\phi}{\to} & Cl(V) \\ \psi & \searrow & \downarrow \widetilde{\psi} \\ & & A \end{array}$$

commute.

By the usual abstract nonsense, this universal property makes it unique upto $\mathbb{R}$-algebra isomorphism. To construct it, let $\mathcal{T}(V) := \oplus_{i=0}^{\infty}(\otimes^i V)$ be the full real tensor algebra on V. Let $1 \in \otimes^0 V = \mathbb{R}$ be its identity element. Let $\mathcal{I}$ be the two-sided ideal generated by the set

$$S := \{v \otimes v + \langle v, v\rangle 1 : \; v \in V = \otimes^1 V \subset \mathcal{T}(V)\}$$

Define $Cl(V) = \mathcal{T}(V)/\mathcal{I}$, and let the map ϕ be the composite:

$$V = \otimes^1 V \hookrightarrow \mathcal{T}(V) \to \mathcal{T}(V)/\mathcal{I}$$

Clearly, by definition, $\phi(v)^2 = (v \otimes v)(\mod \mathcal{I}) = -\langle v, v\rangle 1$, where $1 \in \mathbb{R}$ is the image of $1 \in \mathcal{T}(V)$. It is trivially checked that the universal property of the definition above holds for $\phi : V \to Cl(V)$. We will denote the product of $a, b \in Cl(V)$ as $a.b$ or even ab if no confusion is likely.

Proposition 13.1.2. We have the following facts about the Clifford algebra:

(i): The map $\phi : V \to Cl(V)$ is injective. Hence we may regard V as a subspace of $Cl(V)$.

(ii): With the identification of (i) above,

$$v.w + w.v = -2\langle v, w\rangle 1 \;\text{ for all }\; v, w \in V \subset Cl(V)$$

(iii): If $\{e_i\}_{i=1}^n$ is any $\mathbb{R}$-basis of V, then the products

$$e_I := e_{i_1}.e_{i_2}....e_{i_k}$$

where $I = (i_1 < i_2 < .. < i_k)$ is a multiindex with $0 \leq k \leq n$ (and $e_I := 1$ for the empty multiindex I with $k = 0$), constitute an $\mathbb{R}$-basis for $Cl(V)$. In particular, dim $Cl(V) = 2^n$.

(iv): There is a natural $\mathbb{Z}_2$-grading on $Cl(V)$ defined by setting $Cl^0(V)$ to be the image of the subspace $\oplus_{k=0}^{\infty}(\otimes^{2k}(V)) \subset \mathcal{T}(V)$ and $Cl^1(V)$ to be the image of $\oplus_{k=0}^{\infty}(\otimes^{2k+1}(V)) \subset \mathcal{T}(V)$. With this grading $Cl(V)$ is a so-called *superalgebra*, i.e. satisfies:

$$Cl^i(V).Cl^j(V) \subset Cl^k(V) \quad \text{where} \quad k = i + j(\mod 2)$$

(v): There is a canonical vector space isomorphism (not an algebra homomorphism):

$$Cl(V) \to \Lambda^* V$$

which takes $v.w$ to $v \wedge w$ for all $v, w \in V$.

(vi): For the identically zero inner product $\langle -, - \rangle \equiv 0$, the Clifford algebra $Cl(V)$ *is* the exterior algebra $\Lambda^* V$.

Proof: To see (i), define the *degree* deg x of an element $x \in \mathcal{T}(V)$ by expanding into homogeneous components

$$x = \oplus_i x_i, \qquad x_i \in \otimes^i V$$

to be the largest i such that $x_i \neq 0$. Clearly, $\deg(x \otimes y) = \deg x + \deg y$, and hence the degree of every element in the ideal $\mathcal{I}$ is at least 2. Thus $V \cap \mathcal{I} = \{0\}$ in $\mathcal{T}(V)$, and the map $\phi : V \to Cl(V)$ is injective. This proves (i). We may therefore write v instead of $\phi(v)$ for $v \in V$.

To see (ii), note that for $v, w \in V \subset Cl(V)$, we have by the definition of $Cl(V)$:

$$-(\langle v, v \rangle + \langle w, w \rangle + 2 \langle v, w \rangle)1 = -\langle v + w, v + w \rangle 1 = (v + w)^2 = v.v + w.w + v.w + w.v$$

from which it follows that $v.w + w.v = -2 \langle v, w \rangle 1$.

To see (iii), we use (ii) to see that e_I of the form stated are a spanning set for $Cl(V)$, since any word $e_{j_1}.e_{j_2}.....e_{j_k}$ of any length may be reduced, by using the commutation relations:

$$e_i.e_j + e_j.e_i = -2 \langle e_i, e_j \rangle 1$$

to a word of length at most n. Their linear independence is left as an exercise. (iv), (v) and (vi) are also straightforward, and their proof is omitted. □

Notation: From now on, when we write $Cl(V)$, it will be understood that V is an inner product space with a *positive definite inner product* $\langle -, - \rangle$. Hence, we may always choose an orthonormal basis $\{e_i\}_{i=1}^n$ of V, and the commutation relations for the basis elements will read:

$$e_i.e_j + e_j.e_i = -2\delta_{ij} \quad 1 \leq i, j \leq n$$

Example 13.1.3. If we take $V = \mathbb{R}$, with its usual euclidean inner product $\langle x, y \rangle = xy$, then $Cl(\mathbb{R}) = \mathbb{C}$, for it is generated as an $\mathbb{R}$-algebra by e_1 satisfying $e_1^2 = -1$.

If we take $V = \mathbb{R}^2$ with its usual euclidean inner product, then $Cl(V)$ is generated as an $\mathbb{R}$-algebra by $\{e_1, e_2\}$, satisfying:

$$e_1^2 = e_2^2 = -1, \quad e_1 e_2 = -e_2 e_1$$

Setting $e_1 = i, e_2 = j, e_1 e_2 = k$, we find that

$$Cl(V) = \mathbb{R}.1 \oplus \mathbb{R}.i \oplus \mathbb{R}.j \oplus \mathbb{R}.k$$

subject to the relations $i^2 = j^2 = k^2 = -1$, and $ij = k,\ jk = i,\ ki = j$. Thus $Cl(\mathbb{R}^2) = \mathbb{H}$, the (non-commutative) algebra of quaternions.

Exercise 13.1.4 (Some Clifford Algebras)**.**

(i): Show that for $V = \mathbb{R}^3$, with its usual euclidean inner product, we have $Cl(V) \simeq \mathbb{H} \oplus \mathbb{H}\eta$ where $\eta := e_1 e_2 e_3$. The first summand $\mathbb{H}$ is the span of $1,\ i := e_1 e_2,\ j := e_2 e_3,\ k := e_3 e_1$, and the second summand is the span of $\eta,\ i\eta,\ j\eta,\ k\eta$. Multiplication is given by:

$$(a + b\eta)(c + d\eta) = (ac + bd) + (ad + bc)\eta \quad a, b, c, d \in \mathbb{H}$$

(ii): Prove that $Cl(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}(2)$, the algebra of 2×2 complex matrices. Explicitly, the isomorphism is given by:

$$i \otimes 1 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}; \;\; j \otimes 1 \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}; \;\; k \otimes 1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where the matrices on the right are the *Pauli spin matrices*.

Remark 13.1.5. It is possible to write down a complete list of all the real Clifford algebras $Cl(\mathbb{R}^n)$, because of the remarkable periodicity theorem which states that:

$$Cl(\mathbb{R}^{n+8}) = Cl(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{R}(16)$$

where $\mathbb{R}(n)$ denotes the matrix algebra of $n \times n$ real matrices. This reduces us to finding out $Cl(\mathbb{R}^n)$ for $n = 1, .., 8$, whose list is as below:

$n:$	1	2	3	4	5	6	7	8
$Cl(\mathbb{R}^n):$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}\eta$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

For a proof of this fact, see the paper "Clifford Modules" by Atiyah- Bott-Shapiro.

We need a little more machinery associated with a Clifford algebra. The first is the involution $*$ defined as follows:

Definition 13.1.6 (The involution $*$)**.** Let V be a real positive definite inner product space, and $Cl(V)$ its Clifford algebra. There is an involution $*$ on the full tensor algebra $\mathcal{T}(V)$ whose effect on decomposable tensors is:

$$(a_1 \otimes a_2 \otimes .... \otimes a_k)^* = a_k \otimes a_{k-1} \otimes .... \otimes a_2 \otimes a_1$$

This involution clearly preserves the set $S = \{v \otimes v + \langle v, v \rangle 1 : v \in V\}$ defined in the beginning of this section, and since $(\alpha \otimes \beta)^* = \beta^* \otimes \alpha^*$, we see that $*$ preserves the two-sided ideal $\mathcal{I}$ generated by S. Hence it descends to an involution of $Cl(V) = \mathcal{T}(V)/\mathcal{I}$. If we let $\{e_i\}_{i=1}^n$ be an orthonormal basis for V with respect to $\langle -, - \rangle$, then for the basis of $Cl(V)$ introduced in (iii) of Proposition 13.1.2, we have:

$$(e_{i_1}.e_{i_2}.....e_{i_k})^* = e_{i_k}.e_{i_{k-1}}.....e_{i_2}.e_{i_1}$$

Clearly, $*$ is the unique involution of $Cl(V)$ satisfying:

$$v^* = v \;\text{ for }\; v \in V \subset Cl(V) \;\text{ and }\; (a.b)^* = b^*.a^* \quad \text{for all }\; a, b \in Cl(V)$$

As an exercise, the reader may explicitly compute the involution $*$ on the Clifford algebras $Cl(\mathbb{R})$, $Cl(\mathbb{R}^2)$ and $Cl(\mathbb{R}^3)$ that were determined above.

Definition 13.1.7 (Supercommutators)**.** For a superalgebra $A = A^0 \oplus A^1$ (such as the Clifford algebra), define the *supercommutator* of two *homogeneous* elements $x, y \in A$ by:

$$[x, y]_s := xy - (-1)^{(\deg x)(\deg y)} yx$$

Extend to arbitrary elements of A by linearity in each slot. For example, if $A = \Lambda^* V = \Lambda^{ev} \oplus \Lambda^{odd}$, then the supercommutator of any two elements is 0.

For a superalgebra A as above, define the *supercentre* of A by:

$$Z_s(A) := \{x : \; [x, y]_s \equiv 0 \;\text{ for all }\; y \in A\}$$

Lemma 13.1.8. Let V be a positive definite inner product space. Then the supercentre of the Clifford algebra $Cl(V)$ consists of the scalars $\mathbb{R}.1$.

Proof: It is clear that $\mathbb{R}.1 \subset Z_s(Cl(V))$, since the supercommutator of any scalar with any element is just the usual commutator, and the scalars commute with everything in $Cl(V)$. On the other hand, we claim that if $[x,v]_s = 0$ for all $v \in V$, then x is a scalar. For, write $x = x_0 + x_1$, with $x_i \in Cl^i(V)$ in terms of its homogeneous components. Then $[x,v]_s = [x_0,v]_s + [x_1,v]_s$, and since v has homogeneous degree 1, we have $[x_0,v]_s$ is homogeneous of degree 1, and $[x_1,v]_s$ is homogeneous of degree 0. Thus both $[x_0,v]_s$ and $[x_1,v]$ are individually 0, for all $v \in V$. So it is enough to prove that if $x \in Z_s(Cl(V))$ is homogeneous and $[x,v]_s = 0$ for all $v \in V$, then $x = \lambda.1$.

Let $\{e_i\}$ be an orthonormal basis of V. Write the homogeneous element x as $x = a + e_1.b$, where a and b are independent of e_1 (by using the basis e_I of $Cl(V)$ constructed in (iii) of 13.1.2). Then deg $a =$ deg $x =$ deg b+1. Hence

$$\begin{aligned} [x.e_1]_s &= [a,e_1]_s + [e_1 b, e_1]_s = ae_1 - (-1)^{\deg a} e_1 a + e_1 b e_1 - (-1)^{\deg a} b e_1^2 \\ &= ae_1 + (-1)^{\deg a}(-1)^{\deg a} ae_1 + (-1)^{\deg b} b e_1^2 + (-1)^{\deg a+1} b e_1^2 \\ &= (-1)^{\deg b} 2 b e_1^2 = (-1)^{\deg a} 2b \end{aligned}$$

So that $[x,e_1]_s = 0$ implies that $b = 0$. Thus $x = a + e_1 b = a$ is independent of e_1. By the same reasoning, it is independent of e_i for all i, and hence a scalar. This proves the lemma. □

Remark 13.1.9. Note that the usual *centre* of $Cl(V)$ is usually much larger than the scalars. For example, in $Cl(\mathbb{R}) = \mathbb{C}$, the centre is all of $Cl(\mathbb{R})$.

13.2. **The Groups Pin and Spin.**

Let $V = \mathbb{R}^n$ with its usual positive definite euclidean inner product. Recall the involution $*$ introduced in Definition 13.1.6.

Definition 13.2.1 (Pin(n) and Spin(n))**.** Define the group

$$\text{Pin}(n) := \{x \in Cl(V) : x \text{ is homogeneous}, \ \ xx^* = x^*x = (-1)^{\deg x}, \ xVx^* \subset V\}$$

Further define

$$\text{Spin}(n) = \text{Pin}(n) \cap Cl^0(V)$$

Note that by definition, $x^* = x^{-1}$ for all $x \in \text{Spin}(n)$. Also, since the group $Cl^\times(V)$ of invertible elements in $Cl(V)$ is an open subset of the euclidean space $Cl(V)$, and the operation of Clifford multiplication is algebraic (by using a basis) and hence smooth, it follows that the closed conditions defining Pin(n) and Spin(n) make them closed subgroups of $Cl^\times(V)$. Hence both are Lie groups by Cartan's theorem.

Note that by definition, there is an action of Pin(n) on $V = \mathbb{R}^n$ given by:

$$\rho : \text{Pin}(n) \to GL(n,\mathbb{R})$$

where $\rho(x)v = xvx^*$. We have the following proposition.

Proposition 13.2.2 (Basic facts on Pin(n), Spin(n) and ρ)**.**

(i): $\rho(\text{Pin}(n)) \subset O(n,\mathbb{R})$. The sequence:

$$1 \to \mathbb{Z}_2 \to \text{Pin}(n) \xrightarrow{\rho} O(n,\mathbb{R}) \to 1$$

is exact. (Here $\mathbb{Z}_2 = \{+1,-1\} \subset \text{Spin}(n) \subset \text{Pin}(n)$)

(ii): Any element $x \in \text{Pin}(n)$ may be expressed as a Clifford product:

$$x = v_1 v_2 ... v_k$$

where v_i are some unit vectors in V.

(iii): $\rho(\text{Spin}(n)) \subset SO(n)$ and the sequence

$$1 \to \mathbb{Z}_2 \to \text{Spin}(n) \xrightarrow{\rho} SO(n) \to 1$$

is exact. An element $x \in \text{Spin}(n)$ iff it is a Clifford product $x = v_1...v_k$ with v_i unit vectors in V, and k is even.

(iv): $\text{Spin}(n)$ is connected.

(v): The Lie algebra map $\dot{\rho}$ maps the element $\frac{1}{4}\sum_{i\neq j} a_{ij}e_ie_j \in Cl(V)$ to skew-symmetric matrix $[a_{ij}]$ in the lie algebra $\text{Lie}(\text{Spin}(n)) = \mathfrak{so}(n)$ thus identifying the above Lie algebra with a subspace of $Cl(V)$.

Proof: Note that for $v \in V \subset Cl(V)$, we have $\|v\|^2 .1 = -v^2$. Further, for $x \in \text{Pin}(n)$, we have $\rho(x)v \in V$ as well, so that

$$\begin{aligned} \|\rho(x)v\|^2 .1 &= -(\rho(x)v)^2 = -(xvx^*xvx^*) = -(-1)^{\deg x}xv^2x^* \\ &= (-1)^{\deg x}x\|v\|^2 .1.x^* = (-1)^{2\deg x}\|v\|^2 .1 = \|v\|^2 .1 \end{aligned}$$

which proves the first assertion of (i).

If $\rho(x) = Id_V$ for $x \in \text{Pin}(n)$, then $xvx^* = v$ for all $v \in V$. That is, $xv = (-1)^{\deg x}vx$. That is, the supercommutator $[x, v]_s = 0$ for all $v \in V$. In the proof of Lemma 13.1.8, we remarked that this forces $x = \lambda 1$ and $\deg x = 0$. Thus $x^* = \lambda.1$, and $xx^* = (-1)^{\deg x}.1 = 1$ implies that $\lambda^2 = 1$, or $\lambda = \pm 1$. So $\ker \rho = \mathbb{Z}_2$.

If we let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $V = \mathbb{R}^n$, we note that $e_i^* = e_i$, and hence $e_ie_i^* = -1 = (-1)^{\deg e_1}.1$. Clearly $\rho(e_i)e_i = e_ie_ie_i = -e_i$ Also we have:

$$\rho(e_i)e_j = e_ie_je_i = -e_i^2e_j = e_j \text{ for } j \neq i$$

Thus $e_iVe_i^* \subset V$, and $e_i \in \text{Pin}(n)$ for all i. The above calculation shows that $\rho(e_i)$ is orthogonal reflection about the hyperplane $(\mathbb{R}e_i)^\perp$ in V. Since each unit vector $v \in V$ can be completed to an orthonormal basis, it follows that every unit vector $v \in V \subset Cl(V)$ is in $\text{Pin}(n)$, and $\rho(v)$ is just the reflection about the hyperplane $(\mathbb{R}v)^\perp \subset V$. Since the group $O(n, \mathbb{R})$ is generated by relections about hyperplanes, it follows that $\rho : \text{Pin}(n) \to O(n, \mathbb{R})$ is surjective. This proves the exact sequence of (i), and (i) follows.

For any $x \in \text{Pin}(n)$, we have $\rho(x) \in O(n, \mathbb{R})$, and indeed we saw in the last paragraph that $\rho(x) = \rho(v_1...v_n)$ for some unit vectors v_i. This means that $x = \pm v_1...v_k = (\pm v_1)..v_k$, and (ii) follows.

Since $\deg(v_1...v_k) = k \mod 2$, from (ii) it follows that an element $x \in \text{Pin}(n)$ lies in $Cl^0(V)$ iff x can be expressed as a Clifford product of an *even* number of unit vectors. Since the set of elements in $O(n, \mathbb{R})$ expressible as products of an even number of reflections is precisely $SO(n)$, all the assertions of (iii) follow trivially from (i) and (ii).

Since an element $x \in \text{Spin}(n)$ is expressible as a Clifford product

$$x = v_1....v_{2m}$$

where $v_i \in V$ are unit vectors, to connect x by a path in $\text{Spin}(n)$ to the identity element 1, it is enough to connect the pairwise doublet elements $v_{2i-1}v_{2i}$ to 1 by a path $y_i(t)$ in $\text{Spin}(n)$ (so that $\prod_{i=1}^m y_i(t)$ will be required path connecting x to 1). So let v, w be unit vectors in V, and let us find a path in $\text{Spin}(n)$ connecting $v.w$ to 1. If w is linearly dependent on v, then $v.w = \pm 1$, and it is trivial to connect it to 1. So assume w and v are linearly indpendent. Let e be a unit vector in the span $\mathbb{R}v + \mathbb{R}w$ which is perpendicular to v. Then letting $e(t)$ be a path of unit vectors in V joining w to e, we see that $v.w$ can be joined to $v.e$ by the path $v.e(t)$ in $\text{Spin}(n)$.

Hence, we may assume without loss of generality that the unit vectors v and w are orthogonal, and exhibit a path joining $v.w$ to 1 in $\text{Spin}(n)$. Consider the path:

$$x(t) = (\cos t)1 + (\sin t)v.w$$

we clearly have $x^*(t) = (\cos t)1 + (\sin t)w.v$ and

$$\begin{aligned} x(t)x^*(t) &= [(\cos t)1 + (\sin t)v.w][(\cos t)1 + (\sin t)w.v] \\ &= (\cos^2 t + \sin^2 tv.w.w.v)1 + \sin t\cos t(v.w + w.v) = (\cos^2 t + (-1)^2\sin^2 t)1 + \cos t\sin t(2\langle v, w\rangle) \\ &= (\cos^2 t + \sin^2 t)1 + \cos t\sin t(0) = 1 \end{aligned}$$

It is also easy to check that $x(t)Vx(t)^* \subset V$, (In fact $\rho(v.w)$ is some planar rotation, and $\rho(x(t))$ joins that planar rotation to the identity element of $SO(n)$). Thus $x(t)$ is the required path joining $v.w$ to 1, and (iv) follows.

To see (v), let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $V = \mathbb{R}^n$. For $i \neq j$, note that

$$(e_ie_j)^2 = -e_ie_ie_je_j = -(-1)(-1) = -1$$

so that $(e_ie_j)^{2m} = (-1)^m$ and $(e_ie_j)^{2m+1} = (-1)^m e_ie_j$. Hence if we take the exponential:

$$\begin{aligned} \exp(te_ie_j) &= 1 + te_ie_j + \frac{t^2}{2!}(e_ie_j)^2 + ... \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + ..\right)1 + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + ..\right)e_ie_j \\ &= (\cos t)1 + (\sin t)e_ie_j \end{aligned}$$

We have seen above that this is precisely the path joining 1 to e_ie_j in Spin(n). We can compute its derivative at $t = 0$ as

$$\frac{d(\exp(te_ie_j)}{dt}_{|t=0} = (-\sin t.1 + (\cos t)e_ie_j)_{|t=0} = e_ie_j$$

which shows that all these elements e_ie_j for $i \neq j$ lie in the Lie algebra of Spin(n). Since the span of $\{e_ie_j\}_{i<j}$ is of dimension $\frac{n(n-1)}{2}$, which is precisely the dimension of $\mathfrak{so}(n) = \text{Lie}(\text{Spin}(n))$, it follows that this last Lie algebra *is* the linear span of $\{e_ie_j\}_{i<j}$. To get the isomorphism even more explicitly, note that

$$\begin{aligned} \rho(\exp(te_ie_j))e_i &= (\cos t + \sin te_ie_j)e_i(\cos t + \sin te_je_i) = (\cos^2 t)e_i + (\sin^2 t)(e_ie_je_ie_je_i) + 2\sin t\cos t(e_ie_je_i) \\ &= (\cos^2 t - \sin^2 t)e_i + (2\sin t\cos t)e_j = (\cos 2t)e_i + (\sin 2t)e_j \end{aligned}$$

Similarly, one verifies that

$$\rho(\exp(te_ie_j))e_j = (-\sin 2t)e_i + (\cos 2t)e_j$$

and also that since e_ie_j commutes with e_k for all $k \neq i, k \neq j$, we have $\rho(\exp(te_ie_j))e_k = e_k$ for all $k \neq i$, $k \neq j$. As a consequence,

$$\rho(\exp(te_ie_j)) = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}$$

where the matrix on the right is a rotation in the e_i, e_j plane of V. Thus

$$\dot\rho(e_ie_j) = \frac{d(\rho(\exp(te_ie_j))}{dt}_{|t=0} = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So that $\dot\rho\left(\sum_{i<j} a_{ij}e_ie_j\right) = 2[a_{ij}]$ and thus

$$\dot\rho\left(\frac{1}{4}\sum_{i\neq j} a_{ij}e_ie_j\right) = [a_{ij}]$$

for a skew-symmetric real matrix $[a_{ij}]$. This proves (v), and the proposition follows. □

Example 13.2.3. It is easy to check that $\text{Pin}(1) = \mathbb{Z}_2$, and $\text{Spin}(1) = \{1\}$. Note that $Cl(\mathbb{R}^2) = \mathbb{H}$, and the operation $*$ on $Cl(\mathbb{R}^2)$ is the map defined by $i^* = e_1^* = e_1 = i$, $j^* = e_2^* = e_2 = j$ and $k^* = (e_1e_2)^* = e_2e_1 = -e_1e_2 = -k$. Also $Cl^0(\mathbb{R}^2) = \mathbb{R}1 + \mathbb{R}k$, and $Cl^1(\mathbb{R}^2) = \mathbb{R}i + \mathbb{R}j$. If $x = a1 + bk \in Cl^0$ (resp. $x = ai + bj \in Cl^1$), then $xx^* = a^2 + b^2$ (resp. $-a^2 - b^2$), and also $x(\alpha e_1 + \beta e_2)x^* \in V$ in both cases. Hence $\text{Pin}(2) = S^1 \times \mathbb{Z}_2$, and $\text{Spin}(2) = S^1 = \{a1 + bk : a^2 + b^2 = 1\}$. It is also verified easily that for $x = (\cos t)1 + (\sin t)k \in \text{Spin}(2)$:

$$\rho\left((\cos t)1 + (\sin t)k\right) = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}$$

so that the map $\rho : \text{Spin}(2) = S^1 \to SO(2) = S^1$ is just the squaring map.

Finally, since $Cl^0(\mathbb{R}^3) = \mathbb{H} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ where $i = e_1e_2$, $j = e_2e_3$ and $k = e_3e_1$ (see Exercise 13.1.4), we have $1^* = 1$, $i^* = -i$, $j^* = -j$ and $k^* = -k$. Thus, for a quaternion $x = a1 + bi + cj + dk \in Cl^0$, $x^* = a1 - bi - cj - dk$, the conjugate quaternion, and $xx^* = 1$ implies $a^2 + b^2 + c^2 + d^2 = 1$, viz., x is a unit length quaternion. It is again clear that $xVx^* \subset V$, so that Spin(3) is the group of unit quaternions, homeomorphic to S^3. Further, one easily computes that the homomorphism $\rho : \text{Spin}(3) \to SO(3)$ is given by:

$$\rho(a + bi + cj + dk) = \begin{pmatrix} (a^2 + c^2) - (b^2 + d^2) & 2(cd - ab) & 2(bc + ad) \\ 2(ab + cd) & (a^2 + d^2) - (b^2 + c^2) & 2(bd - ac) \\ 2(bc - ad) & 2(bd - ac) & (a^2 + b^2) - (c^2 + d^2) \end{pmatrix}$$

Note that $\rho(-x)$ and $\rho(x)$ are the same, as they should be. Also, recalling the central element $\eta = e_1e_2e_3 \in Cl^1(\mathbb{R}^3)$ we see that $\eta^* = e_3e_2e_1 = -\eta$, so that $\rho(\eta)e_i = -\eta e_i \eta = -\eta\eta e_i = -e_i$. Thus $\rho(\eta) = -I \in O(3, \mathbb{R})$, and also $\text{Pin}(3) = \text{Spin}(3) \coprod \text{Spin}(3)\eta = \text{Spin}(3) \times \mathbb{Z}_2$, since η is central.

13.3. **Spin structures on manifolds.** Let M be a connected oriented Riemannian manifold of dimension n. There is the *orthonormal oriented frame bundle*:

$$SO(n) \to P \to M$$

whose fibre is

$$P_x = \{\text{oriented orthonormal frames in } T_xM\} \simeq SO(n)$$

Definition 13.3.1. We say that M has a *spin structure* if there exists a principal $\text{Spin}(n)$ bundle $\widetilde{P} \to M$ and a double-covering map $\rho : \widetilde{P} \to P$ so that the following diagram commutes:

$$\begin{array}{ccccc} \text{Spin}(n) & \to & \widetilde{P} & \to M \\ \rho_x \downarrow & & \downarrow \rho & Id \downarrow \\ SO(n) & \to & P & \to M \end{array}$$

There is a handy criterion for the existence of a spin structure on M, as also a way of parametrising all possible spin structures on M. Namely,

Proposition 13.3.2. The oriented Riemannian manifold M as above has a spin structure iff the second Stiefel-Whitney class $w_2(M) = 0$. Furthermore, if there does exist a spin structure on M, then the set of all spin-structures on M is in bijective correspondence with $H^1(M, \mathbb{Z}_2)$.

Proof: Let $SO(n) \xrightarrow{i} P \to M$ be the principal $SO(n)$ bundle as above, and consider the Serre spectral sequence of this fibration with $\mathbb{Z}_2$ coefficients. Then:

$$E_2^{p,q} = H^p(M, H^q(SO(n), \mathbb{Z}_2)) \Rightarrow H^{p+q}(P, \mathbb{Z}_2)$$

Note that we have the exact sequence:

$$0 \to E_3^{0,1} \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to E_3^{2,0} \to 0$$

since $d_2 : E_2^{-2,2} \to E_2^{0,1}$ and $d_2 : E_2^{2,0} \to E_2^{4,-1}$ are zero maps, the spectral sequence being first quadrant. For the same reason, $d_r : E_r^{0,1} \to E_r^{r,2-r}$ and $d_r : E_r^{2,0} \to E_{r+2,1-r}$ are zero maps for $r \geq 3$, and so $E_3^{0,1} = E_\infty^{0,1}$ and $E_3^{2,0} = E_\infty^{2,0}$. Since

$$E_\infty^{0,1} = F^0(H^1(P, \mathbb{Z}_2))/F^1(H^1(P, \mathbb{Z}_2))$$

and $F^0(H^1(E,\mathbb{Z}_2)) = H^1(E,\mathbb{Z}_2)$, we have a natural quotient map $H^1(E,\mathbb{Z}_2) \to E_\infty^{0,1} = E_3^{0,1}$. Noting that $E_2^{0,1} = H^0(M, H^1(SO(n),\mathbb{Z}_2)) = H^1(SO(n),\mathbb{Z}_2)$ and $E_2^{2,0} = H^2(M, H^0(SO(n),\mathbb{Z}_2)) = H^2(M,\mathbb{Z}_2)$, we have the exact sequence:

$$H^1(P,\mathbb{Z}_2) \xrightarrow{i^*} H^1(SO(n),\mathbb{Z}_2) \xrightarrow{\delta} H^2(M,\mathbb{Z}_2) \tag{42}$$

The first map is i^* by applying the functoriality of the Serre spectral sequence to the inclusion of a point into M, and called an "edge homomorphism". The image $\delta(1)$ of the generator $1 \in H^1(SO(n),\mathbb{Z}_2) = \mathbb{Z}_2$ is the Stiefel-Whitney class $w_2(M)$. Also, $\ker\delta = \operatorname{Im} i^*$ by exactness of (42).

M has a spin structure iff there is a double cover $\widetilde{P} \xrightarrow{\rho} P$ which makes the diagram of Definition 13.3.1 commute.

Double covers of P are in 1-1 correspondence with index 2 subgroups of $\pi_1(P)$, which is in bijective correspondence with $\hom_{\mathbb{Z}}(\pi_1(P),\mathbb{Z}_2)$. But this last group is precisely $H^1(P,\mathbb{Z}_2)$. Hence ρ is an element of $H^1(P,\mathbb{Z}_2)$. Since the diagram of Definition 13.3.1 commutes, the restriction of the double cover $\rho : P \to M$ to a point $x \in M$ must correspond to the nontrivial double-cover $\rho_x : \mathrm{Spin}(n) \to SO(n)$. Now the double cover ρ_x is represented by the unique generating element $1 \in \hom_{\mathbb{Z}}(\pi_1(SO(n)),\mathbb{Z}_2) = H^1(SO(n),\mathbb{Z}_2) = \mathbb{Z}_2$. By functoriality, it follows that $i^*(\rho) = 1$. Now, there exists such a $\rho \in H^1(P,\mathbb{Z}_2)$ satisfying $i^*(\rho) = 1$ iff $\delta(1) = w_2(M) = 0$, by the exactness of the sequence (42). This proves that M has a spin structure iff $w_2(M) = 0$, and the first part of the proposition follows.

From the previous para, it also follows that spin structures on M are in 1-1 correspondence with the inverse image $(i^*)^{-1}(1) \in H^1(P,\mathbb{Z}_2)$, where $1 \in H^1(SO(n),\mathbb{Z}_2) = \mathbb{Z}_2$ is the generator. But $(i^*)^{-1}(1)$ is the set-theoretic complement of $(i^*)^{-1}(0) = \ker i^*$ in $H^1(P,\mathbb{Z}_2)$, and has the same cardinality as $\ker i^*$. We claim that this kernel is isomorphic to $H^1(M,\mathbb{Z}_2)$.

Consider the tail-end of the homotopy exact sequence of the fibration $SO(n) \to P \to M$, we have:

$$\pi_1(SO(n)) \xrightarrow{i_*} \pi_1(P) \xrightarrow{\pi_*} \pi_1(M) \to 1$$

so that taking $\hom_{\mathbb{Z}}(-,\mathbb{Z}_2)$ of this sequence, and noting $\hom_{\mathbb{Z}}(\pi_1(X),\mathbb{Z}_2) = H^1(X,\mathbb{Z}_2)$ by Hurewicz and Universal Coefficient Theorems, we have the exact sequence:

$$0 \to H^1(M,\mathbb{Z}_2) \to H^1(P,\mathbb{Z}_2) \xrightarrow{i^*} H^1(SO(n),\mathbb{Z}_2)$$

This shows that $\ker i^* \simeq H^1(M,\mathbb{Z}_2)$, and the proposition is proved. □

Corollary 13.3.3. Every 2-connected Riemannian manifold is an orientable spin manifold.

Example 13.3.4 (Real projective spaces)**.** The real projective space $\mathbb{RP}(n)$ is spin iff $n \equiv 3 \mod 4$. It is well known that $T(\mathbb{RP}(n)) \oplus \epsilon^1 \simeq (\gamma^{1\,*})^{n+1}$, where γ^1 is the tautological line bundle on $\mathbb{RP}(n)$, and ϵ^1 the trivial line bundle. Thus the total Steifel-Whitney class of $\mathbb{RP}(n)$ is given by

$$w(\mathbb{RP}(n)) = (1+x)^{n+1}$$

where $x \in H^1(\mathbb{RP}(n),\mathbb{Z}_2)$ is the generator, and the first Stiefel-Whitney class of $\gamma^{1\,*}$. So

$$w_2(\mathbb{RP}(n)) = \frac{(n+1)n}{2}x^2$$

Now $\mathbb{RP}(n)$ is orientable iff $n = 2k+1$, and in this event $w_2(\mathbb{RP}(n)) = (k+1)(2k+1)x^2$. This is zero iff k is odd, i.e. iff $n = 2(2m+1)+1 = 4m+3$.

Example 13.3.5 (Complex projective spaces)**.** The complex projective space $\mathbb{CP}(n)$ is spin iff n is odd. For, there is again the equivalence of complex vector bundles:

$$T(\mathbb{CP}(n)) \oplus \epsilon^1_{\mathbb{C}} \simeq (\gamma^{1\,*})^{n+1}$$

where γ^1 is the complex tautological line bundle on $\mathbb{CP}(n)$. Thus the total Chern class of $\mathbb{CP}(n)$ is given by

$$c(\mathbb{CP}(n)) = (1+x)^{n+1}$$

where $x \in H^2(\mathbb{CP}(n), \mathbb{Z})$ is the generator, and the first Chern class of $\gamma^{1\,*}$. This shows that the first Chern class

$$c_1(\mathbb{CP}(n)) = (n+1)x$$

It is a fact that w_2 of a complex vector bundle considered as a real bundle is the $\mod 2$ reduction of its first Chern class. Hence $w_2(\mathbb{CP}(n)) = 0$ iff $(n+1)$ is even, i.e. iff n is odd.

Exercise 13.3.6. Using the identity $T(G_k(\mathbb{R}^n)) \simeq \hom(\gamma^k, \gamma^{k,\perp})$, and arguments similar to the ones above, investigate which real grassmannians are spin. Likewise for complex grassmannians.

14. Representations

14.1. **Clifford Modules.** Let V be a real inner product space with positive definite inner product $\langle -,- \rangle$. Let $Cl(V)$ be the corresponding Clifford algebra.

Definition 14.1.1. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that an $\mathbb{F}$-vector space S is an $\mathbb{F}$-*Clifford module* if there is a unital $\mathbb{R}$- algebra homomorphism:

$$\rho : Cl(V) \to \hom_{\mathbb{F}}(S,S)$$

Example 14.1.2. Letting $S = Cl(V)$, and letting $\rho(x)y = x.y$ (left Clifford multiplication by x) or $\rho(x)y = y.x^*$ (right Clifford multiplication by x^*) turns $CL(V)$ into an $\mathbb{R}$- Clifford module. These are called the left (resp. right) regular representations.

A very important $\mathbb{R}$-Clifford module over $Cl(V)$ is the exterior algebra Λ^*V. To describe it, we let $\{e_i\}_{i=1}^n$ be an orthonormal basis of V with respect to $\langle -,- \rangle$. We also have the $\mathbb{R}$-linear *Hodge-star operator*

$$* : \Lambda^k(V) \to \Lambda^{n-k}(V)$$

which is defined on the basis elements of $\Lambda^k(V)$ by:

$$*(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = (-1)^\sigma e_{j_1} \wedge e_{j_2} \ldots \wedge e_{j_{n-k}}$$

where $\{i_1, \ldots, i_k, j_1, .., j_{n-k}\} = \{1, 2, .., n\}$, and $(-1)^\sigma$ is the sign of the permutation $\sigma = (i_1\, i_2 \ldots, j_1, .., j_{n-k})$. We note that with this definition,

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle\, \omega_V$$

where $\omega_n = e_1 \wedge .. \wedge e_n \in \Lambda^n(V)$ is the oriented volume element of V, and $\langle \alpha, \beta \rangle$ is the canonical inner product on $\Lambda^k(V)$ induced by $\langle -,- \rangle$ on V (it is the inner product which makes $\{e_{i_1} \wedge e_{i_2} \ldots . \wedge e_{i_k}\}$ an orthonormal basis for $\Lambda^k(V)$). It is readily checked that this inner product on $\Lambda^k(V)$ and the oriented volume element ω_V do not depend on the choice of orthonormal basis, and hence the $*$-operator is invariantly defined. It is easily checked that the square $* \circ *$ is scalar multiplication by $(-1)^{k(n-k)}$ on $\Lambda^k(V)$.

Definition 14.1.3 (Interior multiplication)**.** For $v \in V$, and $\alpha \in \Lambda^k(V)$, define the element:

$$v \lrcorner\, \alpha := (-1)^{nk+n}(*(v \wedge *\alpha)) \in \Lambda^{k-1}(V)$$

Lemma 14.1.4. Interior multiplication above satisfies the following:

(i): $\langle v\wedge\alpha,\beta\rangle = \langle\alpha, v\lrcorner\beta\rangle$ for $\alpha\in\Lambda^k(V)$, $\beta\in\Lambda^{k+1}(V)$, and $v\in V$.

(ii): The composite:
$$\Lambda^k(V)\stackrel{v\lrcorner}{\to}\Lambda^{k-1}(V)\stackrel{v\lrcorner}{\to}\Lambda^{k-2}(V)$$
is zero.

(iii): For a vector $v\in V$, and $\alpha\in\Lambda^k(V)$, we have:
$$v\wedge(v\lrcorner\alpha)+v\lrcorner(v\wedge\alpha)=\langle v,v\rangle\,\alpha$$

Proof: By the discussion on the $*$-operator, we have
$$\begin{aligned}\langle v\wedge\alpha,\beta\rangle\,\omega_V &= (v\wedge\alpha)\wedge *\beta = (-1)^k\alpha\wedge v\wedge *\beta\\ &= (-1)^k(-1)^{(n-k)k}\alpha\wedge(**)(v\wedge *\beta) = (-1)^{nk}\alpha\wedge *[*(v\wedge *\beta)]\\ &= (-1)^{nk}(-1)^{n(k+1)+n}\alpha\wedge *(v\lrcorner\beta) = \langle\alpha, v\lrcorner\beta\rangle\,\omega_V\end{aligned}$$
which proves (i). Thus interior multiplication $v\lrcorner(-)$ is the adjoint to $v\wedge(-)$ with respect to $\langle -,-\rangle$ on $\Lambda^*(V)$.

Since $v\wedge(v\wedge\alpha)\equiv 0$, the adjoint formula (i) implies (ii).

We note that for the basis vector e_1, and an element $\beta\in\Lambda^*$ such that β does not involve e_1 anywhere, $e_1\lrcorner\beta$ is orthogonal to all γ not involving e_1, since $\langle e_1\lrcorner\beta,\gamma\rangle = \langle\beta, e_1\wedge\gamma\rangle = 0$. Further $\langle e_1\lrcorner\beta, e_1\wedge\gamma\rangle = \langle\beta, e_1\wedge e_1\wedge\gamma\rangle = 0$. It follows that $e_1\lrcorner\beta = 0$, if β does not involve e_1. On the other hand $e_1\lrcorner(e_1\wedge\gamma)=\gamma$ for all γ not involving e_1, as is easily checked again by taking inner products of both sides with various τ, and using (i). Now, for a general $\alpha\in\Lambda^k(V)$, write $\alpha=\gamma+e_1\wedge\beta$, where β and γ do not involve e_1. Then $e_1\wedge\alpha = e_1\wedge\gamma$. Also $e_1\lrcorner\alpha=\beta$. Hence
$$e_1\lrcorner(e_1\wedge\alpha)+e_1\wedge(e_1\lrcorner\alpha) = e_1\lrcorner(e_1\wedge\gamma)+e_1\wedge\beta = \gamma+e_1\wedge\beta=\alpha$$
Since any unit vector v can be completed to an orthonormal basis, the above formula is true for all unit vectors $v\in V$. For a general v, apply this formula to $\frac{v}{\|v\|}$, and (iii) follows. □

Proposition 14.1.5. The exterior algebra $\Lambda^*(V)$ is an $\mathbb{R}$-Clifford module over $Cl(V)$. The action is uniquely determined by the action of $v\in V\subset Cl(V)$, and that in turn is given by:
$$v.\alpha = v\wedge\alpha - v\lrcorner\alpha \quad\text{for}\ \ \alpha\in\Lambda^*(V)$$
Finally, the action of $v\in V$ above is skew-symmetric with respect to the natural inner-product $\langle -,-\rangle$ on $\Lambda^*(V)$.

Proof: We define the action:
$$v.\alpha := v\wedge\alpha - v\lrcorner\alpha$$
To extend this action to all of $Cl(V)$, by the universal property of Clifford algebras, we need to check that $v^2\alpha = v.v.\alpha = -\langle v,v\rangle\,\alpha$ for all $\alpha\in\Lambda^*(V)$. However:
$$\begin{aligned}v.v.\alpha &= v\wedge(v\wedge\alpha - v\lrcorner\alpha) - v\lrcorner(v\wedge\alpha - v\lrcorner\alpha)\\ &= -v\wedge(v\lrcorner\alpha) - v\lrcorner(v\wedge\alpha) = -\langle v,v\rangle\,\alpha\end{aligned}$$
by using (ii) and (iii) of the previous Lemma 14.1.4.

Finally, by using (i) of the previous Lemma 14.1.4, we have
$$\langle v.\alpha,\beta\rangle = \langle v\wedge\alpha,\beta\rangle - \langle v\lrcorner\alpha,\beta\rangle = \langle\alpha, v\lrcorner\beta\rangle - \langle\alpha, v\wedge\beta\rangle = -\langle\alpha, v.\beta\rangle$$
which shows that the action of $v\in V$ is skew symmetric with respect to $\langle -,-\rangle$. Hence the proposition. □

Exercise 14.1.6. Show that:

$$v \lrcorner (w_1 \wedge w_2 \wedge ... \wedge w_k) = \sum_{i=1}^{k} (-1)^i \langle v, w_i \rangle \, w_1 \wedge w_2 \wedge ... \widehat{w_i} ... \wedge w_k$$

where the hat denotes omission. (Simplest to just use a basis, or the adjointness formula $\langle v \wedge \alpha, \beta \rangle = \langle \alpha, v \lrcorner \beta \rangle$ in (i) of 14.1.4.)

Lemma 14.1.7. The representation of $Cl(V)$ on $\Lambda^*(V)$ above has the following further properties:

(i): The map

$$\begin{aligned} \sigma : Cl(V) &\rightarrow \Lambda^*(V) \\ x &\mapsto x.1 \end{aligned}$$

is an $\mathbb{R}$-vector space isomorphism, called the *symbol map*. For a multiindex $I = (i_1 < i_2 < ... < i_k)$, we have $\sigma(e_{i_1} e_{i_2} .. e_{i_k}) = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$.

(ii): The inverse of σ is the $\mathbb{R}$-linear map $c : \Lambda^*(V) \rightarrow Cl(V)$ called the *quantisation map*. It obeys $c(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}) = e_I$.

(iii): The representation above complexifies to a representation:

$$\mathbb{C}l(V) := Cl(V) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \Lambda^*_{\mathbb{C}}(V) = \Lambda^*(V) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^*(V_{\mathbb{C}})$$

and the symbol and quantisation maps extend to the complexifications.

(iv): [Action of the volume element] Give $\mathbb{R}^n$ its usual euclidean inner product and denote the corresponding Clifford algebra $Cl(\mathbb{R}^n)$ as Cl_n. Let $\omega_n = e_1 ... e_n$ be the *volume element* of Cl_n. Then:

(a): $\omega_n v + v \omega_n = 0$ for n even and $\omega_n v = v \omega_n$ for n odd and all $v \in \mathbb{R}^n$. Hence, for n odd, ω_n commutes with everything and is a central element in Cl_n. For n even, ω_n commutes with Cl_n^0 and anticommutes with Cl_n^1.

(b): $\omega_n^2 = (-1)^p$ where $p = \left[\frac{n+1}{2}\right]$, the integral part of $\frac{n+1}{2}$. Hence $\omega_n^2 = -1$ for $n \equiv 1, 2 \mod 4$ and $\omega^2 = 1$ for $n \equiv 0, 3 \mod 4$.

(c): The action of ω_n on $\Lambda^*(V)$ is related to the Hodge-star operator by:

$$\omega.\phi = (-1)^{nk + \frac{k(k-1)}{2}} * \phi \quad \text{for} \quad \phi \in \Lambda^k(V)$$

(v): [Chirality element] In the complexification $\mathbb{C}l_n = CL_n \otimes_{\mathbb{R}} \mathbb{C}$, define the *complex volume element* or *chirality* element:

$$\tau_n := (\sqrt{-1})^p \omega_n \qquad \text{where} \ \ p = \left[\frac{n+1}{2}\right]$$

the box brackets denoting the greatest integer part. By (b) of (iv) above

$$\tau_n^2 = 1 \ \text{ for all } \ n$$

Since ω_n is central for all $n \equiv 1 \mod 2$, we have τ_n is central for all $n \equiv 1 \mod 2$. τ_n is related to the Hodge-star operator by:

$$\tau_n.\phi = i^{p + k(2n+k-1)} * \phi \quad \text{for} \ \ \phi \in \Lambda^k_{\mathbb{C}}(\mathbb{R}^n)$$

In particular, if $n = 4m$ and $k = 2m$, we have $\tau_n \phi = *\phi$. (Chirality coincides with Hodge-star on middle dimension for $n = 4m$).

Proof: Note that, denoting the $Cl(V)$ action with a dot, we have:

$$e_i.1 = e_i \wedge 1 - e_i \lrcorner 1 = e_i \text{ for } 1 \in \Lambda^0(V)$$

and so for $I = (i_1 < i_2 < ... < i_k)$, it follows that:

$$\begin{aligned} e_I.1 &= e_{i_1}e_{i_2}..e_{i_k}.1 = (e_{i_1}e_{i_2}..e_{i_{k-1}}).e_{i_k} = (e_{i_1}e_{i_2}..e_{i_{k-2}}).(e_{i_{k-1}} \wedge e_{i_k} - e_{i_{k-1}} \lrcorner e_{i_k}) \\ &= (e_{i_1}e_{i_2}..e_{i_{k-2}}).(e_{i_{k-1}} \wedge e_{i_k}) = ... = e_{i_1} \wedge e_{i_2} \wedge .... \wedge e_{i_k} \end{aligned}$$

because $e_l \lrcorner (e_{j_1} \wedge e_{j_2} \wedge ..e_{j_m}) = 0$ if $l \neq j_i$ for all i. This proves (i).

(ii) follows immediately from (i), since $c = \sigma^{-1}$. c is called the *quantisation map* because all supercommutators are 0 in $\Lambda^*(V)$, but not in $Cl(V)$, and c puts a "non- supercommuting" algebra structure on the supercommutative algebra $\Lambda^*(V)$.

(iii) is obvious from definitions.

For (iv), note that for any n, $e_ie_j + e_je_i = 2\delta_{ij}$ implies that

$$e_i\omega_n = (-1)^{n-1}\omega_n e_i$$

so we have (a) of (iv). We also have:

$$\omega_n^2 = \omega_{n-1}e_n\omega_{n-1}e_n = (-1)^{n-1}\omega_{n-1}^2 e_n^2 = (-1)^n\omega_{n-1}^2$$

So that $\omega_{4k+4}^2 = \omega_{4k+3}^2 = -\omega_{4k+2}^2 = -\omega_{4k+1}^2 = \omega_{4k}^2 = ...\omega_0^2 = 1$, and (b) follows.

To see (c) of (iv), note that if $e_I = e_{i_1}..e_{i_k} \in Cl_n$, and J is any multi-index with $J = \{1, 2, .., n\} \setminus I$, then $e_Ie_J = (-1)^\sigma\omega_n$ where σ is the permutation

$$\sigma = (i_1, i_2, ..., j_1, .., j_{n-k})$$

Also note that $e_Ie_I = (-1)^{\frac{k(k+1)}{2}}$. Now by (a)

$$\begin{aligned} \omega_n e_I &= (-1)^{k(n-1)}e_I.\omega_n = (-1)^{k(n-1)+\sigma}e_Ie_Ie_J \\ &= (-1)^{k(n-1)+\frac{k(k+1)}{2}}[(-1)^\sigma e_J] = (-1)^{nk+\frac{k(k-1)}{2}}[(-1)^\sigma e_J] \end{aligned}$$

Now apply both sides to $1 \in \Lambda^0$ to get (c).

(v) follows immediately from the definition of τ_n and (iv). When $n = 4m$, $p = \left[\frac{n+1}{2}\right] = 2m$, and for $k = 2m$ the exponent

$$p + k(2n + k - 1) = 2m + 2m(8m + 2m - 1) = 16m^2 + 4m \equiv 0 \mod 4$$

so that $i^{p+k(2n+k-1)} = 1$, and $\tau_{4m}\phi = *\phi$ for $\phi \in \Lambda_{\mathbb{C}}^{2m}(\mathbb{R}^n)$. This proves the lemma. □

Corollary 14.1.8. If W is a $\mathbb{R}$-Clifford module over Cl_n, and $n \equiv 0, 3 \mod 4$, then $W = W^+ \oplus W^-$ as an $\mathbb{R}$-vector space, where $W^\pm = (1 \pm \omega_n)W$ is the (± 1)-eigenspace of the volume element action $\omega_n.(\ )$. If $n \equiv 3 \mod 4$, the centrality of ω_n ensures that $W^\pm$ are both $\mathbb{R}$-Clifford submodules of W. If $n \equiv 0 \mod 4$, then $W^\pm$ are $\mathbb{R}$-modules over Cl_n^0.

Analogously, since $\tau_n^2 = 1$, every $\mathbb{C}$-Clifford module W over Cl_n splits into (± 1)-eigenspaces $W^\pm$ of the chirality element τ_n for all n, both being $\mathbb{C}$-subspaces. Again, if $n \equiv 3 \mod 4$, τ_n is central, and $W^\pm$ are $\mathbb{C}$-Clifford submodules. If $n \equiv 0 \mod 4$, the subspaces $W^\pm$ are modules over Cl_n^0.

Proof: Obvious from (b) of (iv) and (v) of the Lemma 14.1.7 above. When $n \equiv 0 \mod 4$, we note that ω_n (resp. τ_n) anticommutes with all $v \in V$, and hence commutes with Cl^0. Thus $W^\pm$ is a $\mathbb{R}$ (resp. $\mathbb{C}$) module over Cl_n^0. □

Definition 14.1.9. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We will say that an $\mathbb{F}$-Clifford module over $Cl(V)$ is *irreducible* if the only $\mathbb{F}$- Clifford submodules of S are S and $\{0\}$.

Remark 14.1.10. If $n \equiv 3 \mod 4$, then for an irreducible $\mathbb{R}$ (resp. $\mathbb{C}$)-Clifford module W over Cl_n, the volume element ω_n (resp. chirality τ_n) either acts as $+1$ (viz. $W = W^+$) or as -1 (viz. $W = W^-$), and W^+ and W^- (if both exist) are *distinct* irreducible modules. This is obvious because of the Corollary 14.1.8 which asserts that $W^{\pm}$ are $\mathbb{F}$-Clifford submodules of W when $n \equiv 3 \mod 4$, by the centrality of ω_n (resp. τ_n). Also, since Cl_n module equivalence will preserve the sign of $\omega_n.(\ )$ (resp. $\tau_n.(\ )$), it follows that modules on which these operators act as $+1$ are not isomorphic to those on which they act by -1.

Lemma 14.1.11 (Complete reducibility of Clifford modules)**.** Every $\mathbb{F}$-Clifford module is a direct sum of irreducible $\mathbb{F}$-Clifford submodules.

Proof: We define the *Clifford group* in $Cl(V)$ to be the group:

$$\Gamma_n := \{\pm e_I : \ I = (i_1 < i_2 < ... < i_k); 0 \le k \le n\}$$

which is of order 2^{n+1}. For example, Γ_2 is the Hamilton group $\{\pm 1, \pm i, \pm j, \pm k\}$. Denote the element $-1 \in \Gamma_n$ by ν. If we let $\mathbb{R}[\Gamma_n]$ denote the real group algebra over Γ_n, we have a surjective $\mathbb{R}$- algebra homomorphism:

$$\begin{aligned} \rho : \mathbb{R}[\Gamma_n] &\to Cl(V) \\ e_I &\mapsto e_I \\ \nu &\mapsto -1 \end{aligned}$$

Thus there is a 1-1 correspondence between $\mathbb{F}$-modules over $Cl(V)$ and $\mathbb{F}$-modules over $\mathbb{R}[\Gamma_n]$ on which the element ν acts as -1. So let S be a $\mathbb{F}$-module over $Cl(V)$. Then, via ρ, S is a $\mathbb{F}$-module over the algebra $\mathbb{R}[\Gamma_n]$, i.e. a $\mathbb{F}$-module over the Clifford group Γ_n. By averaging over the finite group Γ_n, there alway exists a Γ_n-invariant positive definite real (if $\mathbb{F} = \mathbb{R}$), resp. complex sesquilinear (if $\mathbb{F} = \mathbb{C}$) inner product $\langle -, - \rangle$. on the $\mathbb{F}$-module S. Thus every Γ_n $\mathbb{F}$-submodule will have a Γ_n-invariant orthogonal complement with respect to $\langle -, - \rangle$. It follows that S decomposes into the orthogonal direct sum of finitely many irreducible Γ_n $\mathbb{F}$-submodules S_i. Thus S_i are irreducible $\mathbb{R}[\Gamma_n]$ $\mathbb{F}$-submodules. Since ν is acting as -1 on S, it is acting as -1 on each S_i, so each S_i is a $Cl(V)$ $\mathbb{F}$-submodule. It is clearly irreducible over $Cl(V)$ since it is irreducible over $\mathbb{R}[\Gamma_n]$. The lemma follows. □

So it remains to identify what the irreducible $Cl(V)$ $\mathbb{F}$- modules are. This will be addressed in the following proposition.

Proposition 14.1.12. For $n \equiv 0, 1, 2 \mod 4$, there is exactly one irreducible $\mathbb{R}$-module over Cl_n. For $n \equiv 3 \mod 4$, there are two distinct irreducible $\mathbb{R}$-modules over Cl_n. They are distinguished by the fact that on one the volume element ω_n acts as $(+1)$, and on the other as (-1). The dimensions of these modules are readable from the following list:

$n:$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$	$8k+8$
$d_n:$	2^{4k+1}	2^{4k+2}	2^{4k+2}	2^{4k+3}	2^{4k+3}	2^{4k+3}	2^{4k+3}	2^{4k+4}

For $n \equiv 0 \mod 2$, there is exactly one irreducible $\mathbb{C}$-module over Cl_n, of $\mathbb{C}$-dimension $2^{n/2}$. For $n \equiv 1 \mod 2$, there are exactly two irreducible $\mathbb{C}$-modules over Cl_n, each of $\mathbb{C}$-dimension $2^{\frac{n-1}{2}}$. They are distinguished by the fact that on one the chirality element τ_n acts as $(+1)$ and on the other as (-1).

Proof: We recall the list:

$n:$	1	2	3	4	5	6	7	8
$Cl_n:$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

and the fact that $Cl_{n+8} \simeq Cl_n \otimes_{\mathbb{R}} \mathbb{R}(16)$ from the Remark 13.1.5. Also note that by (i) of Exercise 13.1.4, we have $Cl_3 = \mathbb{H} \oplus \omega\mathbb{H}$, with $\omega^2 = (e_1e_2e_3)^2 = 1$ and ω a central element. This algebra may be rewritten as $(1 + \omega)\mathbb{H} \oplus (1 - \omega)\mathbb{H}$, where $(1 + \omega)(1 - \omega) = 0$, so that $Cl_3 = \mathbb{H} \oplus \mathbb{H}$. Note that $\omega(1 \pm \omega) = (1 \pm \omega)$, so that the two summands in Cl_3 are distinguished by the sign of the action of ω.

The corresponding fact is also true of Cl_7, though we haven't computed it thus far. However, assuming that $Cl_6 = \mathbb{R}(8)$, it is easy to check that $Cl_7^0 \simeq Cl_6$, by taking $e_I e_7 \mapsto \pm e_I$ and $e_J \mapsto e_J$ for all subsets $I, J \subset \{1, 2, .., 6\}$. Now it is easy to check that $Cl_7^1 = \omega Cl_7^0$, and $Cl_7 = Cl_7^0 \oplus \omega Cl_7^0 = (1+\omega)\mathbb{R}(8) \oplus (1-\omega)\mathbb{R}(8) \simeq \mathbb{R}(8) \oplus \mathbb{R}(8)$. So again, the two summands of Cl_7 are distinguished by the sign of Clifford multiplicaton by ω.

From this list it follows that Cl_n is a matrix algebra $\mathbb{K}(k)$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ for $n = 1, 2, 4, 5, 6, 8$ and a sum of two copies of the same matrix algebra $\mathbb{K}(k)$ for $n = 3, 7$. Also, since $\mathbb{K}(k) \otimes_{\mathbb{R}} \mathbb{R}(m) = \mathbb{K}(mk)$, it follows by the 8-periodicity above that Cl_n is a matrix algebra $\mathbb{K}(k)$ for $n \equiv 0, 1, 2, 4, 5, 6 \mod 8$, i.e. $n \not\equiv 3 \mod 4$, and a direct sum of two identical matrix algebras $\mathbb{K}(k)$ for $n \equiv 3, 7 \mod 8$, i.e. $n \equiv 3 \mod 4$.

It is well known that the $\mathbb{K}$-matrix algebra $\mathbb{K}(k)$ is simple, and has *exactly one* irreducible $\mathbb{R}$-module over it, namely $\mathbb{K}^k$, with the obvious left action by matrix multiplication. The direct sum of two copies of $\mathbb{K}(k)$ has *two* distinct irreducible modules over it, viz. $\mathbb{K}^k$ with one action from the first summand, and the other action from the second summand. Thus by the foregoing, the two irreducible modules for $n \equiv 3 \mod 4$ are distinguished by the sign of the action of $\omega_n = e_1 ... e_n$. Letting d_n denote the $\mathbb{R}$-dimension of these irreducible modules, we have the following table:

$n:$	1	2	3	4	5	6	7	8
$d_n:$	2	4	4	8	8	8	8	16

It follows that $Cl_{n+8} = Cl_n \otimes \mathbb{R}(16)$ will have exactly one irreducible $\mathbb{R}$-module for $n \not\equiv 3 \mod 4$ and two distinct ones for $n \equiv 3 \mod 4$. The dimensions of these modules are read off from the above table, and the inductive formula $d_{n+8} = 16 d_n$ arising out of periodicity.

Denote an irreducible $\mathbb{R}$-Clifford module over Cl_n as W_n. Then by the remarks above and the list at the top we have:

n	1	2	3	4	5	6	7	8
W_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$	$\mathbb{R}^{16}$

where the subscript $\pm$ on W_3 and W_7 signifies two distinct irreducible modules, both isomorphic as vector spaces to the entry in that slot. This implies by the periodicity $W_{n+8} = W_n \otimes_{\mathbb{R}} \mathbb{R}^{16}$ that we have the following list of irreducible $\mathbb{R}$-Clifford modules W_n over Cl_n whose $\mathbb{R}$-dimension is d_n:

$n:$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$	$8k+8$
$W_n:$	$\mathbb{C}^{2^{4k}}$	$\mathbb{H}^{2^{4k}}$	$\mathbb{H}_\pm^{2^{4k}}$	$\mathbb{H}^{2^{4k+1}}$	$\mathbb{C}^{2^{4k+2}}$	$\mathbb{R}^{2^{4k+3}}$	$\mathbb{R}_\pm^{2^{4k+3}}$	$\mathbb{R}^{2^{4k+4}}$
$d_n:$	2^{4k+1}	2^{4k+2}	2^{4k+2}	2^{4k+3}	2^{4k+3}	2^{4k+3}	2^{4k+3}	2^{4k+4}

The complex modules are much simpler to describe. Noting that an $\mathbb{R}$- algebra homomorphism:

$$\rho : Cl(V) \to \hom_{\mathbb{C}}(W, W)$$

automatically extends to the complexification $\mathbb{C}l(V) := Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$, we see that a $\mathbb{C}$-Clifford module becomes a $\mathbb{C}l(V)$ module. Noting that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}(2)$ (see (ii) of Exercise 13.1.4), we get the following list of complex Clifford algebras from the real list above:

$n:$	1	2	3	4	5	6	7	8
$\mathbb{C}l_n:$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

which means we have:

$$\begin{aligned} \mathbb{C}l_n &= \mathbb{C}(2^k) \oplus \mathbb{C}(2^k) \text{ for } 1 \le n = 2k+1 \le 8 \\ &= \mathbb{C}(2^k) \text{ for } 1 \le n = 2k \le 8 \end{aligned}$$

Note that for $n = 3, 7$, the two summands in $\mathbb{C}l_n$ are distinguished by the sign of multiplication by the central volume element ω_n. Note also that the chirality elements (see definition in (v) of Lemma 14.1.7) are given by $\tau_3 = -\omega_3$ and $\tau_7 = \omega_7$. Hence, for $n = 3, 7$, the two summands in $\mathbb{C}l_n$ are distinguished by the sign of the action of multiplication by the chirality τ_n.

Since $\mathbb{R}(16)\otimes_{\mathbb{R}}\mathbb{C}=\mathbb{C}(16)$, it follows that

$$\begin{aligned}\mathbb{C}l_{n+8} &= Cl_{n+8}\otimes_{\mathbb{R}}\mathbb{C}=Cl_n\otimes_{\mathbb{R}}\mathbb{R}(16)\otimes_{\mathbb{R}}\mathbb{C}=Cl_n\otimes_{\mathbb{R}}\mathbb{C}(16)=Cl_n\otimes_{\mathbb{R}}(\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}(16))\\ &= (Cl_n\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}\mathbb{C}(16)=\mathbb{C}l_n\otimes_{\mathbb{C}}\mathbb{C}(16)\end{aligned}$$

Thus there is exactly one irreducible $\mathbb{C}$-Clifford module over Cl_n when n is even, and two inequivalent irreducible $\mathbb{C}$- Clifford modules over Cl_n when n is odd. Again, when n is odd, the two inequivalent modules are distinguished by the sign of the action of the chirality τ_n.

Combining the two facts above, we obtain:

$$\begin{aligned}\mathbb{C}l_n &= \mathbb{C}(2^{\frac{n-1}{2}})\oplus\mathbb{C}(2^{\frac{n-1}{2}}) \text{ for } \quad n\equiv 1 \mod 2\\ &= \mathbb{C}(2^{\frac{n}{2}}) \text{ for } \quad n\equiv 0 \mod 2\end{aligned}$$

Since the matrix algebra $\mathbb{C}(k)$ is simple, there is exactly one irreducible $\mathbb{C}$-module over it, viz. $\mathbb{C}^k$ with the obvious action by matrix multiplication. Similarly, over the direct sum algebra $\mathbb{C}(k)\oplus\mathbb{C}(k)$, there are exactly two irreducible ones (each isomorphic as a $\mathbb{C}$-vector space to $\mathbb{C}^k$), coming from the action of the two distinct summands. As noted above, the two summands are distinguished by the sign of the chirality oeprator τ_n. This proves the proposition. □

Remark 14.1.13. Note that from the proposition above, since $d_n<2^n$ for all $n\geq 3$, it follows that the $\mathbb{R}$-Clifford module $\Lambda^*(\mathbb{R}^n)$ of dimension 2^n described in Proposition 14.1.5 is irreducible iff $n=1$ or $n=2$. For the same reason the left and right regular representations of Cl_n on itself is irreducible iff $n=1$ or 2.

To further analyse the real and complex representations of Cl_n, we introduce the notion of a *graded* Clifford module. That is,

Definition 14.1.14. Say that W is a $\mathbb{Z}_2$*-graded* $\mathbb{F}$-Clifford module over $Cl(V)$ (or a $Cl(V)$ $\mathbb{F}$*-supermodule*) if $W=W^0\oplus W^1$, with W^i as $\mathbb{F}$-vector subspaces satisfying:

$$Cl^i(V)W^j\subset W^k \text{ where } k=i+j \mod 2$$

A $\mathbb{C}$-supermodule over $Cl(V)$ can be naturally regarded as a $\mathbb{C}l(V):=Cl(V)\otimes_{\mathbb{R}}\mathbb{C}$ $\mathbb{C}$-supermodule.

Example 14.1.15. If we regard $Cl(V)=Cl^0(V)\oplus Cl^1(V)$ as a module over itself via left regular representation (or right regular multiplication), it becomes a $Cl(V)$ supermodule. Analogously, the decomposition $\mathbb{C}l(V)=\mathbb{C}l(V)^0\oplus\mathbb{C}l(V)^1$ makes $\mathbb{C}l(V)$ a $\mathbb{C}$-supermodule over $\mathbb{C}l(V)$ via left or right regular representation.

Example 14.1.16 (The exterior algebra again)**.** We noted in Proposition 14.1.5 that the exterior algebra $\Lambda^*(V)$ is a $Cl(V)$ module. Hence the summands $\Lambda^e(V)=\oplus_{i=0}^n\Lambda^{2i}(V)=Cl^0V.1$ and $\Lambda^o(V)=\oplus_{i=0}^n\Lambda^{2i+1}(V)=Cl^1(V).1$ gives $\Lambda^*(V)$ the structure of a $Cl(V)$-supermodule, by considering the foregoing example.

In entirely analogous fashion, $\Lambda^*_{\mathbb{C}}(V):=\Lambda^*(V)\otimes_{\mathbb{R}}\mathbb{C}$ becomes a $\mathbb{C}l(V)$ supermodule via the decomposition $\Lambda^*_{\mathbb{C}}(V)=\Lambda^e_{\mathbb{C}}(V)\oplus\Lambda^o_{\mathbb{C}}(V)$ into even and odd degree forms.

Example 14.1.17. Let $n\equiv 0$ or $n\equiv 3 \mod 4$. By the Corollary 14.1.8 the left $\mathbb{R}$-Clifford module Cl_n decomposes into the $(+1)$ and (-1) eigenspaces of ω_n. We denote this decomposition as:

$$Cl_n=Cl_n^+\oplus Cl_n^- \quad\text{where}\quad Cl_n^{\pm}:=(1\pm\omega_n)Cl_n \quad\text{and}\quad n=4m,4m+3$$

This is a *different* $\mathbb{Z}_2$ *grading from the earlier* $Cl^0\oplus Cl^1$ *grading.* Indeed, for $n=4m+3$, the element $(1+\omega_{4m+3})$ is in Cl^+, but not in Cl^0 or Cl^1, since $1\in Cl^0$ and $\omega_{4m+3}\in Cl^1$. Similarly, for $n=4m$, the element $(1-\omega_{4m})\in Cl^-$ but not in Cl^1.

When $n=4m$, we have $e_i\omega_{4m}=-\omega_{4m}e_i$ for all i, and so $a\omega_{4m}=-\omega_{4m}a$ for all $a\in Cl^1$, and $a\omega_{4m}=\omega_{4m}a$ for all $a\in Cl^0$. Thus, for $n=4m$, we have that $Cl_{4m}=Cl_{4m}^+\oplus Cl_{4m}^-$ is a Cl_{4m} $\mathbb{R}$-supermodule. (Unfortunately, the corresponding fact is untrue for $n = 4m+3$ since ω_{4m+3} is central, so Cl^1 preserves both Cl^+ and Cl^- instead of interchanging them). However, the above grading on Cl_{4m} has some bearing on $Cl(\mathbb{R}^{4m+1})$, as we shall see soon.

Example 14.1.18 (Signature grading)**.** Let $V = \mathbb{R}^{4m}$, and consider the $\mathbb{R}$-Clifford module $\Lambda^*(\mathbb{R}^{4m})$ over Cl_{4m}. By the previous example, $Cl_{4m} = Cl^+_{4m} \oplus Cl^-_{4m}$ becomes a Cl_{4m} supermodule via action of left Clifford multiplication, the decomposition being determined by the sign of multiplication by ω_{4m}. Since $\Lambda^*(\mathbb{R}^{4m}) = Cl_{4m}.1$, it follows that:

$$\Lambda^*(\mathbb{R}^{4m}) = \Lambda^+(\mathbb{R}^{4m}) \oplus \Lambda^-(\mathbb{R}^{4m})$$

where $\Lambda^\pm(\mathbb{R}^{4m}) := Cl^\pm_{4m}.1$. This makes $\Lambda^*(\mathbb{R}^{4m})$ a Cl_{4m} supermodule, by the previous example. Similar considerations apply to $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{4m})$ which becomes a $\mathbb{C}l_{4m}$ $\mathbb{C}$-supermodule via the grading:

$$\Lambda^*_{\mathbb{C}}(\mathbb{R}^{4m}) = \Lambda^+_{\mathbb{C}}(\mathbb{R}^{4m}) \oplus \Lambda^-_{\mathbb{C}}(\mathbb{R}^{4m})$$

This last grading is called *the signature* grading because the Clifford action of τ_{4m} coincides with the Hodge-star operator in the middle dimension $\Lambda^{2m}_{\mathbb{C}}$, by the last statement in (v) of Lemma 14.1.7.

It is helpful to have an explicit model for the complex Clifford modules. This is the content of the next proposition.

Proposition 14.1.19 (The irreducible complex $\mathbb{C}l_{2m}$ modules)**.** Let $V = \mathbb{R}^{2m}$ with the usual euclidean inner product $\langle -,-\rangle$. Extend this inner product *by complex linearity* to $\langle -,-\rangle$ on the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{2m}$ (i.e. this inner product is *not* positive definite on $V_{\mathbb{C}}$, being $\mathbb{C}$-linear in both variables). Let P be the complex subspace of $V_{\mathbb{C}}$ defined by:

$$P := \mathbb{C} - \mathrm{span}\{e_{2j-1} - ie_{2j} : 1 \le j \le m\}$$

Then set $S = \Lambda^*(P)$, a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} S = 2^m$. Then $S = S^+ \oplus S^-$ is a $\mathbb{C}l_{2m}$ $\mathbb{C}$-supermodule which is irreducible. $S^\pm$ are the ± 1-eigenspaces with respect to the (non-central) chirality element τ_{2m}, and turn out to be $S^+ = \Lambda^{ev}P$ and $S^- = \Lambda^o P$. Finally:

$$\mathbb{C}l_{2m} = Cl_{2m} \otimes_{\mathbb{R}} \mathbb{C} = \hom_{\mathbb{C}}(S,S)$$

Proof: First note that $V_{\mathbb{C}}$, being a complexification, comes with the natural complex conjugation $v\otimes\lambda \mapsto v\otimes\overline{\lambda}$ for $v \in V$. Also, P is a real-form of $V_{\mathbb{C}}$, i.e.

$$V_{\mathbb{C}} = P \oplus \overline{P}$$

where $\overline{P}$ denotes the complex conjugation of P inside $V_{\mathbb{C}}$. Now we claim that the subspace P is *isotropic*, i.e. $\langle v,w\rangle \equiv 0$ for all $v,w \in P$. For,

$$\langle e_{2j-1} - ie_{2j}, e_{2k-1} - ie_{2k}\rangle = \langle e_{2j-1}, e_{2k-1}\rangle + i^2\langle e_{2j}, e_{2k}\rangle = \delta_{jk} - \delta_{jk} \equiv 0 \ \text{ for all } \ 1 \le j \le m$$

Now define a basis of P by:

$$f_j := \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j}) \quad 1 \le j \le m$$

so that

$$\langle f_j, \overline{f}_k\rangle = \frac{1}{2}(\langle e_{2j-1}, e_{2k-1}\rangle + \langle e_{2j}, e_{2k}\rangle) = \frac{1}{2}(\delta_{jk} + \delta_{jk}) = \delta_{jk}$$

which shows that $\{\overline{f}_j\}$ is a basis of $\overline{P}$ which is dual to the basis $\{f_j\}$ of $\overline{P}$. This identifies $\overline{P}$ with the complex dual P^*.

Define the action of P on Λ^*P by

$$v \circ \phi := \sqrt{2} v \wedge \phi \quad \text{for} \ \ v \in P,\ \phi \in \Lambda^*P$$

Note that $(v \circ (v \circ \phi)) = 2(v \wedge v \wedge \phi) \equiv 0 = \langle v,v\rangle\,\phi$ for all $v \in P$, and $\phi \in \Lambda^*P$.

Define the action of P^* on $S = \Lambda^*P$ by duality:

$$\langle \overline{v} \circ \phi, \psi\rangle := -\langle \phi, v \circ \psi\rangle = -\sqrt{2}\,\langle \phi, v \wedge \psi\rangle = -\sqrt{2}\,\langle v \lrcorner \psi, \phi\rangle$$

Hence setting $\overline{v}\circ\phi = -\sqrt{2}v\lrcorner\phi$ defines an action of $P^* = \overline{P}$ on $\Lambda^* P$, which also satisfies $\overline{v}\circ(\overline{v}\circ\phi) \equiv 0 = -\langle \overline{v}, \overline{v}\rangle \phi$.

Now, we need to verify the Clifford relations. We have already seen that $f_i \circ f_i \circ \phi \equiv 0 = \langle f_i, f_i\rangle \phi$ for all i. Also $f_i \circ f_j \circ \phi + f_j \circ f_i \circ \phi = 2(f_i \wedge f_j + f_j \wedge f_i) \wedge \phi = 0 = -2\langle f_i, f_j\rangle \phi$. Similar relations hold for $\overline{f}_i$'s. We just need to check the mixed relations, viz.,

$$\begin{aligned}
f_k \circ (\overline{f_j} \circ \phi) + \overline{f}_j \circ (f_k \circ \phi) &= -2f_k \wedge (f_j \lrcorner\phi) - 2f_j \lrcorner(f_k \wedge \phi)\\
&= -(e_{2k-1} - ie_{2k}) \wedge [(e_{2j-1} - ie_{2j})\lrcorner\phi] - (e_{2j-1} - ie_{2j}) \wedge [(e_{2k-1} - ie_{2k})\lrcorner\phi]\\
&= -[e_{2k-1} \wedge e_{2j-1}\lrcorner + e_{2j-1} \wedge e_{2k-1}\lrcorner]\phi - [e_{2k} \wedge e_{2j}\lrcorner + e_{2j} \wedge e_{2k}\lrcorner]\phi\\
&+ i[e_{2k} \wedge e_{2j-1}\lrcorner + e_{2j-1} \wedge e_{2k}\lrcorner]\phi + i[e_{2j} \wedge e_{2k-1}\lrcorner + e_{2k-1} \wedge e_{2j}\lrcorner]\phi\\
&= -\langle e_{2k-1}, e_{2j-1}\rangle \phi - \langle e_{2j}, e_{2k}\rangle \phi = -2\delta_{kj}\phi = -2\langle f_k, \overline{f}_j\rangle \phi
\end{aligned}$$

since the relation (iii) in Lemma 14.1.4 implies that $(v \wedge w\lrcorner\phi + w \wedge v\lrcorner\phi) = \langle v, w\rangle \phi$. Similarly one checks for $\overline{f}_k \circ f_j \circ \phi$

This shows that the action "$\circ$" makes $\Lambda^* P$ a $Cl(V_{\mathbb{C}}) = \mathbb{C}l(V)$ $\mathbb{C}$-module. This module, call it S, is irreducible, because its complex dimension is the complex dimension of $\Lambda^* P$, i.e. 2^m. In the second part of Proposition 14.1.12, we saw that the dimension of the unique $\mathbb{C}l_{2m}$ Clifford $\mathbb{C}$-module is 2^m. Hence this must be that module, provided we check that the action is not trivial, and that is obvious.

We also recall that $\tau_{2m}^2 = 1$, and this module $\Lambda^* P$ will split into the (± 1)-eigenspaces $\Lambda^\pm P$ of the chirality element τ_{2m}. Since τ_{2m} anticommutes with all $v \in V_{\mathbb{C}}$, τ_{2m} commutes with $\mathbb{C}l^0_{2m}$ and anticommutes with $\mathbb{C}l^1_{2m}$. Hence $\mathbb{C}l^0_{2m} \circ \Lambda^\pm P \subset \Lambda^\pm P$ and $\mathbb{C}l^1_{2m} \circ \Lambda^\pm P \subset \Lambda^\mp P$. In other words, the grading $\Lambda^* P = \Lambda^+ P \oplus \Lambda^- P$ makes $\Lambda^* P$ a $\mathbb{C}$- supermodule over $\mathbb{C}l_{2m}$. That is $S = S^+ \oplus S^-$, with $S^\pm := \Lambda^\pm P$.

It is also useful to identify $\Lambda^+ P$ and $\Lambda^- P$ explicitly. To compute the action of the chirality element τ_{2m}, first note that

$$f_j\overline{f}_j = 2^{-1}(e_{2j-1} - ie_{2j})(e_{2j-1} + ie_{2j}) = 2^{-1}(-1 - 1 + 2ie_{2j-1}e_{2j}) = (-1 + ie_{2j-1}e_{2j})$$

and similarly $\overline{f}_j f_j = -1 - ie_{2j-1}e_{2j}$ it follows that $ie_{2j-1}e_{2j} = \frac{1}{2}(f_j\overline{f}_j - \overline{f}_j f_j)$, so that

$$\tau_{2m} = i^m(e_1e_2)(e_2e_3)..e_{2m-1}e_{2m} = 2^{-m}\prod_{j=1}^{m}(f_j\overline{f}_j - \overline{f}_j f_j)$$

Write a k-form $\phi \in \Lambda^k P$ as

$$\phi = \alpha_1 + f_1 \wedge \beta_1$$

where α_1 and β_1 are independent of f_1. Then $f_1 \circ \phi = \sqrt{2}f_1 \wedge \alpha_1$, and $\overline{f}_1 \circ \phi = -\sqrt{2}f_1\lrcorner(f_1 \wedge \beta_1) = -\sqrt{2}\beta_1$. Thus:

$$\overline{f}_1 \circ (f_1 \circ \phi) = -2f_1\lrcorner(f_1 \wedge \alpha_1) = -2\alpha_1$$

and

$$f_1 \circ (\overline{f}_1 \circ \phi) = -2f_1 \wedge \beta_1$$

Thus

$$(f_1\overline{f}_1 - \overline{f}_1 f_1)(\alpha_1 + f_1 \wedge \beta_1) = 2(\alpha_1 - f_1 \wedge \beta_1)$$

Identical formulae hold for f_j and $\overline{f}_j$, so that we have the following consequence for a decomposable form $\phi = f_I := f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_k}$:

$$\begin{aligned}
(f_j\overline{f}_j - \overline{f}_j f_j)f_I &= -2f_I \quad \text{whenever } j \in I\\
(f_j\overline{f}_j - \overline{f}_j f_j)f_I &= 2f_I \quad \text{whenever } j \notin I
\end{aligned}$$

It follows that $\tau_{2m} \circ f_I = 2^{-m}.2^m(-1)^k(+1)^{m-k}f_I = (-1)^k f_I$. Hence τ_{2m} acts as $(-1)^k$ on $\Lambda^k P$. Thus $S^+ = \Lambda^+ P = \Lambda^{ev} P$ and $S^- = \Lambda^- P = \Lambda^o P$.

Finally, consider the map:

$$\begin{aligned}\rho : \mathbb{C}l_{2m} &\rightarrow \hom_{\mathbb{C}}(S,S)\\ x &\mapsto x \circ (\,)\end{aligned}$$

Note that:

$$\begin{aligned}\hom_{\mathbb{C}}(S,S) &= (\Lambda^* P)^* \otimes \Lambda^* P = \Lambda^* P^* \otimes \Lambda^* P\\ &= \Lambda^* \overline{P} \otimes \Lambda^* P = \Lambda^*(\overline{P} \oplus \overline{P}) = \Lambda^*(V_{\mathbb{C}})\end{aligned}$$

Both sides have complex dimension 2^{2m}, and it is easy to check that ρ has no kernel (exercise!). This proves that ρ is an isomorphism and the proposition follows. □

By a magical occurrence, the graded pieces of the unique irreducible $\mathbb{C}l_{2m}$ supermodule S above *are* the two distinct irreducible modules over $\mathbb{C}l_{2m-1}$. More precisely:

Corollary 14.1.20 (Irreducible $\mathbb{C}$-modules over $\mathbb{C}l_{2m-1}$)**.** There is an isomorphism $Cl^0_{n+1} \simeq Cl_n$ of $\mathbb{R}$-algebras which complexifies to an isomorphism $\mathbb{C}l^0_{n+1} \simeq \mathbb{C}l_n$. If we consider the graded pieces $S^{\pm}$ of the irreducible $\mathbb{C}l_{2m}$ $\mathbb{C}$-supermodule S of the previous Proposition 14.1.19, we have that $S^{\pm}$ are both $\mathbb{C}$-modules over $\mathbb{C}l^0_{2m}$. Under the isomorphism above, they are $\mathbb{C}$-modules over $\mathbb{C}l_{2m-1}$. Their complex dimensions are 2^{m-1}, and they are precisely the two distinct irreducible $\mathbb{C}$-modules over $\mathbb{C}l_{2m-1}$.

Proof: The map $f : \mathbb{R}^n \rightarrow Cl^0_{n+1}$ is defined by $e_i \mapsto e_i e_{n+1}$ for $i = 1, 2, .., n$. Now $f(e_i)^2 = (e_i e_{n+1})^2 = -1$, and for $i \neq j$

$$f(e_i)f(e_j) + f(e_j)f(e_i) = e_i e_{n+1} e_j e_{n+1} + e_j e_{n+1} e_i e_{n+1} = e_i e_j + e_j e_i = 0$$

So by the universal property of Clifford algebras, it extends to a $\mathbb{R}$- algebra homomorphism $Cl_n \rightarrow Cl^0_{n+1}$. It is an isomorphism because it is clearly injective and both sides have the same dimension. Likewise for the complexifications.

Note that under the isomorphism $f : \mathbb{C}l_{2m-1} \rightarrow \mathbb{C}l^0_{2m}$, we have for the chirality element:

$$\begin{aligned}f(\tau_{2m-1}) &= i^m f(e_1 ... e_{2m-1}) = i^m (e_1 e_{2m})(e_2 e_{2m}) ... (e_{2m-2} e_{2m})(e_{2m-1} e_{2m})\\ &= (-1)^{m-1} i^m e_1 e_2 (e_{2m})^2 e_3 e_4 (e_{2m})^2 ..., e_{2m-3} e_{2m-2} (e_{2m})^2 (e_{2m-1} e_{2m})\\ &= (-1)^{m-1}(-1)^{m-1} i^m e_1 .... e_{2m} = i^m e_1 e_2 ... e_{2m} = \tau_{2m}\end{aligned}$$

Hence the module S^+ over $\mathbb{C}l^0_{2m}$ becomes a module over $\mathbb{C}l_{2m-1}$ via the isomorphism f, and since $f(\tau_{2m-1}) = \tau_{2m}$, it follows that τ_{2m-1} acts as $+1$ on S^+. Similarly, τ_{2m-1} acts as (-1) on S^-. Since $\dim_{\mathbb{C}} S = 2^m$, and $\dim_{\mathbb{C}} S^{\pm} = \frac{1}{2} \dim_{\mathbb{C}} S = 2^{m-1}$, it follows that $S^{\pm}$ are the two inequivalent $\mathbb{C}l_{2m-1}$ irreducible $\mathbb{C}$-modules. The corollary follows. □

Notation: Let us denote the two distinct irreducible $\mathbb{C}l_{2m-1}$ $\mathbb{C}$-modules by S^+_{2m-1} and S^-_{2m-1}, both of complex dimension 2^{m-1}. Let us denote the unique irreducible $\mathbb{C}l_{2m}$ $\mathbb{C}$-module (which is a supermodule) by S_{2m}, of complex dimension 2^m. We note by the Corollary 14.1.20 that the graded pieces $S^{\pm}_{2m}$ (both of complex dimension 2^{m-1}) are precisely $S^{\pm}_{2m-1}$ as $\mathbb{C}$-vector spaces, and their module structure over $\mathbb{C}l^0_{2m}$ is precisely their module structure over $\mathbb{C}l_{2m-1}$ under the identification $\mathbb{C}l_{2m-1} \simeq \mathbb{C}l^0_{2m}$.

14.2. **Complex spin representations.**

We first note that since $\mathrm{Spin}(n) \subset Cl_n \subset \mathbb{C}l_n$, any $\mathbb{C}l_n$ $\mathbb{C}$-module will give a $\mathbb{C}$-module over $\mathrm{Spin}(n)$ by restricting the action, because the group mutiplication on $\mathrm{Spin}(n)$ is the Clifford multiplication in $\mathbb{C}l_n$. A similar remark applies to $\mathrm{Pin}(n)$, but they are of less concern to us here.

Proposition 14.2.1. On the spinor group $\mathrm{Spin}(2m)$, there are two inequivalent irreducible $\mathbb{C}$-modules (=complex representations). They are denoted by $\Delta^{\pm}_{2m}$, and are distinguished by the sign of the chirality element $i^m \omega_{2m}$ (where ω_{2m} resides in $\mathrm{Spin}(2m)$). Both are of complex dimension 2^{m-1}, and are called the *half-spin representations*. They do not descend to $SO(2m)$.

On the spinor group $\mathrm{Spin}(2m-1)$, there is exactly one irreducible $\mathbb{C}$-module, of dimension 2^{m-1}, and is denoted Δ_{2m-1}. It does not descend to $SO(2m-1)$.

Proof: First note that $\omega_{2m} \in \mathbb{C}l^0_{2m}$, and since it is a product of unit vectors e_i, lies in $\mathrm{Pin}(2m)$. Thus $\omega_{2m} \in \mathrm{Spin}(2m)$, and for a $\mathbb{C}$-module over $\mathrm{Spin}(2m)$, the action of $i^m\omega_{2m}$ makes sense. Now, by the Proposition 14.1.19, there is the unique $\mathbb{C}$-supermodule S_{2m} over $\mathbb{C}l_{2m}$, with graded pieces $S^{\pm}_{2m}$. Both of these graded pieces are $\mathbb{C}$-modules over $\mathbb{C}l^0_{2m}$. Hence both are modules over $\mathrm{Spin}(2m) \subset \mathbb{C}l^0_{2m}$. Call them $\Delta^{\pm}_{2m}$. They are distinguished by the sign of the chirality action $i^m\omega_{2m}$, (or $i^m\rho(\omega_{2m})$ to be more precise, where ρ is the representation on S_{2m}).

It is clear that $\mathbb{C}l^0_{2m}$ is generated as an algebra by elements of $\mathrm{Spin}(2m)$ (indeed all elements e_I with $| I |$ even are in $\mathrm{Spin}(2m)$), it follows that if these modules $\Delta^{\pm}_{2m}$ are reducible as $\mathrm{Spin}(2m)$ modules, they will be reducible as $\mathbb{C}l^0_{2m}$ modules. That is $S^{\pm}_{2m}$ will be reducible as $\mathbb{C}l_{2m-1}$ modules. But we have seen in Corollary 14.1.20 that they are precisely the two irreducible $\mathbb{C}l_{2m-1}$ modules. Thus $\Delta^{\pm}_{2m}$ are both irreducible $\mathbb{C}$-modules over $\mathrm{Spin}(2m)$. Their dimensions are given by:

$$\dim_{\mathbb{C}} \Delta^{\pm}_{2m} = \dim_{\mathbb{C}} S^{\pm}_{2m} = 2^{m-1}$$

It is also clear from the construction of the $\mathbb{C}$-supermodule $S = \Lambda^* P$ over $\mathbb{C}l_{2m}$ in Proposition 14.1.19 that $-1 \in \mathbb{C}l_{2m}$ acts as $(-Id_S)$ on S, and hence $(-1) \in \mathrm{Spin}(2m)$ acts as $-Id$ on both $S^{\pm}_{2m}$, so neither representation $\Delta^{\pm}_{2m}$ descends to $SO(2m)$.

For the odd spin representations, we start out with the two distinct irreducible $\mathbb{C}$-modules $S^{\pm}_{2m-1}$ over $\mathbb{C}l_{2m-1}$. This time around, the volume element ω_{2m-1} is of odd parity, and lives in Cl^1_{2m-1}. Hence ω_{2m-1} does not live in $\mathrm{Spin}(2m-1)$. Hence the action of $\mathrm{Spin}(2m-1)$ is completely determined by the action of Cl^0_{2m-1} on $S^{\pm}_{2m-1}$.

We claim that the action of $\mathbb{C}l^0_{2m-1}$ is identical on both the irreducibles $S^{\pm}_{2m-1}$. Indeed if we let $\alpha_n : \mathbb{C}l_n \to \mathbb{C}l_n$ be the involution defined by extending the map $v \to (-v)$ of $V = \mathbb{R}^n$ to $\mathbb{C}l_n$, we have $\mathbb{C}l^0_n$ (resp. $\mathbb{C}l^-_n$ is the $(+1)$ (resp. (-1))-eigenspace of α_n. Since $\omega_{2m-1} \in \mathbb{C}l^1_{2m-1}$, it follows that $\alpha_{2m-1}(\omega_{2m-1}) = -\omega_{2m-1}$. Hence α_{2m-1} interchanges the $+1$ and -1 eigenspaces of τ_{2m-1} on $\mathbb{C}l_{2m-1}$, and so interchanges $\mathbb{C}l^{\pm}_{2m-1}$, the two summands of $\mathbb{C}l_{2m-1}$. Thus $\mathbb{C}l^0_{2m-1}$ is the diagonal subalgebra in the direct sum $\mathbb{C}l_{2m-1} = \mathbb{C}l^+_{2m-1} \oplus \mathbb{C}l^-_{2m-1} = \mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1})$. Hence the two distinct irreducible modules $\mathbb{C}^{2^{m-1}}$, coming from the action of each matrix algebra summand, will receive the same action from the diagonal $\mathbb{C}l^0_{2m-1}$. Hence the claim.

So we may define Δ_{2m-1} to be either S^+_{2m-1} or S^-_{2m-1} (it doesn't matter which) with $\mathrm{Spin}(2m-1)$ action being the restriction of the $\mathbb{C}l_{2m-1}$ action. The proofs of the other statements are similar to the even case above. □

14.3. **Inner products, orthogonality and unitarity.**

Definition 14.3.1. Let W be an $\mathbb{R}$-module (resp. $\mathbb{C}$-module) over $Cl(V)$, and let $(-,-)$ be a positive definite inner product (resp. positive definite hermitian inner product) on W. (We are using a different symbol to distinguish it from the euclidean inner product $\langle -,- \rangle$ on V with respect to which the Clifford algebra $Cl(V)$ is defined.) We say that W is a *self-adjoint module* over $Cl(V)$ if

$$(x.v, w) = (-1)^{\deg x}(v, x^* w) \quad \text{for all } \; v, w \in W, \;\; x \text{ homogeneous } \; \in Cl(V)$$

where $*$ is the anti-isomorphism defined in Definition 13.1.6. This is clearly equivalent to

$$(e.v, w) = -(v, e.w) \quad \text{for all } \; e \in V, \quad v, w \in W.$$

i.e. the Clifford action of vectors should be *skew-adjoint* with respect to $(-,-)$.

Example 14.3.2. By the last remark above, the second part of Proposition 14.1.5 implies that the action of $Cl(V)$ on $\Lambda^*(V)$ is self-adjoint, with the inner product $(-,-)$ on $\Lambda^*(V)$ being the natural inner product $\langle -,- \rangle$, induced by the one on V.

Example 14.3.3. We recall the construction of the unique irreducible $\mathbb{C}$-supermodule S_{2m} over Cl_{2m} (equivalently $\mathbb{C}l_{2m}$) in Proposition 14.1.19. Recall that $V_{\mathbb{C}} = P \oplus \overline{P}$, where $V = \mathbb{R}^{2m}$. We already have the complexification $\langle -, - \rangle$ on $\Lambda^*(V_{\mathbb{C}})$ of the real inner product $\langle -, - \rangle$ on $\Lambda^*(V)$ (alluded to in the foregoing example). This is an inner product on $\Lambda^*(V_{\mathbb{C}}) = \Lambda^*(V) \otimes \mathbb{C}$ which is complex *linear* in both slots. This inner-product satisfies:

$$\left\langle \overline{\phi \otimes \lambda}, \overline{\psi \otimes \mu} \right\rangle = \left\langle \phi \otimes \overline{\lambda}, \psi \otimes \overline{\mu} \right\rangle = \langle \phi, \psi \rangle \, \overline{\lambda} \overline{\mu} = \overline{\langle \phi, \psi \rangle \, \lambda \mu} = \overline{\langle \phi \otimes \lambda, \psi \otimes \mu \rangle} \;\; \text{for all} \;\; \phi, \psi \in \Lambda^*(V)$$

that is,

$$\overline{\langle \phi, \psi \rangle} = \langle \overline{\phi}, \overline{\psi} \rangle \quad \text{for all} \;\; \phi, \psi \in \Lambda^*(V_{\mathbb{C}}) = \Lambda^*(V) \otimes \mathbb{C} \tag{43}$$

We have the complex conjugation $P \to \overline{P}$, which maps $\Lambda^* P \to \Lambda^* \overline{P}$ inside $\Lambda^*(V_{\mathbb{C}})$. So define a *hermitian inner product* on $S_{2m} = \Lambda^* P$ by:

$$(\phi, \psi) := \langle \phi, \overline{\psi} \rangle \quad \text{for} \;\; \phi, \psi \in \Lambda^* P$$

For $e \in V \subset V_{\mathbb{C}}$, we have $e = \overline{e}$. Let $\phi = w_1 \wedge w_2 ... \wedge w_k \in \Lambda^* P$, with $w_i \in P$. Then, by Exercise 14.1.6:

$$\begin{aligned} e \lrcorner \overline{\phi} &= e \lrcorner (\overline{w}_1 \wedge \overline{w}_2 ... \wedge \overline{w}_k) = \sum_i (-1)^i \langle e, \overline{w}_i \rangle \, (\overline{w}_1 \wedge \overline{w}_2 ... \wedge \widehat{\overline{w}_i} \wedge .. \wedge \overline{w}_k) \\ &= \sum_i (-1)^i \overline{\langle \overline{e}, w_i \rangle} \; \overline{(w_1 \wedge w_2 .. \wedge \widehat{w_i} \wedge ... \wedge w_k)} = \overline{e \lrcorner \phi} \end{aligned}$$

using $e = \overline{e}$ and the equation (43) above. Now, using the definition of the Clifford action in Proposition 14.1.19 and (i) of Lemma 14.1.4, we compute:

$$(e.\psi, \phi) = \sqrt{2} \left\langle e \wedge \psi, \overline{\phi} \right\rangle = \sqrt{2} \left\langle \psi, e \lrcorner \overline{\phi} \right\rangle = \sqrt{2} \left\langle \psi, \overline{e \lrcorner \phi} \right\rangle = -(\psi, e.\phi)$$

which shows that Clifford multiplication by elements of V is skew- adjoint with respect to this hermitian inner product $(-, -)$, and hence the module S_{2m} is self-adjoint over Cl_{2m}.

Exercise 14.3.4. Are the irreducible modules $S^{\pm}_{2m-1}$ self- adjoint as Clifford modules over Cl_{2m-1} ?

Here is an important property of self-adjoint Clifford modules.

Proposition 14.3.5. Let W be a self-adjoint $\mathbb{R}$-module (resp. $\mathbb{C}$- module) over Cl_n with respect to the positive definite real (resp. positive definite hermitian) inner product. Then if we consider W as a module over $\mathrm{Spin}(n) \subset Cl_n$, the resulting representation

$$\rho : \mathrm{Spin}(n) \to GL(W)$$

is orthogonal (resp. unitary).

Proof: From (iii) of the Proposition 13.2.2, we have $g \in \mathrm{Spin}(n)$ implies deg $g = 0$ and $g^* g = 1$, so that by self adjointness of W,

$$(g.w_1, g.w_2) = (w_1, g^* g w_2) = (w_1, w_2) \;\; \text{for} \;\; g \in \mathrm{Spin}(n), \;\; w_i \in W$$

which proves the proposition. □

Corollary 14.3.6. The representation of $\mathrm{Spin}(n)$ on $\Lambda^*(\mathbb{R}^n)$ is a (special) orthogonal representation. The two complex half-spin representations $\Delta^{\pm}_{2m}$ of $\mathrm{Spin}(2m)$ are unitary representations.

14.4. **Decomposition formulae for Spin**$(2m)$ **representations.** We would now like to relate the left Spin$(2n)$ module $\mathbb{C}l_{2m}$, the left Spin$(2m)$ module $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$, as well as the lifted representations of $SO(2m)$ modules $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ and its $SO(2m)$-submodules $\Lambda^{ev}_{\mathbb{C}}$, $\Lambda^{o}_{\mathbb{C}}$, $\Lambda^{\pm}_{\mathbb{C}}$ etc., with the irreducible half-spin representations $\Delta^{\pm}_{2m}$ constructed in the Proposition 14.2.1.

Proposition 14.4.1.

(i): Consider $\mathbb{C}l_{2m}$ as a left module over itself, by left multiplication. Then $\mathbb{C}l_{2m}$ decomposes into 2^m irreducible $\mathbb{C}l_{2m}$-modules V_ϵ where $\epsilon = (\epsilon_1, \epsilon_2, .., \epsilon_m)$ with each $\epsilon_i = \pm 1$. Each V_ϵ is isomorphic to the unique irreducible supermodule S_{2m} as a $\mathbb{C}l_{2m}$-module. V_ϵ further decomposes into the two complex subspaces $V^{\pm}_\epsilon$ via the chirality left action of τ_{2m}, so that $V^{\pm}_\epsilon \simeq S^{\pm}_{2m}$.

(ii): Consider $\mathbb{C}l_{2m}$ as a Spin$(2m)$ complex module by the restricted left Clifford multiplication action from $\mathbb{C}l_{2m}$. Then as a Spin$(2m)$ module we have $\mathbb{C}l_{2m} = 2^m \Delta^+_\epsilon \oplus 2^m \Delta^-_\epsilon$, where $\Delta^{\pm}_\epsilon$ are isomorphic to the distinct irreducible half-spin representations $\Delta^{\pm}_{2m}$ respectively, as Spin$(2m)$ modules.

(iii): The complex exterior algebra $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ considered as a $\mathbb{C}l_{2m}$ module as in Proposition 14.1.5 has a decomposition into irreducibles analogous to (i) over $\mathbb{C}l_{2m}$, and a decomposition analogous to (ii) above, as a Spin$(2m)$ module.

Proof: Consider the elements of $\mathbb{C}l_{2m}$ defined by:

$$\alpha_j := ie_{2j-1}e_{2j} \quad j = 1, 2, ..., m$$

Then it easily follows that:

$$\begin{array}{ll} (a) & \alpha_j \alpha_k = \alpha_k \alpha_j \quad \text{for all } 1 \leq k, j \leq m \\ (b) & \alpha_j^2 = 1 \quad \text{for all } 1 \leq j \leq m \end{array}$$

Now consider the *right*-multiplication action of α_j on $\mathbb{C}l_{2m}$. By (a) and (b) above, $\mathbb{C}l_{2m}$ breaks up into simultaneous eigenspaces V_ϵ, where α_j acts by ϵ_j on V_ϵ, and $\epsilon_j = +1$ or -1. Since right and left multiplication commute, each V_ϵ is a left $\mathbb{C}l_{2m}$-submodule of $\mathbb{C}l_{2m}$ under left action. Noting that

$$e_{2j-1}\alpha_j = ie_{2j-1}e_{2j-1}e_{2j} = -ie_{2j-1}e_{2j}e_{2j-1} = -\alpha_{2j}e_{2j-1}$$

it follows that right multiplication by e_{2j-1} will map V_ϵ isomorphically to $V_{\epsilon'}$ as a $\mathbb{C}l_{2m}$-module where $\epsilon'_k = \epsilon_k$ for $k \neq j$ and $\epsilon'_j = -\epsilon_j$. Thus all the V_ϵ are isomorphic to $V_{(+1,+1,..,+1)}$ as $\mathbb{C}l_{2m}$-modules. Thus $\dim_{\mathbb{C}} V_\epsilon = \frac{1}{2^m} \dim_{\mathbb{C}} \mathbb{C}l_{2m} = 2^m$. It follows for reasons of dimension that each V_ϵ is irreducible and $V_\epsilon \simeq S_{2m}$ as a left $\mathbb{C}l_{2m}$-module.

Thus $V_\epsilon = V^+_\epsilon \oplus V^-_\epsilon$, where $V^{\pm}_\epsilon$ are the (± 1)-eigenspaces of left multiplication by chirality τ_{2m}. Clearly $V^{\pm}_\epsilon \simeq S^{\pm}_{2m}$ as $\mathbb{C}l^0_{2m}$- modules.

Now (ii) is clear by setting $\Delta^{\pm}_\epsilon = V^{\pm}_\epsilon$ with Spin$(2m)$ action being restriction of $\mathbb{C}l^0_{2m}$ action and the Proposition 14.2.1.

(iii) follows by noting that $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m}) = \mathbb{C}l_{2m}.1$ where $1 \in \Lambda^0_{\mathbb{C}}(\mathbb{R}^{2m})$. The proposition follows. □

Definition 14.4.2 (Some $\mathbb{C}l_{2m}$-bimodules)**.** We note that $\mathbb{C}l_{2m}$ has both a *left* $\mathbb{C}l_{2m}$-module structure by left multiplication, and a *right* $\mathbb{C}l_{2m}$-module structure by right multiplication, which can be thought of as a left module structure by $x.z := zx^*$. Hence $\mathbb{C}l_{2m}$ may be thought of as a $\mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m}$ left-module, viz. $(x \otimes y) \circ z := x.z.y^*$. Such a thing is called a $\mathbb{C}l_{2m}$-*bimodule.*

Now we recall the algebra isomorphism:

$$\mathbb{C}l_{2m} \to \hom_{\mathbb{C}}(S_{2m}, S_{2m})$$

from Proposition 14.1.19. On the right side, we can again produce two $\mathbb{C}l_{2m}$-module structures. Namely $(x.T)(w) := xT(w)$ for $x \in \mathbb{C}l_{2m}$ and $w \in S_{2m}$, and also $(x \circ T)(w) = T(x^*w)$ for $x \in \mathbb{C}l_{2m}$ and $w \in S_{2m}$. This is again a $\mathbb{C}l_{2m}$ bimodule structure, or left $\mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m}$-module structure given by $(x \otimes y) \circ T = xT(y^* -)$.

We now have the following proposition:

Proposition 14.4.3. The isomorphism $\mathbb{C}l_{2m} \simeq \hom_{\mathbb{C}}(S_{2m}, S_{2m})$ is an isomorphism of $\mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m}$ modules (i.e. $\mathbb{C}l_{2m}$-bimodules). In particular, by restricting to the diagonal subalgebra $\mathbb{C}l_{2m} \subset \mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m}$, we have that the adjoint action of $\mathbb{C}l_{2m}$ on itself is the same as the adjoint action of $\mathbb{C}l_{2m}$ on $\hom_{\mathbb{C}}(S_{2m} \otimes S_{2m})$ by $x.T := x.T.(x^* -)$.

Proof: We note that $\mathbb{C}l_{2m} = \mathbb{C}(2^m)$ as an algebra, and so $\mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m} = \mathbb{C}(2^m) \otimes \mathbb{C}(2^m) = \mathbb{C}(2^{2m})$. But $\mathbb{C}(2^{2m})$ is precisely $\mathbb{C}l_{4m}$. Thus a $\mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m}$ left-module structure (or $\mathbb{C}l_{2m}$-bimodule structure) is precisely a left $\mathbb{C}l_{4m}$-module structure. Since $\dim_{\mathbb{C}} \mathbb{C}l_{2m} = 2^{2m} = \dim_{\mathbb{C}}(S_{2m}, S_{2m})$, and both of these $\mathbb{C}l_{4m}$ modules are non-trivial, it follows that both modules are isomorphic as $\mathbb{C}l_{4m}$-modules to the unique irreducible $\mathbb{C}l_{4m}$-module S_{4m}. That is, they are isomorphic as $\mathbb{C}l_{2m} \otimes \mathbb{C}l_{2m}$-modules, proving the first assertion. The second assertion clearly follows from the first. □

Now we can consider the lifted modules from $SO(2m)$. That is, let

$$\rho : \mathrm{Spin}(2m) \to SO(2m)$$

be the 2-covering defined in the Proposition 13.2.2. Then the modules $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ is a natural $SO(2m)$ module by the action which is defined on decomposables in Λ^k by :

$$g.(v_1 \wedge v_2 \wedge ... \wedge v_k) = (gv_1 \wedge gv_2 ... \wedge gv_k)$$

Clearly this action preserves $\Lambda^{ev}_{\mathbb{C}}$ and $\Lambda^{o}_{\mathbb{C}}$. Also it is easily checked that this action preserves the volume element $e_1 \wedge e_2 \wedge ... \wedge e_n$, as well as the positive definite inner product $\langle -, - \rangle$ on $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$, so it commutes with the Hodge-star operator $*$. Hence $\Lambda^{\pm}_{\mathbb{C}}$ are also $SO(2m)$ submodules of $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$. Thus for W being any of these $SO(2m)$-modules, the composite map:

$$\mathrm{Spin}(2m) \xrightarrow{\rho} SO(2m) \to \hom_{\mathbb{C}}(W, W)$$

makes W into a "lifted" $\mathrm{Spin}(2m)$-module.

Proposition 14.4.4 (Decomposition of lifted $\mathrm{Spin}(2m)$-modules)**.** We have the following identities:

(i): The lifted $\mathrm{Spin}(2m)$ module $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ is isomorphic to $\mathbb{C}l_{2m}$ (with adjoint action of $\mathrm{Spin}(2m)$) as a $\mathrm{Spin}(2m)$-module. It is isomorphic to $\Delta_{2m} \otimes \Delta_{2m}$ (where $\mathrm{Spin}(2m)$ acts by tensor product action $(x.(v \otimes w)) := xv \otimes xw)$. That is, the lifted module $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ has a "*square root*" Δ_{2m}.

(ii): The isomorphism in (i) above maps the $\mathrm{Spin}(2m)$-submodule $\Lambda^+_{\mathbb{C}}$ (resp. $\Lambda^-_{\mathbb{C}}$) of the lifted module $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ isomorphically to the $\mathrm{Spin}(2m)$-submodule $\Delta^+_{2m} \otimes \Delta_{2m}$ (resp. $\Delta^-_{2m} \otimes \Delta$) of $\Delta_{2m} \otimes \Delta_{2m}$.

(iii): The isomorphism of (i) above maps the lifted $\mathrm{Spin}(2m)$-submodule $\Lambda^{ev}_{\mathbb{C}}$ of $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ isomorphically to the submodule $((-1)^m \Delta^+_{2m} \otimes \Delta^+_{2m}) \oplus ((-1)^m \Delta^-_{2m} \otimes \Delta^-_{2m})$ of $\Delta_{2m} \otimes \Delta_{2m}$. Similarly, it maps the submodule $\Lambda^o_{\mathbb{C}}$ isomorphically to $((-1)^m \Delta^+_{2m} \otimes \Delta^-_{2m}) \oplus ((-1)^m \Delta^-_{2m} \otimes \Delta^+_{2m})$ of $\Delta_{2m} \otimes \Delta_{2m}$.

Proof: We note that for the $\mathbb{C}$-basis element $e_{i_1} \wedge e_{i_2} ... \wedge e_{i_k}$ of $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$, the lifted action of $x \in \mathrm{Spin}(2m)$ is given by:

$$x.(e_{i_1} \wedge e_{i_2} ... \wedge e_{i_k}) := \rho(x)e_{i_1} \wedge \rho(x)e_{i_2} \wedge ... \wedge \rho(x)e_{i_k} = xe_{i_1}x^* \wedge xe_{i_2}x^* ... \wedge xe_{i_k}x^*$$

Now, under the $\mathbb{C}$-vector space isomorphism of "quantisation" (see (ii) of Proposition 14.1.7) identifying $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ with $\mathbb{C}l_{2m}$, the element on the right goes to

$$xe_{i_1}x^* xe_{i_2}x^* ... xe_{i_k}x^* = x(e_{i_1}e_{i_2} .... e_{i_k})x^*$$

which is precisely the adjoint action of $\mathbb{C}l_{2m}$ on itself. Hence the lifted $\mathrm{Spin}(2m)$ module $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ is isomorphic to $\mathbb{C}l_{2m}$ with adjoint $\mathrm{Spin}(2m)$ action.

We have seen in the second assertion of the Proposition 14.4.3 above that the $\mathbb{C}l_{2m}$-module $\mathbb{C}l_{2m}$ with adjoint action is isomorphic to the $\mathbb{C}l_{2m}$-module $\hom_{\mathbb{C}}(S_{2m}, S_{2m})$ (also with adjoint action of $\mathbb{C}l_{2m}$.) Restricting both modules to $\mathrm{Spin}(2m)$ shows that $\mathbb{C}l_{2m}$ with adjoint action is isomorphic to $\hom_{\mathbb{C}}(\Delta_{2m}, \Delta_{2m}) = \Delta_{2m} \otimes \Delta^*_{2m}$

as a Spin$(2m)$ module. We can identify the contragredient module Δ_{2m}^* with the right action by x^* $(= x^{-1})$ with the left-module Δ_{2m} with left action by x. This proves (i).

We know that the chirality element τ_{2m} commutes with all elements in $\mathbb{C}l_{2m}^0$, and hence $\tau_{2m}(xyx^*) = x(\tau_{2m}y)x^*$ for $x \in \mathbb{C}l_{2m}^0$, and in particular $x \in \mathrm{Spin}(2m)$. Also $\tau_{2m}^2 = 1$ implies that the splitting of $\mathbb{C}l_{2m}$ as a lifted Spin$(2m)$ module into (±1)-eigenspaces $\mathbb{C}l_{2m}^{\pm}$ makes $\mathbb{C}l_{2m}^{\pm}$ into Spin$(2m)$-submodules. So we need to know the (±1)-eigenspaces of $\Delta_{2m} \otimes \Delta_{2m}$ under left multiplication by τ_{2m}. By the Proposition 14.4.3, this is just the action $\tau_{2m}(x \otimes y) = \tau_{2m}x \otimes y$. Thus the splitting is $\Delta_{2m}^{\pm} \otimes \Delta_{2m}$. So $\mathbb{C}l_{2m}^{\pm} = \Delta_{2m}^{\pm} \otimes \Delta_{2m}$. Using the isomorphism of $\Lambda_{\mathbb{C}}^*(\mathbb{R}^{2m})$ with $\mathbb{C}l_{2m}$, we get (ii).

For (iii), note that the conjugation action of ω_{2m} satisfies

$$\omega_{2m} e_i \omega_{2m}^* = -e_i \omega_{2m} \omega_{2m}^* = -(-1)^{2m} e_i = -e_i$$

which shows that $\rho(\omega_{2m})$ acts as $+1$ on e_I with I of even cardinality and (-1) on I of odd cardinality. Under the identification of $\Lambda_{\mathbb{C}}^*(\mathbb{R}^{2m})$ with $\mathbb{C}l_{2m}$ by (i), we find that $\Lambda_{\mathbb{C}}^{ev}$ is the submodule corresponding to $(+1)$-eigenspace of $\rho(\omega_{2m})$, and $\Lambda_{\mathbb{C}}^{o}$ the (-1)- eigenspace of $\rho(\omega_{2m})$. So it remains to identify, in view of (i), the ±1-eigenspaces of the operator $\omega_{2m} \otimes \omega_{2m}$ on $\Delta_{2m} \otimes \Delta_{2m}$. Note that since ω_{2m} commutes with $\mathbb{C}l_{2m}^0$, it commutes with all of Spin$(2m)$, and so these ±1-eigenspaces are Spin$(2m)$- submodules of $\Lambda_{\mathbb{C}}^*(\mathbb{R}^{2m})$.

Since $i^m\omega_{2m} = \tau_{2m}$, we have $\omega_{2m} \otimes \omega_{2m} = (-1)^m \tau_{2m} \otimes \tau_{2m}$. Using the notation $(-1)^m\Delta_{2m}^+ := \Delta_{2m}^+$ for m even, and Δ_{2m}^- for m odd, (and a similar notation for $(-1)^m\Delta_{2m}^-$) we find that the $+1$-eigenspace (resp. (-1)-eigenspace) of $\omega_{2m} \otimes \omega_{2m}$ is clearly $((-1)^m\Delta_{2m}^+ \otimes \Delta_{2m}^+) \oplus ((-1)^m\Delta_{2m}^- \otimes \Delta_{2m}^-)$ (resp. $((-1)^m\Delta_{2m}^+ \otimes \Delta_{2m}^-) \oplus ((-1)^m\Delta_{2m}^- \otimes \Delta_{2m}^+)$). This proves (iii) and the proposition follows. □

There is a fact about the "derived" adjoint action we shall need later on:

Proposition 14.4.5. The vector subspace spanned by $\{e_ie_j : i < j\}$ inside the real Clifford algebra $CL(V)$ is denoted by $C^2(V)$ (Recalling the quantisation map c of (ii) in Proposition 14.1.7, $C^2(V) = c(\Lambda^2(V))$. Then

(i): $C^2(V)$ is a Lie algebra under the commutator $[x, y] = xy - yx$ in $Cl(V)$.

(ii): The map $\tau : C^2(V) \to \mathfrak{so}(V)$ defined by $\tau(a)v = [a, v]$ is an isomorphism of Lie algebras.

(iii): Define the exponential map of $C(V)$ by:

$$\begin{aligned} \exp_C : C(V) &\to C(V) \\ x &\mapsto 1 + x + \frac{x^2}{2\,!} + \ldots + \frac{x^k}{k\,!} + \ldots \end{aligned}$$

Then $\exp_C(C^2(V)) = \mathrm{Spin}(V)$.

Proof: By directly using $e_i^2 = -1$ and $e_ie_j = -e_je_i$ we compute:

$$\begin{aligned} [e_ie_j, e_ke_l] &= 0 \quad \text{if } i < j,\ k < l,\ \{i, j\} \cap \{k, l\} = \phi \text{ or } (i, j) = (k, l) \\ &= -2e_ie_l \quad \text{if } i < j = k < l \\ &= 2e_je_l \quad \text{if } i = k,\ j \neq l \end{aligned}$$

which shows that $C^2(V)$ is a Lie algebra and (i) follows.

Note also that

$$\begin{aligned} [e_ie_j, e_k] &= -2e_i \quad \text{for } i < j = k \\ &= 2e_j \quad \text{for } k = i < j \\ &= 0 \quad \text{for } k \neq i,\ k \neq j \end{aligned}$$

This clearly shows that $\tau(e_ie_j)$ for $i < j$ preserves $V = \mathrm{span}_{\mathbb{R}}\{e_i\}$, and hence maps to $gl(V)$. Since $\tau(e_ie_j) = 2(E_{ji} - E_{ij})$, ($E_{ij}$ being the matrix with 1 in the (ij)- spot and zeros elsewhere), and since the combinations the set $\{(E_{ji}-E_{ij}) : i<j\}$ constitutes a basis of $\mathfrak{so}(V)$, it follows that τ is a vector space isomorphism. That it is a Lie algebra isomorphism follows easily from the fact that

$$[\tau(x),\tau(y)]v = \tau(x)([y,v]) - \tau(y)([x,v]) = [x,[y,v]] - [y,[x,v]] == -[v,[x,y]] = \tau([x,y])v$$

by the Jacobi identity. This proves (ii).

In the course of proving (v) of Proposition 13.2.2, we found that $\exp_C(te_ie_j) = \cos t.1 + \sin t(e_ie_j)$ for $i \neq j$, and consequently Lie(Spin(V)) was precisely $C^2(V)$. Now the exponential of $C^2(V)$ is going to be a connected Lie- subgroup $G \subset Cl^\times$, and of dimension $\frac{n(n-1)}{2}$. Also its Lie algebra is $C^2(V)$. Since a connected compact Lie group is precisely the exponential of its Lie algebra, it follows that $G = \exp_C V = \text{Spin}(V)$, and (iii) follows.

There is another crucial proposition which allows us to recover any $\mathbb{C}l_{2m}$-supermodule as a tensor product with the irreducible $\mathbb{C}l_{2m}$ supermodule S_{2m}.

Proposition 14.4.6. Let W be any $\mathbb{C}l_{2m}$-module with chirality grading $W^\pm$. Then there exists a $\mathbb{C}$-vector space V such that $W \simeq S_{2m} \otimes_{\mathbb{C}} V$ as a $\mathbb{C}l_{2m}$-supermodule. This V is uniquely determined by W, and is called the *twisting space* for the supermodule W.

Proof: In the statement, we are treating V as an *ungraded* $\mathbb{C}$-vector space, and equipping $S_{2m} \otimes_{\mathbb{C}} V$ with the obvious left $\mathbb{C}l_{2m}$-module structure defined by. $x.(s\otimes v) = xs \otimes v$. The supermodule structure on $S_{2m} \otimes_{\mathbb{C}} V$ is defined by $(S_{2m} \otimes_{\mathbb{C}} V)^0 := S^+_{2m} \otimes V$ and $(S_{2m} \otimes_{\mathbb{C}} V)^1 := S^-_{2m} \otimes V$ (chirality grading). That this is a left $\mathbb{C}l_{2m}$-supermodule structure on $S_{2m} \otimes_{\mathbb{C}} V$ follows from the corresponding fact about S_{2m}.

Consider the functor $\mathcal{F}$ from the category $\mathcal{C}$ of finite dimensional $\mathbb{C}l_{2m}$-supermodules to itself, defined by $W \mapsto S_{2m} \otimes_{\mathbb{C}} \hom_{\mathbb{C}l_{2m}}(S_{2m}, W)$. Here $\hom_{\mathbb{C}l_{2m}}(S_{2m}, W)$ is the *ungraded* $\mathbb{C}$-vector space of $\mathbb{C}l_{2m}$-module morphisms of $S_{2m} \to W$, and the tensor product $S_{2m} \otimes_{\mathbb{C}} \hom_{\mathbb{C}l_{2m}}(S_{2m}, W)$ is made into a $\mathbb{C}l_{2m}$-supermodule as in the last paragraph. There is the natural transformation of functors $\phi : \mathcal{F} \to Id_{\mathcal{C}}$ defined by

$$\begin{aligned} \phi_W : \mathcal{F}(W) = S_{2m} \otimes_{\mathbb{C}} \hom(S_{2m}, W) &\to W \\ s \otimes T &\mapsto T(s) \end{aligned}$$

Note that both functors $\mathcal{F}$ and $Id_{\mathcal{C}}$ are additive with respect to direct sums in $\mathcal{C}$. Also, on an *irreducible* $\mathbb{C}l_{2m}$-supermodule W, we have $\hom_{\mathbb{C}l_{2m}}(S_{2m}, W) \simeq \mathbb{C}\phi_W$, where $\phi_W : S_{2m} \to W$ is the *unique* $\mathbb{C}l_{2m}$-supermodule isomorphism between $S_{2m} \to W$, since $\mathbb{C}l_{2m}$ has a unique irreducible module S_{2m}, and the only $\mathbb{C}l_{2m}$-module maps between these finite dimensional irreducibles are $\{\lambda\phi_W\}_{\lambda\in\mathbb{C}}$ (these statements follow from the Schur lemma). Thus the natural transformation of functors $\mathcal{F} \to Id_{\mathcal{C}}$ is a natural equivalence on the full subcategory of irreducibles.

By Lemma 14.1.11 asserting complete reducibility of all $\mathbb{C}l_{2m}$-modules, and the additivity of both functors $\mathcal{F}$ and $Id_{\mathcal{C}}$, it follows that ϕ_W is a natural equivalence of functors on all of $\mathcal{C}$. Also, the isomorphism ϕ_W : $\mathcal{F}(W) \to W$ is explicitly given by $s \otimes T \mapsto T(w)$, by the definition of ϕ_W. □

Example 14.4.7. For instance, we saw in Proposition 14.4.3 that as a left $\mathbb{C}l_{2m}$-module, $\mathbb{C}l_{2m} \simeq \hom_{\mathbb{C}}(S_{2m}, S_{2m})$. This last module may be rewritten as $S_{2m} \otimes_{\mathbb{C}} S^*_{2m}$, so that the twisting space in this case is S^*_{2m}. By the Proposition 14.4.6 above, there follows the curious fact that $\hom_{\mathbb{C}l_{2m}}(S_{2m}, \mathbb{C}l_{2m}) \simeq S^*_{2m} = \hom_{\mathbb{C}}(S_{2m}, \mathbb{C})$ as a $\mathbb{C}$-vector space.

14.5. **Supertraces.** A useful book-keeping device, which walks the bridge between an index and a trace, is the supertrace.

Definition 14.5.1. Let W be a $\mathbb{C}l_{2m}$-module. Recall the chirality element $\tau_{2m} \in \mathbb{C}l_{2m}$ defined by $i^m \omega_{2m}$.

Give W the $\mathbb{Z}_2$-grading $W^{\pm} = (\pm 1)$- eigenspace of τ_{2m}, which is the same grading as in its supermodule structure over $\mathbb{C}l_{2m}$. We have seen that $\mathbb{C}l^0_{2m} W^{\pm} \subset W^{\pm}$ since $\tau_{2m}(av) = a\tau_{2m}(v)$ for all $v \in W, a \in \mathbb{C}l^0_{2m}$. Similarly, $\mathbb{C}l^1_{2m} W^{\pm} \subset W^{\mp}$, since $\tau_{2m}(av) = -a\tau_{2m}v$ for all $v \in W, a \in \mathbb{C}l^1_{2m}$. For $a \in \mathbb{C}l_{2m}$, consider the endomorphism $a.(-)$ of W, and define the *supertrace*

$$\begin{aligned} \mathrm{str}_W(a) = \mathrm{tr}_W(\tau_{2m} a) &= \mathrm{tr}_{W^+} a - \mathrm{tr}_{W^-} a \ \text{ if } \ a \in \mathbb{C}l^0_{2m} \\ &= 0 \ \text{ if } \ a \in \mathbb{C}l^1_{2m} \end{aligned}$$

The formulas on the right for homogeneous elements in $\mathbb{C}l^0_{2m}$ or $\mathbb{C}l^1_{2m}$ follow from the fact that for $a \in \mathbb{C}l^+_{2m}$, $\tau_{2m} a$ acts as $a : W^+ \to W^+$, and as $(-a) : W^- \to W^-$, whereas for $a \in \mathbb{C}l^-_{2m}$, $\tau_{2m} a$ acts as $(-a) : W^+ \to W^-$ and $a : W^- \to W^+$, and is "off-diagonal".

Note that for any $\mathbb{C}l_{2m}$-module W, the supertrace str_W gives a linear functional on $\mathbb{C}l_{2m}$.

The following lemma characterises all supertraces on $\mathbb{C}l_{2m}$. We define $T : \Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m}) \to \mathbb{C}$ be the projection into the top degree forms (as a multiple of ω_{2m}). Also recall the symbol map $\sigma : \mathbb{C}l_{2m} \to \Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$

Lemma 14.5.2. Let W be any $\mathbb{C}l_{2m}$-module. If $\psi : \mathbb{C}l_{2m} \to \mathbb{C}$ is any linear functional which vanishes on all supercommutators in $\mathbb{C}l_{2m}$, then $\psi = \lambda(\mathrm{str}_W)$ for some $\lambda \in \mathbb{C}$. Finally:

$$\mathrm{str}_W(a) = (-i)^m (\dim_{\mathbb{C}} W)(T \circ \sigma(a))$$

Proof: Recall that the grading on the module W is given by the (± 1)-eigenspaces of τ_m, viz. $W^{\pm}$. Since τ_{2m} commutes with $\mathbb{C}l^0_{2m}$ and anticommutes with $\mathbb{C}l^1_{2m}$, it follows that this grading makes W a supermodule. The supercommutator of $a, b \in \mathbb{C}l_{2m}$ was defined in Definition 13.1.7. Since str_W is linear, it suffices to show that str_W vanishes on supercommutators of homogeneous elements. If $a \in \mathbb{C}l^0_{2m}$ (resp. $b \in \mathbb{C}l^1_{2m}$), we can write it as a block-matrix in the $W^+ \oplus W^-$ decomposition as:

$$a = \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix} \quad \text{resp.} \quad b = \begin{pmatrix} 0 & b^- \\ b^+ & 0 \end{pmatrix}$$

Now if $a \in \mathbb{C}l^0_{2m}$ and $b \in \mathbb{C}l^1_{2m}$, then the supercommutator $[a, b]_s \in \mathbb{C}l^1_{2m}$, and will have supertrace 0, by the definitions above. Similarly for $a \in \mathbb{C}l^1_{2m}$ and $b \in \mathbb{C}l^0_{2m}$. So assume both $a, b \in \mathbb{C}l^0_{2m}$, or both $a, b \in \mathbb{C}l^1_{2m}$. Then, in the first case, $[a, b]_s = ab - ba$, which has the block matrix expression:

$$[a, b]_s = \begin{pmatrix} [a^+, b^+] & 0 \\ 0 & [a^-, b^-] \end{pmatrix}$$

which implies $\mathrm{str}_W[a, b]_s = \mathrm{tr}_{W^+}([a^+, b^+]) - \mathrm{tr}_{W^-}([a^-, b^-]) = 0$. In the second case, when both $a, b \in \mathbb{C}l^1_{2m}$, then $[a, b]_s = ab + ba$, which has the matrix expression:

$$[a, b]_s = \begin{pmatrix} a^- b^+ + b^- a^+ & 0 \\ 0 & a^+ b^- + b^+ a^- \end{pmatrix}$$

so that:

$$\begin{aligned} \mathrm{str}_W[a, b]_s &= \mathrm{tr}_{W^+}(a^- b^+ + b^- a^+) - \mathrm{tr}_{W^-}(a^+ b^- + b^+ a^-) \\ &= \mathrm{tr}_{W^+}(a^- b^+) - \mathrm{tr}_{W^-}(b^+ a^-) + \mathrm{tr}_{W^+}(b^- a^+) - \mathrm{tr}_{W^-}(a^+ b^-) = 0 \end{aligned}$$

noting that both W^+ and W^- are isomorphic as $\mathbb{C}$-vector spaces. (Left action by any e_i interchanges W^+ and W^-.)

Thus

$$\mathrm{str}_W[a, b]_s \equiv 0 \quad \text{for all} \quad a, b \in \mathbb{C}l_{2m}$$

Define $C_k := \sum_{i=0}^{k} c(\Lambda^k_{\mathbb{C}}(\mathbb{R}^{2m}))$. That is, C_k is the subspace of $\mathbb{C}l_{2m}$ spanned by all basis elements e_I with $|I| \leq k$. We now claim that $C_{2m-1} \subset [\mathbb{C}l_{2m}, \mathbb{C}l_{2m}]_s$. For if e_I is any basis element with $|I| \leq 2m-1$, then there exists a j such that $j \notin I$. Letting $|I| = k$, we compute:

$$[e_j, e_j e_I]_s = e_j^2 e_I - (-1)^{1.(k+1)} e_j e_I e_j = -e_I - (-1)^{2k+1} e_I e_j^2 = -2e_I$$

which shows that every e_I with $|I| \leq 2m-1$ is a supercommutator, and the claim follows. Hence the supertrace satisfies str $(C_{2m-1}) \equiv 0$. Since the quotient $\mathbb{C}l_{2m}/C_{2m-1} \simeq \mathbb{C}$ is one dimensional, it follows that str_W descends to this 1-dimensional quotient.

Since $\mathrm{str}_W \tau_{2m} = \mathrm{tr}_W(\tau_{2m}^2) = \dim W \neq 0$, it follows that the supertrace str_W gives an isomorphism of $\mathbb{C}l_{2m}/C_{2m-1} \to \mathbb{C}$. It also follows, since str_W is not the zero map, that the dimension of $\mathbb{C}l_{2m}/[\mathbb{C}l_{2m}, \mathbb{C}l_{2m}]_s$ cannot be zero, and since it is $\leq \dim_{\mathbb{C}} \mathbb{C}l_{2m}/C_{2m-1} = 1$, must be 1. Thus $[\mathbb{C}l_{2m}, \mathbb{C}l_{2m}] = C_{2m-1}$.

The second assertion of the statement is now clear, since any linear functional annihilating all supercommutators descends to the 1-dimensional space $\mathbb{C}l_{2m}/[\mathbb{C}l_{2m}, \mathbb{C}l_{2m}]_s$.

Now note that $T \circ \sigma : \mathbb{C}l_{2m} \to \mathbb{C}$ is a linear functional on $\mathbb{C}l_{2m}$, and annihilates C_{2m-1}, since $\ker T = \sum_{i \leq 2m-1} \Lambda^i = \sigma(C_{2m-1})$. Hence it annihilates $[\mathbb{C}l_{2m}, \mathbb{C}l_{2m}]_s$, and by the last para $T \circ \sigma$ and str_W are scalar multiples of each other on $\mathbb{C}l_{2m}$. Indeed, by evaluating both on τ_{2m}, we saw that

$$\mathrm{str}_W(\tau_{2m}) = \mathrm{tr}_W(\tau_{2m}^2) = \dim W$$

whereas $T \circ \sigma(\tau_{2m}) = i^m$. This implies that $\mathrm{str}_W = (\dim W)(-i)^m (T \circ \sigma)$. The lemma follows. □

Corollary 14.5.3. The proof above showed that $\mathbb{C}_{2m-1} = [\mathbb{C}l_{2m}, \mathbb{C}l_{2m}]_s$.

15. Clifford Bundles and Dirac operators

From now on, let M be a compact oriented Riemannian manifold of dimension $2m$. $P_{SO} \to M$ will denote its oriented orthonormal frame bundle, with structure group $SO(2m)$.

15.1. Clifford bundles, Clifford modules and the Spinor bundle.

Definition 15.1.1. The *Clifford bundle of* M is the complex vector bundle $\pi : \mathbb{C}l(M) \to M$ whose fibre at $x \in M$ is the complex Clifford algebra $\mathbb{C}l(T_x^* M)$, where $T_x^*(M)$ is given the real positive definite inner product $\langle -, - \rangle_x$ from the Riemannian metric induced on the cotangent bundle. It can be viewed as the associated vector bundle:

$$P_{SO} \times_{SO(2m)} \mathbb{C}l(\mathbb{R}^{2m})$$

where $SO(2m)$ has the obvious action on $\mathbb{C}l(\mathbb{R}^{2m*})$ (defined by $e_i^* \mapsto f_i^* := g.e_i^*$ for $g \in SO(2m)$, where e_i^* is an orthonormal basis for $\mathbb{R}^{2m*}$).

Since the vector bundle $\Lambda^*_{\mathbb{C}}(T^*M) \to M$ is the associated bundle:

$$P_{SO} \times_{SO(2m)} \Lambda^*(\mathbb{R}^{2m*}) \to M$$

and the symbol map σ and quantisation map c are $SO(2m)$- equivariant, we get global vector bundle maps:

$$\sigma : \mathbb{C}l(M) \to \Lambda^*_{\mathbb{C}}(T^*M)$$

called the *symbol map of* M, and

$$c : \Lambda^*_{\mathbb{C}}(T^*M) \to \mathbb{C}l(M)$$

called the *quantisation map of* M.

Remark 15.1.2. We note that the action of $SO(2n)$ on $\mathbb{C}l_{2m}$ is the descended action from the Spin$(2m)$ action on $\mathbb{C}l_{2m}$ by *conjugation*. Hence the fibre of $\mathbb{C}l(M)$ is the module $\Delta_{2m} \otimes \Delta_{2m}$, by (i) of Proposition 14.4.4.

In the light of the Proposition 14.4.6, it is desirable to have a bundle $\Delta \to M$ on M with the fibre Δ_{2m} (or what is the same thing, S_{2m}), so that any bundle of Clifford modules on M (such as $\Lambda^*_{\mathbb{C}}$, $\Lambda^{\pm}_{\mathbb{C}}$, $\Lambda^{ev}_{\mathbb{C}}$, $\Lambda^{o}_{\mathbb{C}}$) can be written as a tensor product $\Delta \otimes_{\mathbb{C}} V$, where V is a twisting bundle.

Unfortunately, this cannot be done unless we assume a Spin$(2m)$ structure on M, because the representation Δ_{2m} of Spin$(2m)$ (or for that matter the representation S_{2m} of $\mathbb{C}l_{2m}$) *does not descend* to a representation of $SO(2m)$. Hence there is no way to start with the principal bundle P_{SO} and get an associated bundle with fibre Δ_{2m} or S_{2m}.

Definition 15.1.3. Let M be a Riemannian oriented manifold of dimension $2m$, and assume it has a spin structure. Let $P_{spin} \to M$ denote the principal Spin$(2m)$-bundle over M, (see Definition 13.3.1). Then consider the associated complex vector bundle of rank 2^m:

$$P_{spin} \times_{\mathrm{Spin}(2m)} \Delta_{2m} \to M$$

where $\Delta_{2m} = \Delta^+_{2m} \oplus \Delta^-_{2m}$ is the irreducible $\mathbb{C}l_{2m}$ supermodule S_{2m} with restricted action of Spin$(2m)$, and $\Delta^{\pm}_{2m}$ are the two irreducible half-spin representations (see Proposition 14.2.1). This is called the *spin bundle* over M, and denoted $\mathcal{S}(M) \to M$. It is the direct sum of the *half spin bundles* $\mathcal{S}^{\pm}(M) \to M$, which are analogously defined as the associated rank 2^{m-1} complex vector bundles:

$$P_{spin} \times_{\mathrm{Spin}(2m)} \Delta^{\pm}_{2m} \to M$$

respectively.

Proposition 15.1.4. We have the following facts about the spin bundles:

(i): There exists a bundle map $c : \mathbb{C}l(M) \otimes_{\mathbb{C}} \mathcal{S}(M) \to \mathcal{S}(M)$ called *Clifford multiplication* whose restriction to fibres is the natural map $\mathbb{C}l(T^*M_x) \otimes S_{2m,x} \to S_{2m,x}$ defining the $\mathbb{C}l(T^*M_x)$-module structure on $S_{2m,x}$. For notational simplicity, we denote $c(a, v)$ as $a.v$. Finally $\mathbb{C}l^0(M).\mathcal{S}^{\pm}(M) \mapsto \mathcal{S}^{\pm}(M)$ and $\mathbb{C}l^1(M).\mathcal{S}^{\pm}(M) \mapsto \mathcal{S}^{\mp}(M)$.

(ii): The spin bundle $\mathcal{S}(M) \to M$ is a hermitian vector bundle with a natural hermitian metric $(-,-)$. The direct sum decomposition $\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$ is orthogonal with respect to $(-,-)$.

(iii): The Clifford action defined in (i) above is self-adjoint in the sense of Definition 14.3.1. In particular, we have:

$$(\alpha_x.v, w) = -(v, \alpha^*_x w) \ \text{ for } \ \alpha_x \in T^*_x(M), \ \ v, w \in \mathcal{S}(M)_x$$

Proof: First note that if we let Spin$(2m)$ act by *conjugation* on $\mathbb{C}l_{2m}$ (call this representation τ), then $\rho(-1) = \rho(+1)$, and so $\tau = \mu \circ \rho$ where $\rho : \mathrm{Spin}(2m) \to SO(2m)$ is the double covering homomorphism, and μ is the representation of $SO(2m)$ on $\mathbb{C}l_{2m}$ described in Definition 15.1.1. Thus

$$P_{spin} \times_\tau \mathbb{C}l_{2m} = P_{spin} \times_\rho SO(2m) \times_\mu \mathbb{C}l_{2m} = P_{SO} \times_\mu \mathbb{C}l_{2m} = \mathbb{C}l(M)$$

Now, there is the map defining Clifford module action on S_{2m}:

$$\mathbb{C}l_{2m} \otimes_{\mathbb{C}} S_{2m} \to S_{2m}$$

which is Spin$(2m)$-equivariant (since $g(x \otimes v) = gxg^* \otimes gv \mapsto gxg^*.gv = g.v$). Hence there is a natural map of vector bundles:

$$(P_{spin} \times_\tau \mathbb{C}l_{2m}) \otimes (P_{spin} \times_{\mathrm{Spin}(2m)} S_{2m}) \to P_{spin} \times_{\mathrm{Spin}(2m)} S_{2m}$$

i.e. a bundle map $\mathbb{C}l(M) \otimes \mathcal{S}(M) \to \mathcal{S}(M)$. It clearly restricts on fibres to what we claimed, by its definition. Also the last statement of (i) follows since S_{2m} is a $\mathbb{C}l_{2m}$- supermodule. This proves (i).

For (ii), construct the metric on each fibre by taking the hermitian metric $(-,-)$ constructed on S_{2m} in the Example 14.3.3. This makes the representation of Spin$(2m)$ on $S_{2m} = \Delta_{2m}$ unitary, by Proposition

14.3.5. Hence the associated bundle $P_{spin} \times_{\mathrm{Spin}(2m)} S_{2m}$ is a hermitian vector bundle. Also note that since $\omega_{2m} \in \mathrm{Spin}(2m)$, we have $\omega_{2m}v = i^{-m}\tau_{2m}v = i^{-m}v$ for $v \in S^+_{2m}$, and $\omega_{2m}w = i^{-m}\tau_{2m}w = -i^{-m}w$ for $w \in S^-_{2m}$. Thus, by the unitarity of Spin$(2m)$ action on S_{2m}, we have:

$$(v,w) = (\omega_{2m}v, \omega_{2m}w) = (i^{-m}v, -i^{-m}w) = -i^{-m}(-i)^{-m}(v,w) = -(v,w)$$

which implies $(v,w) = 0$ for $v \in S^+_{2m}$ and $w \in S^-_{2m}$. Then (ii) follows, because the representation of Spin$(2m)$ on S_{2m} is unitary.

(iii) is a direct consequence of the Example 14.3.3, which showed that S_{2m} is a self-adjoint $\mathbb{C}l_{2m}$ module with respect to $(-,-)$. □

Now we are ready to abstract all the facts proved above into a definition.

Definition 15.1.5. Let $\mathcal{E} \to M$ be complex vector bundle over an oriented Riemannian manifold of dimension $2m$, with a hermitian metric $(-,-)$. Say that this bundle is a *Clifford module over M* if:

(i): There is a $(-,-)$-orthogonal decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ into two complex sub-bundles.

(ii): There is a vector bundle *Clifford multiplication* or *Clifford action* map:

$$c : \mathbb{C}l(M) \otimes \mathcal{E} \to \mathcal{E}$$

such that for each point x, the restriction $c_x : \mathbb{C}l(T^*_xM) \otimes \mathcal{E}_x \to \mathcal{E}_x$ gives $\mathcal{E}_x$ the structure of a $\mathbb{C}l(T^*_xM)$ supermodule, with graded pieces $\mathcal{E}^\pm_x$. (In particular, the ranks of $\mathcal{E}^+$ and $\mathcal{E}^-$ are equal, and $\mathcal{E}$ is a bundle of even rank).

(iii): The action of the Clifford algebra $\mathbb{C}l(T^*_xM)$ on $\mathcal{E}_x$ is self-adjoint with respect to the hermitian inner product $(-,-)$ on $\mathcal{E}_x$.

Example 15.1.6. Clearly, for M a spin manifold of dimension $2m$, the spin bundle $\mathcal{S}(M) \to M$ is a Clifford module over M, by Proposition 15.1.4

Example 15.1.7. Let M be an oriented Riemannian manifold, not ncessarily spin. The complexified exterior algebra bundle $\Lambda^*_{\mathbb{C}}(T^*)(M) \to M$ is a Clifford module over M. For, we define the Clifford action fibre by fibre as the action which extends the action:

$$\begin{aligned} T^*_x(M) \otimes \Lambda^*_{\mathbb{C}}(T^*M) &\to \Lambda^*_{\mathbb{C}}(T^*M) \\ \alpha \otimes \phi &\mapsto \alpha \wedge \phi - \alpha \lrcorner \phi \end{aligned}$$

That this extends to an action of $\mathbb{C}l(T^*_xM)$ is the content of Proposition 14.1.5. One makes the natural hermitian extension of the Riemannian inner product $\langle -,- \rangle$ on the real exterior algebra $\Lambda^*(T^*M)$, setting $(\phi \otimes \lambda, \psi \otimes \mu) = \langle \phi, \psi \rangle \lambda\overline{\mu}$, and appeals to the last part of Proposition 14.1.5 to show that the Clifford action is self-adjoint.

As expected, there are two possible gradings $\mathcal{E}^\pm$ available on this bundle $\mathcal{E} = \Lambda^*_{\mathbb{C}}(T^*M)$. There is the *global volume element* $\omega_M \in C^\infty(M, \mathbb{C}l(M))$, given in a coordinate chart U of x by $\omega_{M,x} := e_1(x).e_2(x)....e_{2m}(x)$ where $\{e_i\}$ is a local orthonormal frame for T^*M on U (this definition is independent of coordinate charts, indeed ω_M corresponds to the Riemannian volume form on M under the symbol isomorphism). Similarly, there is the *global chirality element* $\tau_M := i^m\omega_M$.

The first grading then is the even-odd grading, in which the graded pieces $\Lambda^{ev}_{\mathbb{C}}$ and $\Lambda^{o}_{\mathbb{C}}$ come from the pointwise action of *conjugation* by $\omega_{M,x} \in \mathrm{Spin}(2m) \subset \mathbb{C}l(M)_x$ (see the proof of (iii) in Proposition 14.4.4. Another example comes from taking the graded pieces $\Lambda^+_{\mathbb{C}}$ and $\Lambda^-_{\mathbb{C}}$ corresponding to the ± 1 eigenspaces of $i^{m+k(k-1)}*$ (or pointwise *left action* by $\tau_{M,x}$, if we identify $\mathbb{C}l_{2m}$ with $\Lambda^*_{\mathbb{C}}$).

Example 15.1.8. The Clifford bundle of M, viz $\mathbb{C}l(M) \to M$ is a Clifford module, with Clifford action being left multiplication.

Note also that the Clifford bundle $\mathbb{C}l(M)$ has *two* possible $\mathbb{Z}_2$-gradings as a Clifford bundle (see Example 14.1.17, *both* of which equip the typical fibre $\mathbb{C}l(M)_x$ with the structure of a $\mathbb{C}l(T_x^*)$-supermodule. The obvious is the *chirality* $\mathbb{Z}_2$-grading $\mathbb{C}l(M)^{\pm}$ which corresponds under the symbol isomorphism to the decomposition $\Lambda_{\mathbb{C}}^{\pm}$ fo the exterior algebra bundle (see previous Example). This grading coincides with the chirality coming from left multiplication by τ_M. On the other hand, there is the *parity* $\mathbb{Z}_2$-grading $\mathbb{C}l(M) = \mathbb{C}l^0(M) \oplus \mathbb{C}l^1(M)$ (corresponding to $\Lambda_{\mathbb{C}}^{ev}$ and $\Lambda_{\mathbb{C}}^{o}$ under symbol isomorphism), which comes from *conjugation* by ω_M.

Here is the reason for introducing the spin bundle $\mathcal{S}(M) \to M$

Proposition 15.1.9. Let M be an oriented spin manifold of dimension $2m$, and let $\mathcal{S}(M) \to M$ be the spin bundle on it. Then, for any Clifford module $\mathcal{W} \to M$ on M, there is a hermitian complex *twisting vector bundle* $\mathcal{V} \to M$ such that $\mathcal{W} \simeq \mathcal{S}(M) \otimes_{\mathbb{C}} \mathcal{V}$. Note that this isomorphism is an isomorphism of *Clifford modules on* M, i.e. the graded structure and hermitian structure is also preserved.

Proof: Define the bundle $\mathcal{V} = \hom_{\mathbb{C}l(M)}(\mathcal{S}(M), \mathcal{W})$ and appeal to Proposition 14.4.6. That the Clifford action matches follows from that proposition, because the map of vector spaces:

$$\begin{aligned} \phi_W : S_{2m} \otimes_{\mathbb{C}} \hom_{\mathbb{C}l_{2m}}(S_{2m}, W) &\to W \\ s \otimes T &\mapsto T(s) \end{aligned}$$

being an isomorphism of $\mathbb{C}l_{2m}$-modules, is in particular $\mathrm{Spin}(2m)$ equivariant. Thus it globalises to a vector bundle isomorphism $\phi_{\mathcal{W}}$.

Recall the hermitian metric $(-,-)_{\mathcal{S}}$ on $\mathcal{S}_{2m}$, which was defined in Proposition 15.1.4. We just need to put a bundle metric $(-,-)_{\mathcal{V}}$ on $\mathcal{V}$ so that when the tensor product $\mathcal{S}(M) \otimes_{\mathbb{C}} \mathcal{V}$ is equipped with the *tensor product hermitian metric* $(-,-)_{\mathcal{S}} \otimes (-,-)_{\mathcal{V}}$, the Clifford module isomorphism $\phi_{\mathcal{W}}$ is an isometry with the given hermitian metric $(-,-)_{\mathcal{W}}$ on $\mathcal{W}$.

We note that for a vector space V with *any* hermitian inner product $(-,-)_V$ on it, the tensor product hermitian inner product on $S_{2m} \otimes V$, defined by:

$$(s \otimes S, t \otimes T)_{S_{2m} \otimes V} := (s,t)_S (S,T)_V$$

automatically obeys self-adjointness with respect to Clifford action, because the Clifford action is self-adjoint with respect to the natural metric $(-,-)_S$ on S_{2m}, by the Example 14.3.3.

We note that if W is an *irreducible* $\mathbb{C}l_{2m}$-module with a positive definite hermitian inner-product $(-,-)_W$ with respect to which the $\mathbb{C}l_{2m}$ action is self-adjoint, then we claim that this self-adjointness property determines $(-,-)_W$ uniquely upto a non-zero complex scalar. For if $(-,-)'$ is another hermitian inner-product with respect to which the $\mathbb{C}l_{2m}$ action is self-adjoint, then we have a $\mathbb{C}$-linear isomorphism $A : W \to W$ such that:

$$(w_1, w_2)' = (Aw_1, w_2)_W \quad \text{for all } \ w_1, w_2 \in W$$

Also for $c \in \mathbb{C}l_{2m}$, we have

$$(cAw_1, w_2)_W = (-1)^{\deg c}(Aw_1, c^* w_2) = (-1)^{\deg c}(w_1, c^* w_2)' = (cw_1, w_2)' = (A(cw_1), w_2)_W$$

Thus $A : W \to W$ is a map of $\mathbb{C}l_{2m}$-modules, and by irreducibility of W, must be a scalar (Schur Lemma), and since A is an isomorphism, the scalar must be non-zero.

If W is irreducible, and $\phi_W : S_{2m} \to W$ is an isomorphism of $\mathbb{C}l_{2m}$-modules, it follows that there is a scalar $\alpha_W \neq 0$

$$\alpha_W (w_1, w_2)_W = (\phi_W^{-1}(w_1), \phi_W^{-1}(w_2))_S \quad \text{for all} \quad w_1, w_2 \in W$$

or equivalently

$$(s,t)_S = \alpha_W(\phi_W(s), \phi_W(t))_W \quad \text{for all } \ s, t \in S_{2m}$$

Note that α_W gets determined by the equation:

$$\operatorname{tr}\phi_W^*\phi_W = \sum_{i=1}^{2^m}(\phi_W e_i, \phi_W e_i)_W = \alpha_W^{-1}\sum_{i=1}^{2^m}(e_i, e_i)_S = \alpha_W^{-1}2^m$$

where $\{e_i\}$ is an orthonormal basis of S_{2m} with respect to $(-,-)_S$.

Now let $S = \lambda\phi_W$ and $T = \mu\phi_W \in V = \hom_{\mathbb{C}l_{2m}}(S_{2m}, W) = \mathbb{C}\phi_W$. *Define* the hermitian inner product on V given by:

$$(T,S)_V := \alpha_W^{-1}\lambda\overline{\mu} = 2^{-m}\operatorname{tr}(\phi_W^*\phi_W)\lambda\overline{\mu} = 2^{-m}\operatorname{tr} S^*T$$

Then we have:

$$\begin{aligned}(s\otimes S, t\otimes T)_{S_{2m}\otimes V} &= (s,t)_S(S,T)_V = \alpha_W(\phi_W(s), \phi_W(t))_W\alpha_W^{-1}\lambda\overline{\mu} \\ &= (\lambda\phi_W(s), \mu\phi_W(t))_W = (S(s), T(t))_W\end{aligned}$$

which shows that the isomorphism $S_{2m}\otimes V \to W$ given by $s\otimes S \to S(s)$ is an isometry.

For a general W, break it into irreducibles W_i, and note that $V = \hom_{\mathbb{C}l_{2m}}(S_{2m}, W) = \oplus_i V_i$, and equip each summand $V_i := \hom_{\mathbb{C}l_{2m}}(S_{2m}, W_i)$ with the hermitian inner product above. To globalise to Clifford modules $\mathcal{W}$ is obvious, since the inner product $(T,S) = 2^{-m}\operatorname{tr} S^*T$ (the normalised Hilbert-Schmidt norm) is invariantly defined, independent of frames.

□

Example 15.1.10. Let M be a spin manifold, with the spin bundle $\mathcal{S}(M)\to M$, and its half-spin sub-bundles $\mathcal{S}^{\pm}(M)\to M$. Then, as a direct consequence of the module identities of Proposition 14.4.4, and the fact that the isomorphisms there are isomorphisms of Spin$(2m)$-modules (i.e. Spin$(2m)$-equivariant isomorphisms), there are the following bundle identities of associated vector bundles, indeed, of Clifford modules:

(i): $\mathbb{C}l(M) \simeq \Lambda^*_{\mathbb{C}}(T^*M) \simeq \mathcal{S}(M)\otimes\mathcal{S}(M)$.

(ii): $\Lambda^{\pm}_{\mathbb{C}}(M) \simeq \mathcal{S}^{\pm}(M)\otimes\mathcal{S}(M)$.

(iii): $\Lambda^{ev}_{\mathbb{C}}(M) \simeq (-1)^m\mathcal{S}^+(M)\otimes\mathcal{S}^+(M)\oplus(-1)^m\mathcal{S}^-(M)\otimes\mathcal{S}^-(M)$ and $\Lambda^{o}_{\mathbb{C}}(M) \simeq (-1)^m\mathcal{S}^+(M)\otimes\mathcal{S}^-(M)\oplus(-1)^m\mathcal{S}^-(M)\otimes\mathcal{S}^+(M)$.

Remark 15.1.11.

(i): Note that the identity (i) above says that the spin bundle $\mathcal{S}(M)$ is in some sense the "square-root" bundle of the exterior algebra (or Clifford) bundle on M, if M is a spin manifold.

(ii): The chirality grading on $\mathcal{S}(M)\otimes\mathcal{S}(M)$ comes from left action of τ_M on the first factor, and predictably leads to the grading $\mathbb{C}l(M)^{\pm}$ discussed in (ii) of Example 15.1.8. The other grading, which also restricts fibrewise to a $\mathbb{C}l(T_x^*M)$-supermodule structure corresponds to the *parity* or $\mathbb{C}l^0$, $\mathbb{C}l^1(M)$ grading (coming from *conjugation* by ω_M, see Example 15.1.8 above), and has no simple relation to the chirality grading, as is evidenced by the complicated formula in (iii) of Example 15.1.10 above.

15.2. **Clifford connections.**

Definition 15.2.1 (Levi-Civita connection)**.** If M is an oriented Riemannian manifold of dimension $2m$, there is an $SO(2m)$-connection on the principal $SO(2m)$ bundle $P_{SO} \to M$. This means that:

(i): There is a $\mathfrak{so}(2m)$-valued 1-form $[\omega_{ij}] \in \Lambda^1(P) \otimes \mathfrak{so}(2m) := C^\infty(\Lambda^1(T^*P) \otimes \mathfrak{so}(2m))$. This merely means that $[\omega_{ij}]$ is a $2m \times 2m$ skew-symmetric matrix of 1-forms ω_{ij} on P.

(ii): If we think of P_{SO} has having *right* $SO(2m)$-action, then the matrix of 1-forms $\omega := [\omega_{ij}]$ must satisfy:

$$R_g^* \omega = g\omega g^{-1} = (\mathrm{Ad}g)\omega \quad \text{for all } \; g \in SO(2m)$$

(iii): [Torsion-free condition] Let $\sigma : U \to P_{SO|U}$ be a smooth local section of P over an open set $U \subset M$. For $x \in U$, $\sigma(x)$ is an orthonormal frame at x. So σ is a local orthonormal frame field over U, and can be regarded as a $(2m)$-row vector of 1-forms $\sigma = (\sigma_1, .., \sigma_{2m})$, with $\sigma_{i,x}(X) = X_i$ where X_i the i-th component of $X \in T_xM$ in the frame $\sigma(x)$. Then we require the following identity of 2-forms on U:

$$d\sigma_i + \sum_j (\sigma^* \omega_{ij}) \wedge \sigma_j = 0$$

for each smooth section $\sigma : U \to P_{SO|U}$ over U. This connection is called the *Levi-Civita connection* on P_{SO}.

Definition 15.2.2 (Covariant differentiation in associated bundles)**.** Let $\rho : SO(2m) \to GL_{\mathbb{C}}(V)$ be a complex representation of $SO(2m)$ on a complex vector space V. Let $\mathcal{V} := P \times_\rho V$ be the associated complex vector bundle. For a connection on $P_{SO} \to M$ as above, one gets a *covariant differentiation operator* for every open set $U \subset M$:

$$\nabla : C^\infty(U, \mathcal{V}) \to C^\infty(U, T^*M \otimes \mathcal{V})$$

which is a $\mathbb{C}$-linear map satisfying the *Leibnitz Rule*:

$$\nabla(fs) = f\nabla s + df \otimes s \quad \text{for all } \; f \in C^\infty(U), \; s \in C^\infty(\mathcal{V})$$

To define the above covariant differentiation, it is enough to do it on trivialising neighbourhoods $U \subset M$ for P_{SO} (and of course check that the definition is independent of trivialisations). If we fix a basis $\{e_i\}$ of the vector space V, then for each smooth local section $\sigma : U \to P_{SO}$ over a trivialising neighbourhood $U \subset M$ for P_{SO}, we get a local framing $\widetilde{e}_i := \rho(\sigma)e_i$ of the vector bundle $\mathcal{V}_{|U}$. In view of the Leibnitz Rule, it is enough to define $\nabla \widetilde{e}_j$, and these are defined by:

$$\nabla \widetilde{e}_j := \sum_i \dot{\rho}(\sigma^* \omega)_{ij} \otimes \widetilde{e}_i$$

which is often abbreviated to $\nabla \widetilde{e}_j := \sum_i \omega_{ij} \otimes \widetilde{e}_i$, where $\omega := \dot{\rho}(\omega)$ is a $\mathfrak{gl}(V)$-valued 1-form on U, called the *Cartan connection 1-form.* If $X \in T_x^*(M)$ is a (*real*) tangent vector at x, and s a section of $\mathcal{V}$, we can define;

$$\nabla_X s := X \lrcorner \nabla s$$

In a local trivialising neighbourhood we have: $\nabla_X \widetilde{e}_j = \sum_i \omega_{ij}(X)\widetilde{e}_i$. We can also define ∇_X for X a real tangent vector field on M.

We finally note that if $e_x \mapsto g_{\alpha\beta}(x)e_x$ is a coordinate change on $U_\alpha \cap U_\beta$ for the principal bundle P_{SO}, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2m)$, then if ω^α and ω^β are the matrix-valued Cartan 1- forms on U_α and U_β respectively, then there is the transformation formula:

$$\omega^\alpha = \mathrm{Ad}(g_{\alpha\beta})\omega^\beta + dg_{\alpha\beta}.g_{\alpha\beta}^{-1}$$

where the product in the second term on the right is a matrix product.

Lemma 15.2.3. If the representation $\rho : SO(2m) \to V$ is unitary with respect to a hermitian inner product $(-,-)_V$ on V, the associated bundle $\mathcal{V} := P_{SO} \times_\rho \mathcal{V}$ is a hermitian vector bundle, with hermitian inner product denoted $(-,-)$. The covariant derivative on $\mathcal{V}$ associated to the Levi-Civita connection on P_{SO} is a *unitary connection*. It satisfies:

$$X(s,t) = (\nabla_X s, t) + (s, \nabla_X t) \quad \text{for } \ s,t \in C^\infty(M,\mathcal{V}), \quad X \in C^\infty(M,TM)$$

Proof: As noted in Example 15.1.7 above, the hermitian inner product $(-,-)$ on $\mathcal{V}$ is defined as follows. Let $[e,v],[e',w] \in \mathcal{V}_x$, with $e = e'g$ and $g \in SO(2m)$. Then define:

$$([e,v],[e',w]) = ([e,v],[e,\rho(g)w]) := (v,\rho(g)w)_V$$

To check this is well-defined, we choose a different representative $[eh^{-1},\rho(h)v]$ for $[e,v]$, with $h \in SO(2m)$, then $e' = (eh^{-1}).(hg)$, and so

$$([eh^{-1},\rho(h)v],[e',w]) = (\rho(h)v,\rho(hg)w)_V = (\rho(h)v,\rho(h)\rho(g)w)_V = (v,\rho(g)w)_V = ([e,v],[e',w])$$

since $\rho(h)$ is a unitary automorphism of V. This shows the definition of $(-,-)$ is independent or representatives in the first slot. Similarly for the second slot.

To check the second fact, note that if we start with the $(-,-)_V$-orthonormal frame $\{e_j\}$ for V, and $s \to \sigma(x).1$ a local section on on some trivialising neighbourhood U for P_{SO}, then the frame $\{\widetilde{e}_{j,x}\} = \{[1,\rho(\sigma(x))e_j]\}$ is orthonormal in $\mathcal{V}_x$ for all $x \in U$ (since $\sigma(x) \in SO(2m)$ and hence $\rho(\sigma(x)) \in U(V)$). Hence, for a smooth vector field $X \in C^\infty(U)$, we have:

$$\begin{aligned}
(\nabla_X\widetilde{e}_i,\widetilde{e}_j) + (\widetilde{e}_i,\nabla_X\widetilde{e}_j) &= \sum_k ((\dot\rho(\omega(X))_{ki}\widetilde{e}_k,\widetilde{e}_j) + (\widetilde{e}_i,\dot\rho(\omega(X))_{kj}\widetilde{e}_k)) \\
&= \sum_k \left(\dot\rho(\omega(X))_{ki}\delta_{kj} + \overline{\dot\rho(\omega(X))_{kj}}\delta_{ki}\right) \\
&= \dot\rho(\omega(X))_{ji} + \overline{\dot\rho(\omega(X))_{ij}} = 0 = X(\delta_{ij}) = X(\widetilde{e}_i,\widetilde{e}_j)
\end{aligned}$$

since $\dot\rho(\omega(X))_{ij}$ is skew-hermitian ($\dot\rho : \mathfrak{so}(2m) \to \mathfrak{u}(V)$). Now write a section $s \in C^\infty(U,\mathcal{V})$ as $s = \sum_i s_i\widetilde{e}_i$ and $t \in C^\infty(U,\mathcal{V})$ as $t = \sum_j t_j\widetilde{e}_j$ for smooth functions $s_i, t_j \in C^\infty(U)$ and use Leibnitz's Rule to conclude the result on $U \subset M$, and hence globally. □

Corollary 15.2.4. Let M be a compact oriented Riemannian manifold of dimension $2m$. Then all of the complex vector bundles associated to the principal bundle P_{SO}, namely $\Lambda^*_{\mathbb{C}}(T^*M)$, $\mathbb{C}l(M)$, $\Lambda^{ev}_{\mathbb{C}}$, $\Lambda^{o}_{\mathbb{C}}$, $\Lambda^{\pm}_{\mathbb{C}}$ carry a natural associated connection or covariant derivative, called the *Levi-Civita connection*. This Levi-Civita connection is a *unitary connection* with respect to the hermitian inner product $(-,-)$ introduced on them as above (see Example 15.1.7), by the foregoing Lemma 15.2.3.

In the sequel, when we write ∇ or ∇_X for any of these bundles without any further decorations, it is understood to mean covariant derivative with respect to the Levi-Civita connection on them.

Remark 15.2.5. There are the following immediate observations:

(i): The volume form $\omega := dV \in \Lambda^n_{\mathbb{C}}(M)$ is covariantly constant, where $\omega = e_1 \wedge e_2 \wedge ... \wedge e_n$ in a local orthonormal frame $\{e_i\}$. This is because the Levi-Civita connection is compatible with the extended hermitian metric on $\Lambda^*_{\mathbb{C}}T^*(M)$. Indeed, by definition of $(-,-)$ on $\Lambda^*_{\mathbb{C}}(T^*_xM)$, we have $(\omega(x),\omega(x))_x \equiv 1$ for all $x \in M$, and also $\omega = \overline{\omega}$, since it is a real differential form. So, for any real tangent vector field $X \in C^\infty(M,TM)$, $\nabla_X\omega$ is also a real differential n-form (i.e. equal to its conjugate). Hence $\nabla_X\omega = f(x)\omega$ for some smooth real valued function f on M. Unitarity of the Levi-Civita connection gives:

$$0 = X(\omega(x),\omega(x))_x = (\nabla_X\omega(x),\omega(x))_x + (\omega(x),\nabla_X\omega(x))_x = 2f(x)$$

which implies $\nabla_X\omega \equiv 0$.

(ii): We have noted following the Definition 15.1.1 the vector bundle isomorphisms given by the symbol and quantisation maps between $\mathbb{C}l(M)$ and $\Lambda^*_{\mathbb{C}}(T^*M)$, which arise out of the $SO(2m)$- equivariant symbol and quantisation maps of $\mathbb{C}l_{2m}$ and $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$. Since this last map is an isometry between the hermitian inner products $\langle -, - \rangle$, on both sides, it follows that the quantisation and symbol maps of bundles are bundle isometries with respect to $(-, -)$. The proof of (i) above can then be repeated verbatim for $\mathbb{C}l(M)$, to show that $\nabla_X(\omega_M) \equiv 0$ and $\nabla_X(\tau_M) \equiv 0$, where ω_M is the global volume element in $\mathbb{C}l(M)$ and τ_M the global chirality element as defined in Example 15.1.7.

(iii): From the fact that the derived representation

$$\dot{\rho} : \mathfrak{so}(2m) \to \operatorname{End}_{\mathbb{C}}(\Lambda^k_{\mathbb{C}}(\mathbb{R}^m)$$

is a derivation, and satisfies:

$$\dot{\rho}(X)(v_1 \wedge v_2 \wedge ... \wedge v_k) = \sum_i (v_{i_1} .... \wedge \dot{\rho}(X)v_i \wedge .. \wedge v_k)$$

one immediately obtains that for $X \in T_x(M)$:

$$\nabla_X(\omega_1 \wedge .... \wedge \omega_k) = \sum_i (\omega_1 \wedge ... \wedge (\nabla_X \omega_i) .. \wedge \omega_k)$$

Analogously, for the Clifford bundle $\mathbb{C}l(M)$, we have the derivation formula:

$$\nabla_X(\omega_1 ... \omega_k) = \sum_i \omega_1 ... (\nabla_X \omega_i) ... \omega_k)$$

Definition 15.2.6 (Clifford connections)**.** Let $\mathcal{E} \to M$ be a Clifford module over M, in the sense of Definition 15.1.5. We say that a connection (i.e. covariant differentiation) $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ is a *Clifford connection* if:

(i): (*Metric compatibility*) It is a unitary connection with respect to the given hermitian inner product $(-, -)$ on $\mathcal{E}$, and

(ii): (*Clifford compatibility*) For all smooth sections c of $\mathbb{C}l(M)$ and s of $\mathcal{E}$, we have:

$$\nabla^{\mathcal{E}}_X(c.s) = (\nabla_X c).s + c.\nabla^{\mathcal{E}}_X s$$

and another way of saying it is that the commutator of the covariant derivative and Clifford multiplication operators:

$$[\nabla^{\mathcal{E}}_X, c.(-)] = (\nabla_X c).(-)$$

where the right hand side denotes Clifford action by the (Levi-Civita) covariant derivative $\nabla_X c$.

Remark 15.2.7. Note that a Clifford connection as above on $\mathcal{E}$ will preserve $\mathcal{E}^{\pm}$, if $\mathcal{E}^{\pm}$ are the (± 1)-eigenspaces from left action by the global chirality $\tau_M \in C^\infty(M, \mathbb{C}l(M))$. For if $s \in \mathcal{E}$ is a smooth section, then by (ii) in the above definition:

$$\nabla^{\mathcal{E}}_X(\tau_M.s) = (\nabla_X \tau_M.s) + \tau_M.(\nabla^{\mathcal{E}})s = \tau_M.(\nabla^{\mathcal{E}} s)$$

since $\nabla_X \tau_M \equiv 0$ by (ii) of Remark 15.2.5 above. Thus covariant differentiation $\nabla^{\mathcal{E}}_X$ commutes with left Clifford action by τ_M, and thus maps the (± 1)-eigenspaces $\mathcal{E}^{\pm}$ of τ_M. In particular, it restricts to connections on $\mathcal{E}^{\pm}$, and these connections are also Clifford compatible.

Example 15.2.8. Regarding the bundle $\mathbb{C}l(M) \to M$ as a Clifford module via left multiplication (with either the chirality grading $\mathbb{C}l^{\pm}$, *or* the parity grading $\mathbb{C}l^0, \mathbb{C}l^1$), the Levi-Civita connection defined in Corollary 15.2.3 above is a Clifford connection. The property (i) is metric compatible, as remarked there. The Clifford compatibility comes from the last statement in (iii) of Remark 15.2.5 above. Similarly, the Levi-Civita conection on all of the other Clifford modules discussed in Corollary 15.2.3 is a Clifford connection.

Example 15.2.9 (The Spin-connection on $\mathcal{S}(M)$)**.** We recall the spin bundle $\mathcal{S}(M)$, and the half-spin bundles $\mathcal{S}^{\pm}$, which were introduced in Definition 15.1.3. These are *not* bundles associated to the principal bundle P_{SO}, as we noted earlier. To put a connection on them, we need a connection on the principal Spin$(2m)$-bundle P_{spin}. So assume in this example that M is a compact Riemannian manifold of dimension $2m$ with a spin structure, and let $P_{spin} \to M$ be its principal spin bundle.

In (ii) and (iii) of Proposition 14.4.5, we noted that $\mathrm{Lie}(\mathrm{Spin}(2m)) = C^2(V) = \mathrm{span}_{\mathbb{R}}\{e_ie_j : i < j\}$. Also we saw that the map $\tau : C^2(V) \to \mathfrak{so}(2m)$ satisfies:

$$\tau(e_ie_j) = 2(E_{ji} - E_{ij})$$

Indeed, this τ is precisely the derivative $\dot\rho$ of the map $\rho : \mathrm{Spin}(2m) \to SO(2m)$, because

$$\begin{aligned} \dot\rho(e_ie_j) &= \frac{d}{dt}_{|t=0}(\rho(\exp_C(te_ie_j)) = \frac{d}{dt}_{|t=0}(\rho(\cos t.1 + \sin te_ie_j)) \\ &= \frac{d}{dt}_{t=0} R^{ij}_{2t} = 2(E_{ji} - E_{ij}) \end{aligned}$$

where R^{ij}_θ is the counter-clockwise rotation by θ in the 2-plane $\mathbb{R}e_i \oplus \mathbb{R}e_j$, by using the last paragraph in the proof of (iv) Proposition 13.2.2.

Since $\rho \circ R_g = R_{\rho(g)} \circ \rho : P_{spin} \to P_{SO}$, we have:

$$\rho_* \circ R_{g*} = R_{\rho(g)*} \circ \rho_* : TP_{spin} \to TP_{SO} \tag{44}$$

Denote the map $x \mapsto gxg^{-1}$ on a Lie group G as $\mathrm{Ad}^G g$, and its derivative at $1 \in G$ simply as $\mathrm{Ad}\, g : \mathfrak{g} \to \mathfrak{g}$ where $\mathfrak{g} := \mathrm{Lie}(G) = T_1(G)$. Now, recalling the homomorphism $\rho : \mathrm{Spin}(2m) \to SO(2m)$ (also denoted by the same symbol ρ, in keeping with the definition of a spin structure), we see that the homomorphism:

$$\rho \circ \mathrm{Ad}^{\mathrm{Spin}(2m)} g : \mathrm{Spin}(2m) \to SO(2m)$$

is the same as the homomorphism:

$$\mathrm{Ad}^{SO}\rho(g) \circ \rho : \mathrm{Spin}(2m) \to SO(2m)$$

for all $g \in \mathrm{Spin}(2m)$. By equating the derivative at the identity $1 \in \mathrm{Spin}(2m)$ of both these maps and noting that $\dot\rho = D\rho(1) = \tau$, we have:

$$\tau \circ \mathrm{Ad}\, g = \mathrm{Ad}\,(\rho(g)) \circ \tau \tag{45}$$

Now define a $C^2(V)$-valued 1-form on P_{spin} by:

$$\widetilde\omega := \tau^{-1}(\rho^*\omega)$$

where $\omega \in \Lambda^1(P, \mathfrak{so}(2m))$ is the Levi-Civita connection 1-form on P_{SO}, and $\rho : P_{spin} \to P_{SO}$ is the double covering map. We need to check $\widetilde\omega$ satisfies the correct translation property. Using equations (44) and (45), we have:

$$\begin{aligned} R^*_g\widetilde\omega(v) &= \widetilde\omega(R_{g*}v) = \tau^{-1}\left[(\rho^*\omega)(R_{g*}v)\right] = \tau^{-1}[\omega(\rho_*R_{g*}v)] \\ &= \tau^{-1}[\omega(R_{\rho(g)*}\rho_*v)] = \tau^{-1}\left[(R^*_{\rho(g)}\omega)(\rho_*v)\right] \\ &= \tau^{-1}\mathrm{Ad}\rho(g)[\omega(\rho_*v)] = \mathrm{Ad}(g)\tau^{-1}[\omega(\rho_*v)] \\ &= \mathrm{Ad}(g)[(\tau^{-1}\rho^*\omega)(v)] = \mathrm{Ad}(g)\widetilde\omega(v) \end{aligned}$$

This connection form on P_{spin} is called *the spin connection.*

If $\widetilde\sigma : U \to P_{spin|U}$ is a local section of P_{spin} on a coordinate chart $U \subset M$, then $\sigma := \rho \circ \widetilde\sigma : U \to P_{SO|U}$ will be a local section for P_{SO}. Then

$$\widetilde\sigma^*\widetilde\omega = \tau^{-1}\widetilde\sigma^*\rho^*\omega = \tau^{-1}\sigma^*\omega$$

Let $\sigma^*\omega$ be given by the Cartan connection matrix of 1-forms (see Definition 15.2.2) $[\omega_{ij}]$ on U, so we can write $\sigma^*\omega := \sum_{i<j}\omega_{ij}E_{ij}$, where ω_{ij} are 1-forms on U. Since $\tau^{-1}(E_{ij}) = \frac{1}{2}e_ie_j$, it follows that the Cartan connection 1-form on U for the spin connection $\widetilde{\omega}$ is given on U by:

$$\omega^{sp} = \widetilde{\sigma}^*\widetilde{\omega} = \frac{1}{2}\sum_{i<j}\omega_{ij}e_ie_j \tag{46}$$

as an element of $C^\infty(U, C^2(V))$ where we are making the identification $C^2(V) = \text{Lie}(\text{Spin}(2m))$.

Now that we have a Spin$(2m)$-connection on P_{spin}, all associated vector bundles get a connection by the same procedure as before. That is, if $\mu : \text{Spin}(2m) \to GL(V)$ is any representation, and $\mathcal{V} = P_{spin} \times_\mu V$ is the associated bundle, then giving the Cartan 1-forms on a trivialising coordinate neighbourhood $U \subset M$ for P_{spin} by

$$\omega^{\mathcal{V}}_{ij} := \dot{\mu}(\omega^{sp})_{ij}$$

will define covariant differentiation $\nabla^{\mathcal{V}}$ on $\mathcal{V}$. Again if the representation is unitary, the bundle will be hermitian, and $\dot{\mu} : C^2(V) \to \mathfrak{u}(n)$ implies that the connection $\nabla^{\mathcal{V}}$ will be compatible with this hermitian metric, i.e. will be a unitary connection.

Letting $V := \Delta_{2m} = S_{2m}$, the complex spin-representation defined in Proposition 14.2.1, we have the spin-bundle $\mathcal{S}(M)$ defined in Definition 15.1.3 as the associated bundle $P \times_\mu \Delta_{2m}$. Here we are denoting this representation by μ, and since we saw in (ii) and (iii) of Proposition 15.1.4 that the representation of $\mathbb{C}l_{2m}$ was self-adjoint with respect to the hermitian inner-product of Δ_{2m}, and that this representation is unitary, viz. $\mu : \text{Spin}(2m) \to U(\Delta_{2m})$. Since the connection form $\dot{\mu}(\widetilde{\omega})$ takes values in $\mathfrak{u}(\Delta_{2m})$, the spin connection on $\mathcal{S}(M)$ is a unitary connection, and metric compatibility follows by definition.

To check Clifford compatibility, we need compute the commutator of the Cartan coefficients on an open set U, i.e. the skew-hermitian matrix $[\omega^{\mathcal{S}}] = [\dot{\mu}(\omega^{sp})_{ij}]$ and Clifford multiplication by $c \in \mathbb{C}l(M)_U$. Note that $\mu : \text{Spin}(2m) \to U(\Delta_{2m})$ is the restriction of the Clifford action $\mu : \mathbb{C}l_{2M} \to \text{End}_{\mathbb{C}}(\Delta_{2m})$. Also with the identification of $\text{Lie}(\text{Spin}(2m)) = C^2(V)$, it follows that:

$$\begin{aligned}
[\dot{\mu}(x), \mu(c)] &= \left[\frac{d(\mu(\exp_C tx)}{dt}_{|t=0}, \mu(c)\right] = \frac{d}{dt}_{|t=0}(\mu[\exp_C tx, c]) = \mu\left[\frac{d\exp_C tx}{dt}_{|t=0}, c\right] \\
&= \mu([x, c]) \quad \text{for } x \in C^2(V), c \in \mathbb{C}l_{2m}
\end{aligned}$$

Now, for a section $s \in \mathcal{S}(M)_{|U}$, c a smooth section for $\mathbb{C}l(M)_{|U}$ and X a smooth real vector-field on U, where U is a trivialising neighbourhood for P_{spin}, we have :

$$\begin{aligned}
[\nabla^{\mathcal{S}}_X, c]s &= \nabla^{\mathcal{S}}_X(c.s) - c.\nabla^{\mathcal{S}}_X s = \omega^{\mathcal{S}}(X)(c.s) - c.\omega^{\mathcal{S}}(X)s \\
&= \dot{\mu}(\omega^{sp}(X))\mu(c)s - \mu(c)(\dot{\mu}(\omega^{sp}(X))s = ([\dot{\mu}\omega^{sp}(X), \mu(c)])\, s \\
&= (\mu[\omega^{sp}(X), c])s = \mu(\tau(\omega^{sp}(X))c)s \quad \text{(by (ii) of 14.4.5)} \\
&= \mu\left([\omega^{SO}(X)]c\right) s = (\nabla_X c).s
\end{aligned}$$

which shows Clifford compatibility of the spin connection $\nabla^{\mathcal{S}}$.

Thus the spin connection on $\mathcal{S}(M)$ is a Clifford connection.

Proposition 15.2.10. Let M be a spin manifold of dimension $2m$, and Let $\mathcal{V} \to M$ be any hermitian complex vector bundle with the inner-product $(-,-)_{\mathcal{V}}$, and a unitary connection $\nabla^{\mathcal{V}}$ on it. Then the tensor product bundle:

$$\mathcal{E} := \mathcal{S}(M) \otimes_{\mathbb{C}} \mathcal{V}$$

equipped with the natural Clifford action, and the natural hermitian inner product $(-,-)_{\mathcal{E}} := (-,-)_{\mathcal{S}} \otimes (-,-)_{\mathcal{V}}$ is a Clifford module on M. The tensor product connection $\nabla^{\mathcal{E}}$ of the spin connection $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{V}}$ is a Clifford connection on this bundle.

Proof: The Clifford action is given by:

$$c.(s \otimes v) = c.s \otimes v \quad \text{for } c \in \mathbb{C}l(M), s \in \mathcal{S}(M), v \in \mathcal{V}$$

Clearly the supermodule structure is $\mathcal{E}^{\pm} = \mathcal{S}^{\pm}(M) \otimes_{\mathbb{C}} \mathcal{V}$. The tensor product hermitian inner product is given on decomposable elements by:

$$(s \otimes v, t \otimes w)_{\mathcal{E}} := (s,t)_{\mathcal{S}}(v,w)_{\mathcal{V}}$$

Since $S^+(M)$ and $S^-(M)$ are orthogonal under $(-,-)_{\mathcal{S}}$, it easily follows that $\mathcal{E}^+$ and $\mathcal{E}^-$ are orthogonal under $(-,-)_{\mathcal{E}}$.

To check self-adjointness of Clifford action, it is enough to do it on decomposable elements, and for that we have:

$$(c.(s\otimes v), t\otimes w)_{\mathcal{E}} = (c.s\otimes v, t\otimes w)_{\mathcal{E}} = (c.s,t)_{\mathcal{S}}(v,w)_{\mathcal{V}} = (-1)^{\deg c}(s, c.t)_{\mathcal{S}}(v,w)_{\mathcal{W}} = (-1)^{\deg c}(s\otimes v, c.(t\otimes w))_{\mathcal{E}}$$

by the self-adjointness of the Clifford module $\mathcal{S}(M)$.

The tensor product connection $\nabla^{\mathcal{E}}$ is unitary because we again check on decomposable sections that:

$$\begin{aligned}(\nabla^{\mathcal{E}}_X(s\otimes v), t\otimes w)_{\mathcal{E}} \quad &+ \quad (s\otimes v, \nabla^{\mathcal{E}}(t\otimes w))_{\mathcal{E}} = (\nabla^{\mathcal{S}}_X s\otimes v + s\otimes \nabla^{\mathcal{V}} v, t\otimes w)_{\mathcal{E}} + (s\otimes v, \nabla^{\mathcal{S}}_X t\otimes w + t\otimes \nabla^{\mathcal{V}} w)_{\mathcal{E}} \\ &= \quad \left[(\nabla^{\mathcal{S}}_X s, t)_{\mathcal{S}} + (s, \nabla^{\mathcal{S}}_X t)_{\mathcal{S}}\right](v,w)_V + (s,t)_{\mathcal{S}}\left[(\nabla^{\mathcal{V}}_X v, w)_{\mathcal{V}} + (v, \nabla^{\mathcal{V}}_X w)_{\mathcal{V}}\right] \\ &= \quad [X(s,t)_{\mathcal{S}}](v,w)_{\mathcal{V}} + (s,t)_{\mathcal{S}}[X(v,w)_{\mathcal{W}}] = X((s\otimes v, t\otimes w)_{\mathcal{E}})\end{aligned}$$

To check Clifford compatibility, again:

$$\begin{aligned}\nabla^{\mathcal{E}}_X(c.(s\otimes v)) = \nabla^{\mathcal{E}}_X(c.s\otimes v) \quad &= \quad \nabla^{\mathcal{S}}_X(c.s)\otimes v + c.s\otimes \nabla^{\mathcal{V}} v = (\nabla_X c.s + c.\nabla^{\mathcal{S}}_X s)\otimes v + c.s\otimes \nabla^{\mathcal{V}} v \\ = (\nabla_X c.s\otimes v) + c.(\nabla^{\mathcal{S}}_X s\otimes v \quad &+ \quad s\otimes \nabla^{\mathcal{V}} v) = (\nabla_X c).(s\otimes v) + c.\nabla^{\mathcal{E}}_X(s\otimes v)\end{aligned}$$

using the Clifford compatibility of $\nabla^{\mathcal{S}}$ proved in Example 15.2.9 above. This proves the proposition. □

Corollary 15.2.11. Let M be a spin manifold of dimension $2m$, and let $\mathcal{E} \to M$ be a Clifford module (over the Clifford bundle $\mathbb{C}l(M) \to M$). Then there exists a hermitian complex vector bundle $\mathcal{V} \to M$ with a compatible unitary connection $\nabla^{\mathcal{V}}$ such that the bundle $\mathcal{E} \simeq \mathcal{S} \otimes \mathcal{V}$ as Clifford modules over M, and $\mathcal{E}$ becomes a Clifford module with a Clifford connection given by the tensor-product connection of the spin connection $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{V}}$. Indeed, every Clifford connection on $\mathcal{E}$ is obtained in this way.

Proof: By the Proposition 15.1.9, we have a complex hermitian vector bundle $\mathcal{V} \to M$ such that $\mathcal{E} \simeq \mathcal{S}(M) \otimes \mathcal{V}$ as Clifford modules. Since V has a hermitian metric, it has a compatible unitary connection $\nabla^{\mathcal{V}}$ (by using partitions of unity, for example).

Define the connection $\nabla^{\mathcal{E}}$ as the tensor-product connection of spin-connection $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{V}}$. Then we are done by the Proposition 15.2.10 above.

To see the last assertion, note that for any finite dimensional module E over $\mathbb{C}l_{2m}$, we have an isomorphism of $\mathbb{C}l_{2m}$-modules by Proposition 14.4.6:

$$E \simeq S_{2m} \otimes_{\mathbb{C}} V \quad \text{for } \ V := \hom_{\mathbb{C}l_{2m}}(S_{2m}, E)$$

By breaking up E into irreducibles $E_i \simeq S_{2m}$ as before, and noting that $\hom_{\mathbb{C}l_{2m}}(E_i, E_j)$ is one dimensional, it is trivial to check that the natural map:

$$\begin{array}{rcl} \hom_{\mathbb{C}}(V, V) & \to & \hom_{\mathbb{C}l_{2m}}(E, E) = \hom_{\mathbb{C}l_{2m}}(S_{2m} \otimes_{\mathbb{C}} V, S_{2m} \otimes_{\mathbb{C}} V) \\ \Lambda & \mapsto & Id_{S_{2m}} \otimes \Lambda \end{array}$$

is an isomorphism of complex vector-spaces. Since this isomorphism is canonical (basis-independent), we have an isomorphism of complex vector bundles:

$$\hom_{\mathbb{C}l_{2m}}(\mathcal{E}, \mathcal{E}) \simeq \hom_{\mathbb{C}}(\mathcal{V}, \mathcal{V})$$

Now if $\nabla^{\mathcal{E}}$ denotes the tensor product connection defined above, and $\widetilde{\nabla}^{\mathcal{E}}$ is another Clifford connection, it follows by Leibnitz's rule that:

$$\left(\nabla^{\mathcal{E}} - \widetilde{\nabla}^{\mathcal{E}}\right)(fs) = f\left(\nabla^{\mathcal{E}} - \widetilde{\nabla}^{\mathcal{E}}\right)s \quad \text{for all } \ f \in C^\infty(M), \ s \in C^\infty(M, \mathcal{E})$$

which shows that $\left(\nabla^{\mathcal{E}} - \widetilde{\nabla}^{\mathcal{E}}\right) = \alpha$ for some smooth section $\alpha \in C^\infty(T^*M \otimes \hom_{\mathbb{C}}(\mathcal{E}, \mathcal{E}))$. The Clifford compatibility condition shows that $[\alpha, c] \equiv 0$ for all smooth sections $c \in C^\infty(M, \mathbb{C}l(M))$, i.e. $\alpha \in C^\infty(M, T^*M \otimes \hom_{\mathbb{C}l_{2m}}(\mathcal{E}, \mathcal{E})$. By the above, this last space is isomorphic to $C^\infty(M, T^*M \otimes \hom_{\mathbb{C}l_{2m}}(\mathcal{V}, \mathcal{V}))$, so that $\alpha = 1 \otimes \beta$ for some section $\beta \in C^\infty(M, T^*M \otimes \hom_{\mathbb{C}l_{2m}}(\mathcal{V}, \mathcal{V}))$. This shows that $\widetilde{\nabla}^{\mathcal{E}}$ is given by:

$$\widetilde{\nabla}^{\mathcal{E}} = \nabla^{\mathcal{E}} - \alpha = (\nabla^{\mathcal{S}}) \otimes 1 + 1 \otimes \nabla^{\mathcal{V}} - 1 \otimes \beta = \nabla^{\mathcal{S}} \otimes 1 + 1 \otimes (\nabla^{\mathcal{V}} - \beta)$$

which is the tensor product connection of $\nabla^{\mathcal{S}}$ and $\widetilde{\nabla}^{\mathcal{V}} := \nabla^{\mathcal{V}} - \beta$. This proves the last assertion. $\square$

Definition 15.2.12. We say that a Clifford module $\mathcal{E} \to M$ over M is a *Dirac Bundle* if it has a compatible Clifford connection.

Example 15.2.13. The bundles $\mathbb{C}l(M) \to M$, $\Lambda^*_{\mathbb{C}}(T^*M) \to M$, on an oriented Riemannian manifold of dimension $2m$ are all Dirac bundles, by Example 15.2.8. The spin bundle $\mathcal{S}(M) \to M$ on a spin manifold of dimension $2m$ is a Dirac bundle, by Example 15.2.9 above.

The above Corollary 15.2.11 says that to generate any Dirac bundle on a *spin manifold* M of dimension 2m, it is enough to start with the prototypical spinor bundle $\mathcal{S}(M) \to M$ with its natural structure as a Dirac bundle, and then twist it with various hermitian bundles $\mathcal{V} \to M$ (with compatible unitary connections).

15.3. **Dirac operator on a Dirac bundle.**

Definition 15.3.1. Let $\mathcal{E} \to M$ be a Dirac bundle on M, an Riemannian manifold of dimension $2m$, with chirality grading by $\mathcal{E}^{\pm}$. Let $\mathbb{C}l(M) \to M$ be the Clifford bundle of M. Denote by c the Clifford action on $\mathcal{E}$:

$$T^*(M) \otimes \mathcal{E} \xrightarrow{c} \mathcal{E}$$

Let $\nabla^{\mathcal{E}}$ denote the Clifford connection on $\mathcal{E}$. The *Dirac operator* on $\mathcal{E}$ is the operator D defined by the composite:

$$C^\infty(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^\infty(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} C^\infty(M, \mathcal{E})$$

Since $c(T^*M \otimes \mathcal{E}^{\pm}) \subset \mathcal{E}^{\mp}$, and by definition the Clifford connection preserves the subbundles $\mathcal{E}^{\pm}$, it follows that the Dirac operator is also $\mathbb{Z}_2$-graded, and $D = D^+ \oplus D^-$, where

$$D^{\pm} : C^\infty(M, \mathcal{E}^{\pm}) \to C^\infty(M, \mathcal{E}^{\mp})$$

Remark 15.3.2 (Dirac operator in local coordinates)**.** In a local coordinate chart, we may choose an orthonormal frame $\{e_i\}_{i=1}^{2m}$ of the cotangent bundle T^*M. Then for a smooth section $s \in C^\infty(M, \mathcal{E})$, we have

$$\nabla^{\mathcal{E}} s = \sum_{i=1}^{2m} e_i \otimes \nabla^{\mathcal{E}}_{e_i} s$$

so that the Dirac operator is expressed as

$$Ds = \sum_{i=}^{2m} e_i . \nabla^{\mathcal{E}}_{e_i} s$$

where the dot denotes Clifford action. Since $\nabla^{\mathcal{E}}_{e_i}$ are 1st order differential operators, it follows that D is a 1st order ifferential operator.

Proposition 15.3.3 (Self-adjointness of the Dirac operator)**.** Let M be a compact oriented Riemannian manifold of dimension $2m$, and let $(-,-)$ denote the given hermitian inner-product on a Dirac bundle $\mathcal{E} \to M$. Define the global L^2-inner product on $C^\infty(M, \mathcal{E})$ by $(s,t)_M := \int_M (s(x), t(x))_x dV(x)$. Then Dirac operator $D : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ is formally self-adjoint with respect to $(-,-)_M$. In particular $D^+ : C^\infty(M, \mathcal{E}^+) \to C^\infty(M, \mathcal{E}^-)$ and $D^- : C^\infty(M, \mathcal{E}^-) \to C^\infty(M, \mathcal{E}^+)$ are adjoints of each other.

Proof: Fix a point $x \in M$, and fix a *synchronous* orthonormal frame in a nieghbourhood U of x, i.e. for the Levi-Civita connection we have

$$(a) \quad \nabla e_i(x) \equiv 0 \qquad (b) \quad e_{i,x} = \partial_{i,x} = \frac{\partial}{\partial x_{i\,|x}} \quad \text{for all } \; i = 1, .., 2m$$

for some coordinate system $(x_1, ..., x_{2m})$ on U. By the self-adjointness of Clifford multiplication with respect to the pointwise hermitian inner product, unitarity and Clifford compatibility of the connection, and synchronicity of the frame $\{e_i\}$, we have:

$$\begin{aligned}(e_i \nabla^{\mathcal{E}}_{e_i} s, t)_x &= -(\nabla^{\mathcal{E}}_{e_i} s, e_i.t)_x = -e_i(s, e_i.t)_x + (s, \nabla^{\mathcal{E}}_{e_i}(e_i.t)_x \\ &= -e_i(s, e_i.t)_x + (s, (\nabla_{e_i} e_i).t)_x + (s, e_i.\nabla^{\mathcal{E}}_{e_i} t)_x \\ &= -\partial_i(s, \partial_i.t)_x + (s, e_i.\nabla^{\mathcal{E}}_{e_i} t)_x\end{aligned}$$

Summing over i we find:

$$(Ds, t)_x - (s, Dt)_x = -\delta\sigma(x)$$

where σ is the 1-form $v \mapsto (s, v.t)_x = \sum_i (s, e_i.t) e_i^*$ on U, and $\delta\sigma(x) = \sum_i \partial_i (s, e_i.t)_x = \pm(*d*\sigma)(x)$ (i.e. the divergence of σ). Integrating over M, and noting that $\int_M (\delta\sigma) dV = \int_M \delta\sigma \wedge (*1) == \pm \int_M \sigma \wedge d(*1) = 0$, we have:

$$(Ds, t)_M = (s, Dt)_M$$

and our assertion follows. The last statement is clear from the fact that the restriction of D to $C^\infty(M, \mathcal{E}^{\pm})$ are $D^{\pm}$ respectively. □

Corollary 15.3.4. With the hypothesis of the previous proposition, the second order differential operator:

$$D^2 : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$$

is formally self-adjoint with respect to $(-,-)_M$. Its restrictions, namely the composites D^+D^- and D^-D^+ :

$$C^\infty(M, \mathcal{E}^\pm) \stackrel{D^\pm}{\to} C^\infty(M, \mathcal{E}^\mp) \stackrel{D^\mp}{\to} C^\infty(M, \mathcal{E}^\pm)$$

are self-adjoint.

15.4. **Weitzenbock Formulas.** We need to assert that the square of the Dirac operator on a Dirac bundle is a generalised Laplacian. To this end, we have the following.

Definition 15.4.1. Let $\mathcal{E} \to M$ be a Dirac bundle, and let $\nabla^{\mathcal{E}}$ be its Clifford connection. Then for two real tangent vector fields $X, Y \in C^\infty(M, T_x(M))$, we define a 2-form $\Omega^{\mathcal{E}} \in C^\infty(\Lambda^2 T^*M \otimes \hom_{\mathbb{C}}(\mathcal{E}, \mathcal{E}))$ by:

$$\Omega^{\mathcal{E}}(X, Y) \otimes s = \nabla^{\mathcal{E}}_X \nabla^{\mathcal{E}}_Y s - \nabla^{\mathcal{E}}_Y \nabla^{\mathcal{E}}_X s - \nabla^{\mathcal{E}}_{[X,Y]} s$$

The 2-form $\Omega^{\mathcal{E}}$ is called the *curvature of the Clifford connection* $\nabla^{\mathcal{E}}$ or just the *Clifford curvature of* $\mathcal{E}$.

That the object on the right side of the definition defines a 2-form follows by changing X to fX and Y to gY where f and g are two smooth functions, and calculating by Leibnitz's rule that:

$$\left[\nabla^{\mathcal{E}}_{fX}, \nabla^{\mathcal{E}}_{gY}\right] - \nabla^{\mathcal{E}}_{[fX,gY]} = fg\left(\left[\nabla^{\mathcal{E}}_X, \nabla^{\mathcal{E}}_Y\right] - \nabla^{\mathcal{E}}_{[X,Y]}\right)$$

Exercise 15.4.2 (Clifford curvature in local coordinates)**.** Let $\{s_i\}$ be an orthonormal frame of $\mathcal{E}_{|U}$ with respect to $(-,-)$, the hermitian inner prroduct on $\mathcal{E}$, where $U \subset M$ is a trivialising neighbourhood of $\mathcal{E}$. Then we may write:

$$\nabla^{\mathcal{E}} s_j = \sum_{i=1}^{\mathrm{rk}_{\mathbb{C}}\mathcal{E}} \omega_{ij} s_i$$

where ω_{ij} is the skew-hermitian matrix of Cartan connection 1-forms on U. Apply the definitions to show that $\Omega^{\mathcal{E}}$ is another skew hermitian matrix of 2-forms given by:

$$\Omega^{\mathcal{E}}_{ij} = d\omega_{ij} + \sum_{l=1}^{\mathrm{rk}_{\mathbb{C}}\mathcal{E}} (\omega_{il} \wedge \omega_{lj} - \omega_{lj} \wedge \omega_{li})$$

which is often abbreviated in the notation:

$$\Omega = d\omega + [\omega, \omega]$$

Proposition 15.4.3 (Weitzenbock for a Dirac Bundle)**.** Let $\mathcal{E} \to M$ be a Dirac bundle on an oriented compact Riemannian manifold of dimension $2m$. Then the square of the Dirac operator is given by:

$$D^2 = \nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} + \frac{1}{2}\Omega^{\mathcal{E}}$$

Proof: We again fix a point $x \in M$ and choose a synchronous frame $\{e_i\}$ for $TM_{|U}$ for some neighbourhood U of x. Then note that we have $[e_i, e_j]_x = [\partial_{i,x}, \partial_{j,x}] = 0$ for all $1 \le i, j \le 2m$, and also $(\nabla_{e_i} e_j)(x) = 0$ for the Levi-Civita connection on TM. Then denote$\nabla^{\mathcal{E}}_{e_j}$ by $\nabla^{\mathcal{E}}_j$, and similarly for the Levi-Civita covariant derivative ∇_{e_j} by ∇_j. We first note that:

$$\nabla^{\mathcal{E}} : C^\infty(M, \mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E})$$

has a *global* L_2*-adjoint*:

$$\nabla^{\mathcal{E}*} : C^\infty(M, T^*M \otimes \mathcal{E}) \to C^\infty(M, \mathcal{E})$$

which satisfies:

$$(\nabla^{\mathcal{E}} s, \omega \otimes t)_M = (s, \nabla^{\mathcal{E}*}(\omega \otimes t))_M \ \text{ for all } \ \omega \in \Lambda^1(M, \mathbb{C}), \ \ s, t \in C^\infty(M, \mathcal{E})$$

(Note that the hermitian inner product on $T^*_{\mathbb{C}}M \otimes \mathcal{E}$ is taken to be the tensor product hermitian inner-product of the hermitian inner products on the two factors.) In fact, we claim that in terms of the synchronous frame at x defined above, we have:

$$\nabla^{\mathcal{E}*}(e_i^* \otimes t)(x) = -(\nabla_i^{\mathcal{E}} t)(x) \tag{47}$$

To verify this, we write $\nabla^{\mathcal{E}} s = \sum_j e_j^* \otimes \nabla_j^{\mathcal{E}} s$, then:

$$\begin{aligned}
(\nabla^{\mathcal{E}} s, e_i^* \otimes t)_x &= \sum_j (e_j^* \otimes \nabla_j^{\mathcal{E}} s, e_i^* \otimes t)_x = \sum_j (e_j, e_i)_x (\nabla_j^{\mathcal{E}} s, t)_x \\
&= (\nabla_i^{\mathcal{E}} s, t)_x = e_i(s,t)_x - (s, \nabla_i^{\mathcal{E}} t) \\
&= \sum_j e_j (s, e_j \lrcorner (e_i^* \otimes t))_x + (s, \nabla^{\mathcal{E}*}(e_i^* \otimes t))_x \\
&= \delta\sigma(x) + (s, \nabla^{\mathcal{E}*}(e_i^* \otimes t))_x
\end{aligned}$$

where σ is the 1-form defined by $v_x \mapsto (s, v \lrcorner (e_i^* \otimes t))_x$. Again, integrating over M and noting that $\int_M \delta\sigma dV = 0$ by Stokes Theorem, we have our assertion.

Now we compute for a section $s \in C^\infty(U, \mathcal{E})$ that:

$$\begin{aligned}
D^2 s(x) &= \sum_i e_i \nabla_i^{\mathcal{E}} \left(\sum_j e_j \nabla_j^{\mathcal{E}} s \right) = \sum_{i,j} (e_i \nabla_i e_j) . \nabla_j^{\mathcal{E}} s + \sum_{i,j} e_i e_j \nabla_i^{\mathcal{E}} \nabla_j^{\mathcal{E}} s \\
&= \sum_{i,j} e_i e_j \nabla_i^{\mathcal{E}} \nabla_j^{\mathcal{E}} s \quad \text{since } \ (\nabla_i e_j)(x) = 0 \\
&= -\sum_i \nabla_i^{\mathcal{E}} \nabla_i^{\mathcal{E}} s + \sum_{i<j} e_i . e_j . (\nabla_i^{\mathcal{E}} \nabla_j^{\mathcal{E}} - \nabla_j^{\mathcal{E}} \nabla_i^{\mathcal{E}} s) \quad \text{(by Clifford relations on } e_i) \\
&= \sum_i \nabla^{\mathcal{E}*}(e_i^* \otimes \nabla_i^{\mathcal{E}} s) + \frac{1}{2} \Omega^{\mathcal{E}} s \ \ \text{(since } \ [e_i, e_j](x) = 0 \ \text{ and by using (47) above)} \\
&= \nabla^{\mathcal{E}*} \nabla^{\mathcal{E}} s + \frac{1}{2} \Omega^{\mathcal{E}} s
\end{aligned}$$

where $\Omega^{\mathcal{E}} s(x) = \left[\sum_{i,j} e_i e_j . (\nabla_i^{\mathcal{E}} \nabla_j^{\mathcal{E}} - \nabla_j^{\mathcal{E}} \nabla_i^{\mathcal{E}}) s \right](x)$ in our synchronous frame $\{e_i\}$ around x. This proves the assertion. □

Proposition 15.4.4 (Weitzenbock formula for the Spin Bundle)**.** Let M be a compact spin manifold of dimension $2m$, and let $\mathcal{S}(M) \to M$ be the spin bundle on M with its natural structure as a Dirac bundle with its spin connection $\nabla^{\mathcal{S}}$ (see Examples 15.2.9 and 15.2.13). Then for its Dirac operator D, we have:

$$D^2 = \nabla^{\mathcal{S}*} \nabla^{\mathcal{S}} + \frac{1}{4} \sum_{i,j} k$$

where k is the scalar curvature function of the Riemannian metric on M.

Proof: In view of the Proposition 15.4.3 above, we need to calculate the curvature operator $\Omega^{\mathcal{S}}$ of the spin connection $\nabla^{\mathcal{S}}$. We recall from the Example 15.2.9 that the spin connection 1-form $\widetilde{\omega} \in C^\infty(P_{spin}, T^* P_{spin} \otimes C^2(V))$ is related to the connection 1-form $\omega \in C^\infty(P_{SO}, T^* P_{SO} \otimes \mathfrak{so}(2m))$ by:

$$\widetilde{\omega} = \tau^{-1}(\rho^* \omega)$$

Since pullbacks commute with exterior differentiation and wedge products, and τ is an isomorphism of Lie algebras, it follows that the curvature of $\widetilde{\omega}$ is related to the curvature Ω of ω by:

$$\widetilde{\Omega} = \tau^{-1}(\rho^* \Omega)$$

Now pulling back everything onto a trivialising neighbourhood U for P_{spin} (resp. P_{SO}) via a section $\sigma : U \to P_{spin|U}$ (resp. $(\rho \circ \sigma : U \to P_{SO|U})$, and using that $\tau(e_i e_j) = 2(E_{ji} - E_{ij})$, we have:

$$\begin{aligned} \Omega^{spin} &= \sigma^*(\tau^{-1}.\rho^*\Omega) = \tau^{-1}((\rho \circ \sigma)^*\Omega) \\ &= \tau^{-1}(\sum_{i,j} \Omega_{ij}^{SO} E_{ij}) = \tau^{-1}(\sum_{i<j} \Omega_{ij}^{SO}(E_{ji} - E_{ij})) = \frac{1}{2}\sum_{i<j} \Omega_{ij}^{SO} e_i e_j \\ &= \frac{1}{4}\sum_{i \neq j} \Omega_{ij}^{SO} e_i e_j \end{aligned}$$

where Ω_{ij}^{SO} is the Cartan curvature 2-form on U for P_{SO}.

In terms of the synchronous frame $\{e_i\}$ of T^*M around a point x one knows that the curvature form of the Riemannian connection is related to the Riemannian curvature tensor by:

$$\Omega_{ij}^{SO} = -\sum_{k \neq l} R_{klij} e_k e_l = -R_{klij} e_k e_l$$

where we have used the Einstein repeated summation convention. (The minus sign comes from the fact that the Riemannian connection of the principal bundle P_{SO} of frames in the cotangent bundle T^*M is the negative of that of the tangent bundle). It follows that:

$$\begin{aligned} \Omega^{\mathcal{S}} &= \Omega^{spin} = -\frac{1}{4} R_{klij} e_i e_j e_k e_l \\ &= \frac{1}{4}(e_i e_j e_l R_{klij}) e_k \ \text{ since } \ R_{llij} = 0 \ \text{ and } \ e_k e_l = -e_l e_k \ \text{ for } \ k \neq l \end{aligned}$$

If i, j, l are distinct indices, $e_i e_j e_l = e_l e_i e_j = e_j e_l e_i$, and also by the Bianchi identity, $R_{klij} + R_{kjli} + R_{kijl} = 0$. SO all such terms will drop out of the sum above. Terms with $i = j$ also vanish since R_{klij} is antisymmetric in i, j. So the only terms remaining are those with $i = l \neq j$ and $i \neq l = j$. The sum becomes:

$$\begin{aligned} \Omega^{\mathcal{S}} &= \frac{1}{4}(e_l e_j e_l R_{kllj} + e_i e_l e_l R_{klil}) e_k = \frac{1}{4}(e_j R_{kllj} - e_i R_{klil}) e_k \\ &= \frac{1}{4} e_j (R_{kllj} - R_{kljl}) e_k = -\frac{1}{2}(R_{kljl}) e_j e_k \\ &= -\frac{1}{2} R_{kj} e_j e_k = -\frac{1}{2} R_{ii} e_i^2 = \frac{1}{2} k \quad \text{(since Ricci curvature } R_{ij} \text{ is symmetric)} \end{aligned}$$

which proves the proposition by substituting into the Weitzenbock formula in Proposition 15.4.3. □

Corollary 15.4.5 (Bochner-Lichnerowicz)**.** Let M be a compact spin manifold of dimension $2m$, and with everywhere strictly positive scalar curvature. Then the kernel of the Dirac operator on $C^\infty(M, \mathcal{S}(M))$ is trivial. (That is, M has no "harmonic spinors").

Proof: Let $s \in C^\infty(M, \mathcal{S}(M))$, with $Ds = 0$. By the Weitzenbock formula of Proposition 15.4.4, we have

$$0 = (Ds, Ds)_M = (D^2 s, s) = (\nabla^{\mathcal{S}*}\nabla^{\mathcal{S}} s, s) + \frac{1}{4}(ks, s)_M$$

If $s \neq 0$, the fact that $k > 0$ everwhere implies the right hand side is strictly positive, and we have a contradiction. □

Corollary 15.4.6 (Weitzenbock for a Dirac bundle on a spin manifold)**.** Let M be a spin manifold of dimension $2m$, and let $\mathcal{E} \to M$ be a Dirac bundle on M with Clifford connection $\nabla^{\mathcal{E}}$. By the Corollary 15.2.11, we have that $\mathcal{E} = \mathcal{S} \otimes_{\mathbb{C}} \mathcal{V}$, where $\mathcal{S} \to M$ is the spin bundle on M, and $\nabla^{\mathcal{E}}$ is the tensor product connection of the spin connection $\nabla^{\mathcal{S}}$ on S, and a unitary connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}$. Then, for the Dirac operator $D^{\mathcal{E}}$ we have the Weitzenbock formula:

$$(D^{\mathcal{E}})^2 = \nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} + \frac{1}{4}k + R^{\mathcal{V}}$$

where $R^{\mathcal{V}}$ is the curvature operator of $\mathcal{V}$ defined by

$$R^{\mathcal{V}}(s \otimes \sigma) = \sum_{i<j} e_i.e_j.s \otimes \Omega^{\mathcal{V}}(e_i, e_j)\sigma = \sum_{i<j} c(e_i)c(e_j)R(e_i, e_j)(s \otimes \sigma)$$

and k is the scalar curvature function of M.

Proof: By Proposition 15.4.3 above, we just have to compute the Clifford curvature $\Omega^{\mathcal{E}}$ in terms of the curvatures $\Omega^{\mathcal{S}}$ and $\Omega^{\mathcal{V}}$. First note that by definition of the tensor product connection:

$$\nabla^{\mathcal{E}}_X(s \otimes \sigma) = \nabla^{\mathcal{S}}_X s \otimes \sigma + s \otimes \nabla^{\mathcal{V}}_X \sigma \quad \text{for all} \;\; X \in T_xM, \;\; s \in C^\infty(M, \mathcal{S}), \;\; \sigma \in C^\infty(M, \mathcal{V})$$

For real vector fields X, Y on M, the commutators

$$[\nabla^{S}_X \otimes 1, 1 \otimes \nabla^{\mathcal{V}}_Y] = 0 = [\nabla^{S}_Y \otimes 1, 1 \otimes \nabla^{\mathcal{V}}_X]$$

Hence

$$\begin{aligned}
\Omega^{\mathcal{E}}(X,Y)(s \otimes \sigma) &= \left([\nabla^{\mathcal{E}}_X, \nabla^{\mathcal{E}}_Y] - \nabla^{\mathcal{E}}_{[X,Y]}\right)(s \otimes \sigma) \\
&= \left([\nabla^{\mathcal{S}}_X, \nabla^{\mathcal{S}}_Y]s\right) \otimes \sigma + s \otimes \left([\nabla^{\mathcal{V}}_X, \nabla^{\mathcal{V}}_Y]\sigma\right) - \nabla^{\mathcal{S}}_{[X,Y]} s \otimes \sigma - s \otimes \nabla^{\mathcal{V}}_{[X,Y]}\sigma \\
&= \Omega^{S}(X,Y)s \otimes \sigma + s \otimes \Omega^{\mathcal{V}}(X,Y)\sigma
\end{aligned}$$

Now, in a local orthonormal frame $\{e_i\}$:

$$\begin{aligned}
\Omega^{\mathcal{E}}(s \otimes \sigma) &= \sum_{i,j} c(e_i)c(e_j)\Omega^{\mathcal{E}}(e_i, e_j)(s \otimes \sigma) \\
&= (\sum_{i,j} e_i.e_j\Omega^{\mathcal{S}}_{ij}s) \otimes \sigma + \sum_{i,j} e_i.e_j.s \otimes \Omega^{\mathcal{V}}_{ij}\sigma \\
&= \frac{k}{2} s \otimes \sigma + 2\sum_{i<j}(e_i.e_j.s) \otimes \Omega^{\mathcal{V}}_{ij}\sigma) = \frac{k}{2} s \otimes \sigma + 2R^{\mathcal{V}}(s \otimes \sigma)
\end{aligned}$$

by using the Bochner-Lichnerowicz formula for the spin bundle $\mathcal{S}$ deduced in Corollary 15.4.5. Our corollary now follows from the Weitzenbock formula 15.4.3. □

Corollary 15.4.7 (Bochner's Theorem)**.** Let M be a compact oriented Riemannian manifold of dimension $2m$. For the Dirac bundle $\Lambda^*_{\mathbb{C}}M \to M$, the Dirac operator D is the operator $d + \delta$, (viz. the Dirac operator of the elliptic deRham complex). Furthermore:

(i): On 1-forms, we have the *Bochner formula*:

$$\Delta\phi = \nabla^{T^*M*}\nabla^{T^*M}\phi + R\phi \quad \text{for} \;\; \phi \in \Lambda^1(M, \mathbb{C})$$

where R is the Ricci-curvature operator of M.

(ii): If M has everwhere positive Ricci curvature (viz. R is a real positive definite symmetric matrix at each point of M), then the first Betti number $\beta_1(M) := \dim_{\mathbb{C}} H^1(M, \mathbb{C})$ vanishes (that is M has no nontrivial harmonic 1-forms).

Proof: By definition, in a local orthonormal frame e_i of T^*M, we have:

$$D = \sum_i c(e_i)\nabla_i$$

where $\nabla_i = \nabla_{e_i}$ is with respect to the Levi-Civita connection. If we further assume the frame is synchronous at x, then $\nabla_{e_i,x} = \partial_{i,x}$. The operator $c(e_i)$ of Clifford multiplication by e_i on the Clifford module $\Lambda^*_{\mathbb{C}}T^*M$ is given by (see Example 15.1.7):

$$c(e_i)\alpha = e_i \wedge \alpha - e_i \lrcorner \alpha \quad \alpha \in \Lambda^*(M, \mathbb{C})$$

So, in a synchronous frame at x, the Dirac operator reads as:

$$D\alpha = \sum_i e_i \wedge \partial_{i,x}\alpha - \sum_i e_i \lrcorner\, \partial_{i,x}\alpha = d\alpha + \delta\alpha$$

(Note the minus sign appears because the "L^2-adjoint of ∂_i is $-\partial_i$" from integration by parts.) This proves the first assertion.

To see the Bochner formula in (i), we appeal to the Weitzenbock formula from Proposition 15.4.3, and apply it to the Dirac bundle $\mathcal{E} := \Lambda^*_{\mathbb{C}}(T^*M)$. We note that by the above, $D^2 = (d+\delta)^2 = d\delta + \delta d = \Delta$, the Laplace-Beltrami operator. For the right side, we need to compute the Clifford curvature $\frac{1}{2}\Omega^{\mathcal{E}}$. We continue with the synchronous frame above, and for the sake of convenience, we denote the operator $e_k \wedge (-)$ by e_k, and the operator $e_k \lrcorner (-)$ by i_k. Note that:

$$e_k i_l(e_m) + i_l e_k(e_m) = e_k\delta_{lm} + i_l(e_k \wedge e_m) = e_k\delta_{lm} + \delta_{kl}e_m - e_k\delta_{lm} = \delta_{kl}e_m$$

so that $e_k i_l = -i_l e_k$ for $k \neq l$. Now, the Clifford curvature operator on a 1-form $\phi = \phi_k e_k$ (using the repeated summation convention) is given by:

$$\begin{aligned} \Omega^{\mathcal{E}}\phi &= c(e_k)c(e_l)\Omega^{\mathcal{E}}(e_k, e_l)\phi = -(e_k - i_k)(e_l - i_l)R_{klrs}\phi_r e_s \\ &= (e_k i_l + i_k e_l)R_{klrs}\phi_r e_s = (e_k i_l - i_l e_k)R_{klrs}\phi_r e_s \ \ (\text{since } \ R_{klrs} = -R_{lkrs}) \\ &= 2e_k i_l R_{klrs}\phi_r e_s = 2R_{klrs}\phi_r e_k \delta_{ls} = 2R_{klrl}\phi_r e_k = 2(R_{kr}\phi_r)e_k = 2R\phi \end{aligned}$$

so that $\frac{1}{2}\Omega^{\mathcal{E}}\phi = R\phi$ and the Bochner formula (i) follows.

To see (ii), note that if $\phi \in \Lambda^1(M, \mathbb{C})$ is a harmonic form with $\phi \neq 0$, then $\Delta\phi = 0$ so that by the Bochner formula:

$$0 = (\Delta\phi, \phi) = (\nabla\phi, \nabla\phi) + (R\phi, \phi) > 0$$

by the hypothesis on R, a contradiction. Now, by (i) of the Hodge Theorem,

$$\beta_1(M) = \dim_{\mathbb{C}} H^1(M, \mathbb{C}) = \dim_{\mathbb{C}} \ker \Delta_{\Lambda^1}$$

so (ii) and the Corollary follow. □

Corollary 15.4.8. Let $\mathcal{E} \to M$ be a Dirac bundle, with associated Dirac operator D. Then the operator (called the *Dirac Laplacian of* $\mathcal{E}$):

$$D^2 : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$$

is a generalised laplacian in the sense of Definition 12.2.1. In particular, the operators $D^-D^+ : C^\infty(M, \mathcal{E}^+) \to C^\infty(M, \mathcal{E}^+)$ and $D^+D^- : C^\infty(M, \mathcal{E}^-) \to C^\infty(M, \mathcal{E}^-)$ are both generalised laplacians. The two term complexes $D^\pm : C^\infty(M, \mathcal{E}^\pm) \to C^\infty(M, \mathcal{E}^\mp)$ are both elliptic 2- term complexes in the sense of Definition 9.4.1, and the two operators D^+D^- and D^-D^+ are the Dirac Laplacians of this 2-term elliptic complex.

Proof: By the Weitzenbock formula Proposition 15.4.3, we have:

$$D^2 = \nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} + \frac{1}{2}\Omega^{\mathcal{E}}$$

The last term $\frac{1}{2}\Omega^{\mathcal{E}}$ is a zero-th order operator, locally given as $\frac{1}{2}\sum_{i,j} e_i e_j.\Omega^{\mathcal{E}}_{ij}$, where each $\Omega_{i,j}$ is a 2-form. In the proof of Weitzenbock's formula, we also computed the adjoint of $\nabla^{\mathcal{E}}$ in a synchronous frame $\{e_i\}$at x to be:

$$\nabla^{\mathcal{E}*}(e_i \otimes s)(x) = -\nabla_i s$$

Consider the tensor product of the Levi-Civita connection ∇ and $\nabla^{\mathcal{E}}$, and call it the connection $\nabla^{T^*M\otimes\mathcal{E}}$. We claim that the composite:

$$C^\infty(M, T^*M \otimes \mathcal{E}) \stackrel{\nabla^{T^*M\otimes\mathcal{E}}}{\to} C^\infty(M, T^*M \otimes T^*M \otimes \mathcal{E}) \stackrel{-\mathrm{tr}}{\to} C^\infty(M, \mathcal{E})$$

is the same as

$$\nabla^{\mathcal{E}*} : C^\infty(M, T^*M \otimes \mathcal{E}) \to C^\infty(M, \mathcal{E})$$

(see Lemma 12.2.4). For, if we compute with the synchronous frame at x used in the proof of the Weitzenbock formula in 15.4.3, we have:

$$\nabla^{T^*M\otimes\mathcal{E}}(e_i^* \otimes s)(x) = \nabla e_i^* \otimes s + e_i^* \otimes \nabla^{\mathcal{E}} s = e_i^* \otimes \sum_j e_j^* \nabla_j^{\mathcal{E}} s$$

since $(\nabla e_i)(x) = 0$. Thus:

$$-\mathrm{tr}\nabla^{T^*M\otimes\mathcal{E}}(e_i^* \otimes s)(x) = -\mathrm{tr}(\sum_j e_i^* \otimes e_j^* \nabla_j^{\mathcal{E}} s)(x) = -(\sum_j \delta_{ij} \nabla_j^{\mathcal{E}} s)(x) = -(\nabla_i^{\mathcal{E}} s)(x)$$

We computed in the proof of Weitzenbock that:

$$\nabla^{\mathcal{E}*}(e_i^* \otimes s)(x) = -(\nabla_i^{\mathcal{E}} s)(x)$$

Hence our assertion follows. Thus, in the notation of Lemma 12.2.4,

$$\nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} = -\mathrm{tr}\nabla^{T^*M\otimes\mathcal{E}}\nabla^{\mathcal{E}} = \Delta^{\mathcal{E}}$$

By the Lemma 12.2.4, $\Delta^{\mathcal{E}}$ is a generalised laplacian. Hence $D^2 = \Delta^E + \frac{1}{2}\Omega^{\mathcal{E}}$ is also a generalised laplacian. Since D^+D^- and D^-D^+ are restrictions of D^2 to $C^\infty(M, \mathcal{E}^-)$ and $C^\infty(M, \mathcal{E}^+)$ respectively, they are also elliptic second order differential operators.

Since D^+ and D^- are adjoints of each other by Proposition 15.3.3, the 2-term complex is an elliptic omplex by 9.4.2. $\square$

16. The Atiyah-Singer Index Theorem

The goal now is to write down a formula for the index of a Dirac operator on a Dirac bundle. The idea of the proof of the index theorem is to carefully examine the coefficient of the term independent of t in the asymptotic expansion of the (super)trace of heat kernel for the Dirac laplacian, since integrating this over M would compute the index of D, in view of Proposition 10.2.1. To handle the Dirac operator by bare hands is quite an effort, and was carried out by Patodi, Atiyah-Bott-Patodi and Gilkey for the various classical Dirac bundles. There is however a simple proof due to Getzler, following ideas of the physicists Alvarez-Gaume and Witten, which replaces the Dirac laplacian by a much simpler operator by a scaling procedure.

Before we get into the proof of the index theorem, let us study this simpler operator.

16.1. The Quantum Harmonic Oscillator and Mehler's Formula.

Definition 16.1.1. The quantum harmonic oscillator is the Schrodinger operator defined on $C^\infty(\mathbb{R})$ by:

$$H := -\frac{d^2}{dx^2} + x^2$$

Proposition 16.1.2 (Facts about the Harmonic Oscillator)**.** *H defined above is formally self-adjoint (on compactly supported functions), and has a discrete positive spectrum $\lambda_n = (n + \frac{1}{2})$, corresponding to smooth eigenfunctions ϕ_n in the Schwartz class $\mathcal{S}(\mathbb{R})$ (which are defined in terms of the Hermite functions). Finally, ϕ_n form an orthonormal Hilbert space basis for $L_2(\mathbb{R})$.*

Proof: The formal self-adjointness on $C_c(\mathbb{R})$ is clear since $H = \Delta + x^2$, and both operators on the right are formally self- adjoint. Also since

$$(H\phi, \phi) = (\partial_x\phi, \partial_x\phi) + (x\phi, x\phi)$$

for the $L_2(\mathbb{R})$ inner-product $(-,-)$, it follows that the eigenvalues (if any) of H are non-negative.

To get the discreteness of the spectrum, one uses the *annihilation operator* $A = x + \partial_x$ and its adjoint, the *creation operator* $A^* = x - \partial_x$. It is easily checked that:

$$\begin{aligned} AA^* &= H + I, \quad A^*A = H - I \\ [A, A^*] &= -2I \\ [H, A] &= -2A, \quad [H, A^*] = 2A^* \end{aligned}$$

Then one defines the *ground state* of the oscillator as the function ϕ_0 satisfying $A\phi_0 = 0$ and $\|\phi_0\| = 1$. That is,

$$(\partial_x + x)\phi = 0$$

But this is a simple ODE, and by using the integrating factor of $e^{x^2/2}$, and using the L_2-normalisation, we have

$$\phi_0 = \pi^{-1/4} e^{-x^2/2}$$

Now all the other eigenfunctions are given inductively by applying the creation operator A^* and normalising. More precisely:

$$\phi_k = (2k)^{-1/2} A^* \phi_{k-1}$$

Note that if ϕ_{k-1} corresponds to eigenvalue λ_{k-1}, then

$$\begin{aligned} H\phi_k &= (2k)^{-1/2} HA^*\phi_{k-1} = (2k)^{-1/2} \left(A^*\lambda_{k-1}\phi_{k-1} + [H, A^*]\phi_{k-1} \right) \\ &= (\lambda_{k-1} + 2)\phi_k \end{aligned}$$

So it remains to compute the eigenvalue of the ground state ϕ_0. But

$$(-\partial_x^2 + x^2)e^{-x^2/2} = \partial_x(xe^{-x^2}) + x^2 e^{-x^2/2} = e^{-x^2/2}$$

So $H\phi_0 = \phi_0$, and $\lambda_0 = 1$. This shows that $\lambda_k = (2k+1)$. Since ϕ_0 is is the Schwartz class, so is $\phi_k = C(A^*)^k \phi_0$.

We will skip the proof of the fact that ϕ_k form an orthonormal basis of $L_2(\mathbb{R})$. See standard texts on Quantum Mechanics, which prove that ϕ_k is a polynomial times the Hermite function H_k. □

Corollary 16.1.3. The associated heat operator $(H + \partial_t)$ on $\mathbb{R}$ has a smooth integral kernel $p_t(x, y)$ which satisfies:

(i): $(H_x + \partial_t)p_t(x, y) = 0$ for all $t > 0$, and $x, y \in \mathbb{R}$.

(ii): For all $\phi \in L_2(\mathbb{R})$, and $t > 0$ the function $F(x, t) := e^{-tH}\phi$ is a smooth function, given by the integral $\int_{\mathbb{R}} p_t(x, y)\phi(y)dy$. It satisfies:

$$\lim_{t \to 0} F(x, t) = \lim_{t \to 0} \int_{\mathbb{R}} p_t(x, y)\phi(y)dy = \phi(x)$$

Proof: Follows by defining :

$$p_t(x, y) = \sum_{k=0}^{\infty} e^{-t\lambda_k} \phi_k(x)\phi_k(y)$$

which is a convergent series for all $t > 0$ since the coefficients $e^{-t\lambda_k}$ die faster than all powers of t, and ϕ_k are in the Schwartz class. The proof of the other assertions are analogous to the case of a positive elliptic operator on a compact manifold (see (iii) of the Proposition 10.1.3 and (iii) of Proposition 10.1.6). □

Now we can explicitly compute $u(x, t) = p_t(x, 0)$. Note that by definition, this is a *fundamental solution* to the heat equation, satisfying:

$$(H + \partial_t)u(x, t) = 0, \quad \lim_{t \to 0} u(x, t) = \delta_x$$

where δ_x is the Dirac distribution massed at x.

Proposition 16.1.4 (Mehler's Formula)**.** The function $u(x,t)$ defined by:

$$u(x,t) = (2\pi \sinh 2t)^{-1/2} \exp\left(-\frac{x^2 \coth 2t}{2}\right)$$

is a fundamental solution to the heat equation $(H+\partial_t)u(x,t) = 0$.

Proof: By taking one's cue from the Gaussian, we try:

$$u(x,t) = \alpha(t) \exp\left(-\frac{\beta(t)x^2}{2}\right)$$

Then compute derivatives:

$$\begin{aligned}
-\partial_x u(x,t) &= \alpha\beta x \exp\left(-\frac{\beta x^2}{2}\right)\\
-\partial_x^2 u(x,t) &= \alpha\beta(1-\beta x^2)\exp\left(-\frac{\beta x^2}{2}\right) = \beta(1-\beta x^2)u(x,t)\\
Hu(x,t) &= (-\partial_x^2 + x^2)u(x,t) = (\beta + (1-\beta^2)x^2)u(x,t)\\
\partial_t u(x,t) &= \left[\alpha'(t) - \frac{\alpha\beta'(t)x^2}{2}\right]\exp\left(-\frac{\beta x^2}{2}\right) = \left(\frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)x^2}{2}\right)u(x,t)
\end{aligned}$$

So if we arrange that:

$$\frac{\alpha'}{\alpha} + \beta = 0, \quad \beta'/2 = (1-\beta^2)$$

Then $u(x,t)$ would be a solution to the required heat equation $(H+\partial_t)u(x,t) = 0$. The second equation leads to:

$$\left(\frac{1}{\beta+1} - \frac{1}{\beta-1}\right) d\beta = 4\,dt$$

so that $\log\left(\frac{\beta+1}{\beta-1}\right) = 4t + C$, which implies (by taking $C=0$) that

$$\beta(t) = \coth 2t$$

The other equation now becomes:

$$\alpha'(t) + \alpha(t)\coth 2t = 0$$

which is the same as: $\sinh 2t\alpha'(t) + \cosh 2t\alpha(t) = 0$, which is rewritten as $2\sinh 2t\alpha'(t)\alpha(t) + 2\cosh 2t\alpha^2(t) = 0$. But this implies:

$$\frac{d}{dt}(\alpha(t)^2 \sinh 2t) = 0$$

Thus $\alpha(t) = C(\sinh 2t)^{-1/2}$, for some constant C. Now as $t \to 0$, $\alpha(t) \sim C(2t)^{-1/2}$ and $\beta(t) \sim 1/2t$, so that $u(x,t) \sim C(2t)^{-1/2}e^{-x^2/4t}$ as $t \to 0$. We choose $C = (2\pi)^{-1/2}$, so that $u(x,t)$ approaches the Euclidean heat kernel as $t \to 0$. Hence:

$$u(x,t) = (2\pi \sinh 2t)^{-1/2} \exp\left(-\frac{\coth 2t\, x^2}{2}\right)$$

and we certainly have $(H+\partial_t)u(x,t) = 0$. Also, as $t \to 0$, $u(x,t) \to (4\pi t)^{-1/2}\exp(-x^2/4t)$, the Euclidean heat kernel on $\mathbb{R}$, and we know by the Proposition 11.1.1 that the Euclidean heat kernel tends to δ_x as $t \to 0$. (Actually, more precise estimates are needed to justify these limits in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, but we leave these details to the reader). The proposition follows. □

Corollary 16.1.5 (Mehler's Formula II)**.** Let us define the 2nd order differential operator on $C^\infty(\mathbb{R})$ given by:

$$H := -\partial_x^2 + \frac{a^2x^2}{16} + b$$

where $a, b \in \mathbb{R}$. Then a fundamental solution to $(H+\partial_t)v(x,t) = 0$ is given by:

$$v(x,t) = (4\pi t)^{-1/2}\left(\frac{at/2}{\sinh(at/2)}\right)^{\frac{1}{2}} \exp\left(-(at/2)\coth(at/2)(x^2/4t) - bt\right)$$

Proof: First we try to find a solution to $(H+\partial_t)v(x,t)=0$ by tinkering with the fundamental solution of the foregoing proposition. So let $u(y,s)$ be the fundamental solution satisfying:

$$(-\partial_y^2+y^2+\partial_s)u(y,s)=0 \tag{48}$$

where $u(y,s)$ is as in the statement of Proposition 16.1.4 above. Define:

$$v(x,t)=e^{-bt}u(\lambda^{1/2}x,\lambda t)=e^{-bt}u(y,s) \quad \text{where} \quad y:=\lambda^{1/2}x, \quad s:=\lambda t$$

Then:

$$\partial_t v(x,t)=-be^{-bt}u(y,s)+e^{-bt}\lambda\partial_s u(y,s)$$

which implies

$$(\partial_t+b)v(x,t)=\lambda e^{-bt}\partial_s u(y,s) \tag{49}$$

Now for the space derivatives:

$$\begin{aligned}(-\partial_x^2+\lambda^2x^2)v(x,t) &= \lambda\left(-\frac{1}{\lambda}\partial_x^2+\lambda x^2\right)v(x,t)\\ &= \lambda(-\partial_y^2+y^2)e^{-bt}u(y,s)=\lambda e^{-bt}(-\partial_y^2+y^2)u(y,s)\end{aligned} \tag{50}$$

Adding the equations (49) and (50), we find:

$$(-\partial_x^2+\lambda^2x^2+b+\partial_t)v(x,t)=\lambda e^{-bt}(-\partial_y^2+y^2+\partial_s)u(y,s)=0$$

from equation (48). Thus $v(x,t)$ is a solution to the equation in the statement by setting $\lambda=a/4$. Thus by using the explicit formula for $u(y,s)$ derived in Proposition 16.1.4, the fundamental solution we seek is given by:

$$\begin{aligned}v(x,t) &= Ce^{-bt}u(\frac{a^{1/2}x}{2},at/4)=Ce^{-bt}(2\pi\sinh(at/2))^{-1/2}\exp\left[-\coth(at/2)(ax^2/8)\right]\\ &= \widetilde{C}(4\pi t)^{-1/2}(at/2)^{1/2}(\sinh(at/2))^{-1/2}\exp\left[-(at/2)\coth(at/2)(x^2/4t)-bt\right]\end{aligned}$$

Note that as $\lim_{t\to0}\frac{\sinh(at/2)}{(at/2)}=1$, $\lim_{t\to0}bt=0$ and $\lim_{t\to0}\cosh(at/2)=1$, which implies that $\lim_{t\to0}(at/2)\coth(at/2)=1$. Thus $v(x,t)$ above approaches the Euclidean heat kernel $(4\pi t)^{-1/2}\exp(-x^2/4t)$ if we set $\widetilde{C}=1$ (Again pointwise limits are not good enough, one needs to use Lebesgue's dominated convergence theorem to make these assertions in $\mathcal{S}'(\mathbb{R})$. We leave these matters to the reader.) This proves the corollary. □

We would like to write a multivariate and matrix formulation of the above Mehler formula. First we make a definition.

Definition 16.1.6. Denote by $\mathcal{A}$ the commutative algebra $\Lambda^{ev}_{\mathbb{C}}(\mathbb{R}^{2m})$, with $\wedge$ being the multiplication. Note that any word of $a_1a_2...a_i$ of length $i>m$ and with $a_j\in\oplus_{k\geq1}\Lambda^{2k}_{\mathbb{C}}$ (i.e. no a_j has a constant term) vanishes.

Let R be a skew-symmetric $2m\times2m$ matrix whose entries are in $\Lambda^2_{\mathbb{C}}(\mathbb{R}^{2m})$. Note that R is automatically a nilpotent matrix, by the remark above. Hence all power series in tR for $t\in(0,\infty)$ are actually polynomials in t. For such an $\mathbb{R}$, define the $\mathcal{A}$-valued function:

$$j(R):=\det\left(\frac{e^{R/2}-e^{-R/2}}{R}\right)=\det\left(\frac{\sinh(R/2)}{(R/2)}\right)$$

Note that $e^{R/2}-e^{-R/2}$ is the series (=polynomial) (of $\mathcal{A}$-valued matrices) given by $2\sinh(R/2)$, and involving only odd powers of R, whence $\frac{1}{2}(e^{R/2}-e^{-R/2})=\sinh(R/2)=(R/2)\alpha(R)$, for another polynomial $\alpha(R)$ with leading coefficient 1. Then $j(R)$ is the determinant of $\alpha(R)$.

Indeed, if t is small enough, the series:

$$\alpha(tR)=I+\frac{t^2R^2}{2^23!}+\frac{t^4R^4}{2^45!}+...$$

is a polynomial, and an invertible element, since every term except the first is nilpotent. Its determinant $j(tR) := \det(\alpha(tR))$ is a unit in $\mathcal{A}$, and again a polynomial in t. Then one can define $j(tR)^{-1/2} = [\det(\alpha(tR))]^{-1/2}$ as a formal power series

$$(j(tR))^{-1/2} = 1 + \sum_{i=1}^{\infty} t^i f_i(R)$$

where f_i are polynomials in the entries of R. This formal power series is again a polynomial in t since $j(tR) - I$ is a nilpotent element.

Now we consider the *symmetric matrix*

$$\beta(tR) := (tR/2)\coth(tR/2)$$

Then it defines an $\mathcal{A}$-valued symmetric bilinear form (or quadratic form) on $\mathbb{R}^{2m}$ by the formula:

$$\langle x|(tR/2)\coth(tR/2)|y\rangle := \sum_{i,j=1}^{2m} x_i(tR/2)\coth(tR/2)_{ij}y_j$$

Again, we have a power series expansion for $(tR/2)\coth(tR/2)$ in terms of even powers of t, which starts with I (since $\cosh(tR/2)$, and $\frac{tR/2}{\sinh(tR/2)} = \alpha(tR)^{-1}$ both have even power series starting with I). So the quadratic form above has a power series expansion:

$$\langle x|(tR/2)\coth(tR/2)|x\rangle = \|x\|^2 + \sum_{k=1}^{\infty} t^{2k}c_k\langle x|R^{2k}|x\rangle$$

Since R is nilpotent, this power series is again a polynomial.

Proposition 16.1.7 (Mehler's Formula III)**.** Let R be a skew-symmetric $2m \times 2m$ matrix, and let F be any $N \times N$ matrix, both matrices having coefficients in $\Lambda^2_{\mathbb{C}}(\mathbb{R}^{2m})$. Set $\mathcal{A} := \Lambda^{ev}_{\mathbb{C}}(\mathbb{R}^{2m})$. Note both matrices are *constant* with respect to $x \in \mathbb{R}^{2m}$.

Define the *generalised harmonic oscillator* to be the operator defined on $C^\infty(\mathbb{R}^{2m}, \mathcal{A} \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N))$ by:

$$H(f \otimes T) = -\left[\sum_{i=1}^{2m}\left(\partial_i + \frac{1}{4}\sum_j R_{ij}x_j\right)^2 f\right] \otimes T + f \otimes F.T \quad \text{for } f \in C^\infty(\mathbb{R}^n, \mathcal{A}),\ T \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$$

Then the associated heat operator $(H + \partial_t)$ has a fundamental solution $p_t(x, R, F)$ defined by:

$$p_t(x, R, F) = (4\pi t)^{-m} j(tR)^{-1/2} \exp\left(\frac{-1}{4t}\langle x|(tR/2)\coth(tR/2)|x\rangle\right)\exp(-tF)$$

with $\lim_{t\to 0} p_t(x, R, F) = \delta_x(1 \otimes Id)$. (Note that by the discussion in Definition 16.1.6 above, $j(tR)$, $\langle x|(tR/2)\coth(tR/2)|x\rangle$ and $\exp(-tF)$ are all polynomials in t, by the nilpotency of the matrices R and F).

Proof: As remarked in the discussion following Definition 16.1.6, the power $(j(tR))^{-1/2}$, $\langle x|(tR/2)\coth(tR/2)|s\rangle$, and $\exp(-tF)$ are all *polynomials* in t, whose coefficients are *polynomials* in the coefficients of R and F, because the entries of R and F are in $\Lambda^2_{\mathbb{C}}(\mathbb{R}^{2m})$, and any word $a_1a_2..a_i$ of length $i > m$ with $a_i \in \oplus_{i\geq 1}\Lambda^{2i}_{\mathbb{C}}(\mathbb{R}^{2m})$ vanishes. Suppose we verify the formula:

$$(H + \partial_t)p_t(x, R, F) = 0$$

for R and F matrices with real entries. Since $(H + \partial_t)p_t(x, R, F)$ is an analytic function of F_{ij} and R_{ij}, it will follow that the equation $(H + \partial_t)p_t(x, R, F) = 0$ for all F with complex entries F_{ij} and all skew-symmetric matrices R with complex entries R_{ij}. That is, we will have an identity of power series in R_{ij} and F_{ij}. Hence this identity will hold when we substitute F a nilpotent matrix with entries in $\Lambda^2_{\mathbb{C}}(\mathbb{R}^{2m})$, and R an antisymmetric matrix with entries in $\Lambda^2_{\mathbb{C}}(\mathbb{R}^{2m})$. So we may assume without loss of generality that R is a real antisymmetric matrix, and F is a matrix with real entries. Note that all the power series (in t) occurring above in the expression for $p_t(x, R, F)$ converge for small values of t at least.

Note that $R \in \Lambda^2(\mathbb{R}^{2m}) = \mathfrak{so}(2m)$, and there is a matrix $P \in SO(2m)$ which conjugates R into the Cartan subalagebra of $\mathfrak{so}(2m)$. That is, there is a change of orthonormal basis (given by P) for $\mathbb{R}^{2m}$ so that $PRP^t = S$ is in block diagonal form, where the i-th block of S is:

$$S^i := \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}$$

Since $j(tR)$ is the determinant of $\alpha(tR)$, we will have $j(tR) = j(tS)$. Setting $y = Px$, we find that the quadratic form:

$$\begin{aligned} \langle x|(tR/2)\coth(tR/2)|x\rangle &= \langle (tR/2)\coth(tR/2)x, x\rangle = \langle P(tR/2)\coth(tR/2)x, Px\rangle \\ &= \langle (tS/2)\coth(tS/2)Px, Px\rangle = \langle y|(tS/2)\coth(tS/2)|y\rangle \end{aligned}$$

Finally, note that

$$\begin{aligned} \frac{\partial}{\partial x_i} + \frac{1}{4}\sum_j R_{ij}x_j &= \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} + \frac{1}{4}(Rx)_i = \sum_j P_{ji}\partial_{y,j} + \frac{1}{4}(P^tSPx)_i \\ &= [P^t(\partial_y + \frac{1}{4}Sy)]_i \end{aligned}$$

which implies that the "norm" of the "vector $(\partial_x + \frac{1}{4}Rx)$ is the same as that of $(\partial_y + \frac{1}{4}Sy)$, that is:

$$\sum_i \left(\frac{\partial}{\partial x_i} + \frac{1}{4}\sum_j R_{ij}x_j\right)^2 = \sum_i \left(\frac{\partial}{\partial y_i} + \frac{1}{4}\sum_j S_{ij}y_j\right)^2$$

Of course F will not change, so under the change of variables $x \mapsto y = Px$, the form of the operator H will remain the same, with R replaced by S and x replaced by y. Hence proving that$(H + \partial_t)p_t(x, R, F) = 0$ is equivalent to proving that $(H + \partial_t)p_t(y, S, F)$. Hence we may assume without loss of generality that R is in block diagonal form.

But once R is in block diagonal form, we are reduced to showing the identity for $m = 1$. Indeed, defining the 2×2 block operator:

$$H^i = -\left(\partial_{2i-1} - a_i x_{2i}\right)^2 - \left(\partial_{2i} + a_i x_{2i-1}\right)^2 + \frac{1}{m}F \quad \text{for } \ i = 1, 2, .., m$$

and denoting $x^i := (x_{2i-1}, x_{2i})$, and its fundamental solution by $p_t^i(x^i, S^i, \frac{1}{m}F)$, we note that $p_t(x, S, F) = \prod_{i=1}^m p_t^i(x^i, S^i, \frac{1}{m}F)$ obeys the equation

$$Hp_t = \sum_{i=1}^m (H^i p_t) = \sum_{i=1}^m (p_t^1 ... \widehat{p_t^i} .. p_t^m) H^i p_t^i = -\sum_{i=1}^m (p_t^1 ... \widehat{p^i}_t .. p_t^m)\partial_i p_t^i = -\partial_t p_t$$

Also, as $t \to 0$, each $p^i \to \delta_{x_{2i-1}, x_{2i}}$, and so $p \to \delta_x$, since the Dirac distribution in several variables is the product of the Dirac distributions in each variable. Thus we need to only find the two variable solution p_t^i.

Also note that the expression on the right, viz.

$$(4\pi t)^{-m} j(tS)^{-1/2} \exp\left(\frac{-1}{4t}\langle x|(tS/2)\coth(tS/2)|x\rangle - tF\right)$$

is exactly the expression:

$$= \prod_{i=1}^m \left((4\pi t)^{-1} j(tS^i)^{-1/2} \exp\left(\frac{-1}{4t}\langle x^i|(tS^i/2)\coth(tS^i/2)|x^i\rangle - tF/m\right)\right)$$

since determinants (like $j(tS)$) are mulplicative with respect to direct sum of (2×2)-blocks, and quadratic forms are additive.

Thus we may as well assume that we are in $\mathbb{R}^2$. That is, $m = 1$, and

$$R = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

In this event:

$$\begin{aligned} H &= -(\partial_1 - \frac{1}{4}ax_2)^2 - (\partial_2 + \frac{1}{4}ax_1)^2 + F \\ &= -(\partial_1^2 + \partial_2^2) + \frac{a}{2}(x_2\partial_1 - x_1\partial_2) - \frac{a^2}{16}\|x\|^2 + F \\ &= \left(-\partial_1^2 + \left(\frac{ia}{4}\right)^2 x_1^2 + \frac{F}{2}\right) + \left(-\partial_2^2 + \left(\frac{ia}{4}\right)^2 x_2^2 + \frac{F}{2}\right) + \frac{a}{2}(x_2\partial_1 - x_1\partial_2) \end{aligned} \tag{51}$$

On the other hand, by diagonalising R over $\mathbb{C}$, we have:

$$j(tR) = \det\left(\frac{e^{tR/2} - e^{-tR/2}}{tR}\right) = \left(\frac{\sinh(iat/2)}{(iat/2)}\right)\left(\frac{\sinh(-iat/2)}{(-iat/2)}\right) = \left(\frac{\sinh(iat/2)}{(iat/2)}\right)^2$$

Similarly, the quadratic form:

$$\langle x|(tR/2)\coth(tR/2)|x\rangle = (iat/2)\coth(iat/2)x_1^2 + (-iat/2)\coth(-iat/2)x_2^2 = (iat/2)\coth(iat/2)(x_1^2 + x_2^2)$$

So the function:

$$\begin{aligned} p_t(x_1, x_2) : &= (4\pi t)^{-1} j(tR)^{-1/2} \exp(-\frac{1}{4t}\langle x|(tR/2)\coth(tR/2)|x\rangle - tF) \\ &= \prod_{j=1}^{2}\left[(4\pi t)^{-1/2}\left(\frac{iat/2}{\sinh(iat/2)}\right)^{1/2} \exp\left(-(iat/2)\coth(iat/2)\frac{x_j^2}{4t} - tF/2\right)\right] \end{aligned}$$

is a fundamental solution for the operator

$$(-\partial_1^2 + (ia/4)^2 x_1^2 + F/2) + (-\partial_2^2 + (ia/4)^2 x_1^2 + F/2)$$

by the Corollary 16.1.5. (We have to soup up that Corollary to include all complex a, but that is straightforward). Also, since the function $p_t(x_1, x_2)$ is a function only of $(x_1^2 + x_2^2)$ in the space variables, it is annihilated by the operator $(x_2\partial_1 - x_1\partial_2)$. Hence it is a fundamental solution of H in equation (51). This proves the proposition. □

16.2. **The Heat Kernel and Index Density.**

Proposition 16.2.1. Let $\mathcal{E} \to M$ be a Dirac bundle on a compact Riemannian oriented manifold of dimension $2m$. Then for the two term elliptic complex:

$$\mathrm{Str} D := \mathrm{ind}\,(D^+) = \dim\ker D^+ - \dim\ker D^- = \int_M \mathrm{str}\, k_t(x,x) dV(x) = -\mathrm{ind}\, D^-$$

where $k_t(x,x)$ is a self-adjoint endomorphism of $\mathcal{E}_x$ that maps $\mathcal{E}_x^\pm$ to $\mathcal{E}_x^\pm$. Indeed, $k_t(x,y)$ is the integral kernel which represents the heat-operator e^{-tD^2} for the Dirac laplacian D^2. (Note that for a endomorphism $T \in \hom_{\mathbb{C}}(\mathcal{E}_x, \mathcal{E}_x)$, which *preserves the grading*, we define the supertrace as in Definition 14.5.1, i.e. $\mathrm{str}\, T = \mathrm{tr}\, T^+ - \mathrm{tr}\, T^-$).

Proof: By the Corollary 15.4.8, the two term complex:

$$D^+ : C^\infty(M, \mathcal{E}^+) \to C^\infty(M, \mathcal{E}^-)$$

is an elliptic complex. By (iii) of the Proposition 10.1.3, the infinitely smoothing heat operators $e^{-t\Delta^\pm} = e^{-tD^\mp D^\pm}$ on $C^\infty(M, \mathcal{E}^\pm)$ have integral heat kernels:

$$k_t^\pm(x,y) \in C^\infty(M \times M, \hom_{\mathbb{C}}(\pi_2^*\mathcal{E}^\pm, \pi_1^*\mathcal{E}^\pm))$$

By the McKean-Singer Theorem 10.2.1, we have:

$$\mathrm{ind}(D^+) = \int_M \left(\mathrm{tr}(k_t^+(x,x)) - \mathrm{tr}(k_t^-(x,x)\right) dV(x)$$

Now note that $k_t(x,x) : \mathcal{E}_x \to \mathcal{E}_x$ can be defined as the operator which is $k_t^\pm(x,x)$ on $\mathcal{E}_x^\pm$, in which case, its supertrace:

$$\mathrm{str}\, k_t(x,x) = \mathrm{tr}\, k_t^+(x,x) - \mathrm{tr}\, k_t^-(x,x)$$

by the definition of supertrace above. The proposition follows. □.

Now let us consider the case of a spin-manifold M. We have:

Lemma 16.2.2. Let M be a spin manifold of dimension $2m$, and $\mathcal{E}$ be a Dirac bundle on it. By the Corollary 15.2.11 $\mathcal{E}$ is isomorphic as a Dirac bundle to $\mathcal{S}(M)\otimes_{\mathbb{C}}\mathcal{V}$, where $\mathcal{S}(M)\to M$ is the spin bundle on M with its spin connection $\nabla^{\mathcal{S}}$, and $\mathcal{V}$ is a twisting bundle with some unitary connection $\nabla^{\mathcal{V}}$, and $\nabla^{\mathcal{E}}$ is the tensor product connection of these two connections. Then:

(i): There is an isomorphism of complex vector bundles:

$$\hom_{\mathbb{C}}(\mathcal{E},\mathcal{E})\simeq\Lambda^*_{\mathbb{C}}T^*M\otimes\hom_{\mathbb{C}}(\mathcal{V},\mathcal{V})$$

(ii): For an endomorphism $K:=\alpha\otimes F\in\hom_{\mathbb{C}}(\mathcal{E}_x,\mathcal{E}_x)$ where $\alpha\in\Lambda^*_{\mathbb{C}}T^*_xM$ and $F\in\hom_{\mathbb{C}}(\mathcal{V}_x,\mathcal{V}_x)$, the supertrace:

$$\mathrm{str}_{\mathcal{E}}K=(-2i)^mT(\alpha)\mathrm{tr}_{\mathcal{V}}F$$

where $T:\Lambda^*_{\mathbb{C}}(T^*M)\to\Lambda^{2m}_{\mathbb{C}}(T^*M)$ is the projection into the top-degree forms, introduced in Definition 14.5.1.

Proof: First note that the map:

$$\begin{array}{rcl}\mathbb{C}l_{2m}&\to&\hom_{\mathbb{C}}(S_{2m},S_{2m})\\ c&\mapsto&c.(-)\end{array}$$

is an isomorphism by the last assertion in Proposition 14.1.19. Since this isomorphism is canonical, we have a bundle isomorphism:

$$\mathbb{C}l(M)\simeq\hom_{\mathbb{C}}(\mathcal{S}(M),\mathcal{S}(M))$$

By the symbol map $\mathbb{C}l(M)\simeq\Lambda^*_{\mathbb{C}}(T^*M)$. Finally note that:

$$\begin{array}{rcl}\hom_{\mathbb{C}}(\mathcal{E},\mathcal{E})&\simeq&\hom_{\mathbb{C}}(\mathcal{S}(M)\otimes_{\mathbb{C}}\mathcal{V},\mathcal{S}(M)\otimes_{\mathbb{C}}\mathcal{V})\simeq\hom_{\mathbb{C}}(\mathcal{S}(M),\mathcal{S}(M))\otimes_{\mathbb{C}}\hom_{\mathbb{C}}(\mathcal{V},\mathcal{V})\\ &\simeq&\Lambda^*_{\mathbb{C}}(T^*M)\otimes_{\mathbb{C}}\hom_{\mathbb{C}}(\mathcal{V},\mathcal{V})\end{array}$$

This proves (i).

To see (ii), note that we may view

$$K\in\mathbb{C}l(M)_x\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(\mathcal{V}_x)$$

as the element $c(\alpha)\otimes F$, where c is the quantisation map identifying $\Lambda^*_{\mathbb{C}}T^*_xM$ with $\mathbb{C}l(M)_x$, and $c(\alpha)$ means the element of defined by Clifford multiplication by $c(\alpha)\in\mathrm{End}_{\mathbb{C}}(\mathcal{S}(M)_x)$. Then by definition:

$$\mathrm{str}_{\mathcal{E}}K=\mathrm{str}_{\mathcal{E}}(c(\alpha)\otimes F)=\mathrm{tr}_{\mathcal{E}}(\tau_{2m}\circ(c(\alpha)\otimes F))=\mathrm{tr}_{\mathcal{E}}(\tau_{2m}c(\alpha)\otimes F)=\mathrm{tr}_{\mathcal{S}}(\tau_{2m}c(\alpha)).\mathrm{tr}_{\mathcal{V}}F=\mathrm{str}_{\mathcal{S}}\,c(\alpha).\mathrm{tr}_{\mathcal{V}}F$$

where τ_{2m} is the chirality element in $\mathbb{C}l(M)_x$. Now by the Lemma 14.5.2, we have

$$\mathrm{str}_{\mathcal{S}}c(\alpha)=(-i)^m(\dim_{\mathbb{C}}\mathcal{S})(T\circ\sigma(c(\alpha))=(-2i)^m(T(\alpha))$$

Plugging this into the foregoing equation, we have (ii), and the lemma follows. □

Proposition 16.2.3 (The index density)**.** Let M be a compact spin manifold of dimension $2m$. Let $\mathcal{E}\to M$ be a Dirac bundle on M, and D the corresponding Dirac operator. Note that D^2 is a generalised Laplacian by Corollary 15.4.8. For $a\in M$, there is an asymptotic expansion: 12.3.5:

$$k_t(x,a)\sim(4\pi t)^{-m}\exp\left(-\frac{\delta(x,a)^2}{4t}\right)\left(\sum_{i=0}^{\infty}t^ik_i(x)\right)$$

where $k_0(a)=Id_{\mathcal{E}}$. The the index of the Dirac operator is given by:

$$\mathrm{ind}\,D=\int_M\mathrm{str}_{\mathcal{E}}k_t(a,a)dV(a)=(4\pi)^{-m}\int_M\mathrm{str}_{\mathcal{E}}k_m(a)dV(a)=\int_M\nu(a,\mathcal{E})dV(a)$$

The quantity $\nu(a,\mathcal{E})$ is called the *index density* of the Dirac operator of $\mathcal{E}$, and is is a polynomial in the jets of the metric and the unitary connection $\nabla^{\mathcal{V}}$ on the twisting bundle $\mathcal{V}$ at the point a.

Proof: As remarked earlier, we know from (iii) of the Proposition 10.1.3, that the smooth integral kernel $k_t(x,y)$ exists, and by the Proposition 16.2.1,

$$\operatorname{ind} D = \int_M \operatorname{str}_{\mathcal{E}} k_t(a,a) dV(a)$$

Choose a local framing $\{e_{j,y}\}$ for $\mathcal{E}_{|U}$ and $y \in U$ a neighbourhood of a, and note that for a basis element $e_{j,a} \in \mathcal{E}_a$, $u_t^j(y) := k_t(y,a)e_{j,a}$ is the fundamental solution of e^{-tD^2} with pole at $(a, e_{j,a})$, by Proposition 11.2.2. Thus

$$k_t(y,a) = \sum_{j=1}^{\dim \mathcal{E}} u_t^j(y) \otimes e_{j,a}^*$$

By the Theorem 12.3.5, there is an asymptotic expansion:

$$u_t^j(y) \sim (4\pi t)^{-m} \exp\left(-\frac{\delta(y,a)^2}{4t}\right)\left(\sum_{i=0}^{\infty} t^i u_i^j(y)\right)$$

such that $u_0^j(a) = e_j \in \mathcal{E}_a$, and for all i, the vector $u_j(a) \in \mathcal{E}_a$ is given as a polynomial in the jets of the coefficients of the Clifford connection $\Delta^{\mathcal{E}}$ at a. (See the last statement of Theorem 12.3.5, and note that the Dirac Laplacian is a generalised laplacian whose 0-th and 1st order terms involve the connection coefficients of $\nabla^{\mathcal{E}}$, by 15.4.8). By considering $\sum_j u_t^j(y) \otimes e_{j,y}^*$, we have a corresponding asymptotic expansion for $k_t(y,a)$ given by:

$$k_t(y,a) = (4\pi t)^{-m} \exp\left(-\frac{\delta(y,a)^2}{4t}\right)\left(\sum_{i=0}^{\infty} t^i k_i(y)\right)$$

with $k_0(a) = \sum_j u_0^j(a) \otimes e_{j,a}^* = \sum_j e_{j,a} \otimes e_{j,a}^* = I_{\mathcal{E}_a}$, and $k_i(a)$ all depending polynomially on the jets of the connection coefficients of $\nabla^{\mathcal{E}}$ at a.

By (iii) of the Theorem 12.3.5

$$\left\| u_t^j(y) - S_k(y) \right\|_{l,\infty} \leq C t^{N+1}$$

for a sufficiently long partial sum S_k of the asymptotic series for $u_t^j(y)$, and so a corresponding statement holds for $k_t(y,a)$. Since $\delta(a,a) = 0$, it follows that the difference

$$\left| \int_M \operatorname{str}_{\mathcal{E}} k_t(a,a) dV(a) - (4\pi t)^{-m} \sum_{i=0}^{k} t^i \int_M \operatorname{str}_{\mathcal{E}} k_i(a) \right| < C t^{N+1} < \epsilon$$

for $t < \delta$ and k large enough depending on m and N. The first integral inside the modulus sign is the index of D, and constant in t, so it follows that

$$\operatorname{ind} D = \int_M \operatorname{str}_{\mathcal{E}} k_t(a,a) dV(a) = (4\pi)^{-m} \int_M \operatorname{str}_{\mathcal{E}} k_m(a) dV(a)$$

where $k_m(a) \in \operatorname{End}_{\mathbb{C}}(\mathcal{E}_a)$ depends polynomially on the jets of the connection coefficients of $\nabla^{\mathcal{E}}$ at a.

By (i) of the previous Lemma 16.2.2, we can write the endomorphism $k_m(a) \in \operatorname{End}_{\mathbb{C}}(\mathcal{E}_a)$ as:

$$(4\pi)^{-m} k_m(a) = \sum_{i=1}^{r} \alpha_i(a) \otimes F_i(a) \quad \alpha_i \in \Lambda_{\mathbb{C}}^*(T_a^* M), \quad F_i(a) \in \operatorname{End}_{\mathbb{C}}(\mathcal{V}_a)$$

and by (ii) of the same Lemma,

$$\operatorname{str}_{\mathcal{E}} k_m(a) = (-2i)^m \sum_i T(\alpha_i(a)) \operatorname{tr}_{\mathcal{V}} F_i(a) =: \nu(a, \mathcal{E})$$

Since the Clifford connection $\nabla^{\mathcal{E}}$ is the tensor product of the spin connection on $\mathcal{S}(M)$, and the unitary connection $\nabla^{\mathcal{V}}$, the polynomial dependence of $\nu(a,\mathcal{E})$ on the jets of the metric g on M and connection coefficients of $\nabla^{\mathcal{V}}$ at a is clear from the corresponding fact about $k_m(a)$ stated above. This proves the proposition. □

16.3. **Local expression for $\nabla^{\mathcal{E}}$.** In the light of the Proposition 16.2.3, all we need to do now is to find out $\nu(a, \mathcal{E})$ where

$$\text{str}_{\mathcal{E}} k_m(a) = (-2i)^m (T \otimes \text{tr})(k_m(a)) = \nu(a, \mathcal{E}) dV(a)$$

This is a purely a point-wise problem at each $a \in M$. So we introduce the *geodesic normal coordinates* on a geodesically convex neighbourhood U of a (via the exponential map), with $\exp_a : T_aM \to M$ a diffeomorphism of some neighbourhood W of $0 \in T_aM$ with U, and $\exp_a(0) = a$. This will give a local formula for the Dirac operator $D^{\mathcal{E}}$ on the neighbourhood W. If we can write the asymptotic expansion coefficient $k_m(0)$ for the heat kernel k_t of $D^{\mathcal{E}}$ from this expression, then one can compute its supertrace.

We first need a lemma about synchronous frames (which we have been using in for T^*M in the past).

Lemma 16.3.1 (Synchronous framings)**.** Let $V \to M$ be a complex vector bundle with a connection ∇, M a Riemannian manifold. Then for $a \in M$, there exists a neighbourhood U of a, a trivialisation of $V_{|U}$ by sections $\{s_\alpha\}$, and a coordinate system $\{x_i\}$ on U with $a = (0, .., 0)$ such that:

(i): The Cartan connection coefficients of ∇ are given on U by:

$$\nabla s_\alpha = \omega . s_\alpha = \sum_\beta \omega_{\beta\alpha} s_\beta$$

where ω is a 1-form with values in $\text{End}_{\mathbb{C}}(V)$.

(ii): Denoting the curvature 2-form of ∇ by F, and denoting the curvature coefficients by:

$$F = \sum_{i<j} F_{ij} dx_i \wedge dx_j$$

with $F_{ij}(x) := F(\partial_i, \partial_j)(x) \in \text{End}(V_x)$, we have:

$$\omega(x) = -\frac{1}{2} \sum_{i,j} F_{ij}(0) x_j dx_i + O(|\, x\,|^2)$$

where $O(|\, x\,|^2)$ is a 1-form with values in $\text{End}_{\mathbb{C}}(V_x)$ (i.e. a section in $C^\infty(U, \Lambda^1 T^*M \otimes E)$).

(iii): If ∇ is a unitary connection with respect to a hermitian metric $(-,-)$ on V, we can choose $\{s_\alpha\}$ to be a frame that is orthonormal at each point of U.

Proof: We define the neighbourhood U to be a geodesically convex neighbourhood of a, and the coordinate system via the exponential map. That is $(x_1, .., x_n)$ are the coordinates of $x = \exp_a(x_1, .., x_n)$, so $a = (0, .., 0)$. Choose a frame $\{s_\alpha(0)\}$ of V_a, and for $x = \exp_a(x)$, define the framing $\{s_\alpha(x)\}$ of V_x by parallel transport of $s_\alpha(0)$ to x along the radial geodesic $\exp_a(tx)$. In case the connection is unitary, choose $\{s_\alpha(0)\}$ to be an orthonormal frame of V_a. In this event, since parallel transport preserves inner products, $\{s_\alpha(x)\}$ will be an orthonormal frame at x for all $x \in U$. Hence (i) and (iii) are automatic by this definition. We need to verify (ii)

Clearly for every $v \in T_a(M)$, denoting parallel translation along $\exp_a(tv)$ by P_t^v, we have:

$$(\nabla_v s_\alpha)(0) = \lim_{t \to 0} \frac{P_{-t}^v s_\alpha(\exp_a(tv)) - s_\alpha(0)}{t} = \lim_{t \to 0} \frac{s_\alpha(0) - s_\alpha(0)}{t} = 0$$

it follows that $\omega_{\beta\alpha}(0) = 0$ for all α, β, that is:

$$\omega(0) = 0 \tag{52}$$

Define the *radial vector field* $u := \sum_j x_j \partial_j$, and by i_u the operator $u \lrcorner (-)$. Since $u(x)$ is tangent to the radial geodesic at $\exp_a(tx)$ through x, it follows by the definition of s_α that $\nabla_u s_\alpha \equiv 0$ on U. Thus, for each connection coefficient $\omega_{\beta\alpha}$, we have $i_u \omega_{\beta\alpha} = 0$. That is, $i_u \omega \equiv 0$ on U. Writing $\omega = \sum_i \omega_i dx_i$, we have:

$$0 = i_u \omega(x) = \sum_i x_i \omega_i(x) \quad \text{for all } \ x \in U$$

Let us take the derivative of this last equation with respect to x_j. Then:

$$\sum_i(\delta_{ij}\omega_i(x) + x_i\partial_j\omega_i(x)) = \omega_j(x) + \sum_i x_i\partial_j\omega_i(x) = 0$$

From which it follows by again applying ∂_i that:

$$\partial_i\omega_j(x) + \partial_j\omega_i(x) + \sum_k x_k\partial_i\partial_j\omega_k(x) = 0 \text{ for all } x \in U$$

This shows that

$$\partial_i\omega_j(0) = -\partial_j\omega_i(0)$$

That is, the matrix $\partial_i\omega_j(0)$ is skew-symmetric.

Since $F = d\omega + \omega \wedge \omega$, and $\omega(0) = 0$, it follows that

$$F(0) = \sum_{i<j} F_{ij}(0)dx_i \wedge dx_j = \sum_{i<j} d\omega(0) = \sum_{i<j}(\partial_i\omega_j - \partial_j\omega_i)(0)(dx_i \wedge dx_j) = -2\sum_{i<j}\partial_j\omega_i(0)dx_i \wedge dx_j$$

That is,

$$\partial_j\omega_i(0) = -\frac{1}{2}F_{ij}(0)$$

Now we Taylor expand ω about 0, noting that by (52) that $\omega_i(0) = 0$ for all $i = 1, .., n$. Hence:

$$\omega(x) = \sum_i \omega_i(x)dx_i = \sum_{i,j} x_j\partial_j\omega_i(0)dx_i + O(|\ x\ |^2) = -\frac{1}{2}\sum_{i,j} F_{ij}(0)x_j dx_i + O(|\ x\ |^2)$$

where $O(|\ x\ |^2)$ is a 1-form with values in $\mathrm{End}(V_x)$. This proves the lemma. □

Lemma 16.3.2 (Local expression for $\nabla^{\mathcal{E}}$)**.** Let M be a spin manifold of dimension $2m$, and $\mathcal{E} \to M$ be Dirac bundle on M, with $\mathcal{E} = \mathcal{S}(M) \otimes_{\mathbb{C}} \mathcal{V}$, where $\mathcal{S}$ is the spin bundle on M, with its spin connection $\nabla^{\mathcal{S}}$, $\mathcal{V}$ is the twisting bundle with the twisting connection $\nabla^{\mathcal{V}}$, and $\nabla^{\mathcal{E}}$ the tensor product connection. and let $(x_1, .., x_{2m})$ denote the geodesic coordinate system given by $\exp_a$ in a geodesically convex neighbourhood U of $a = (0, .., 0)$. Also, for $x \in U$, let the bundle $\mathcal{S}$ be trivialised over U by *parallel transport* of an orthonormal frame s_α of $\mathcal{S}_a$ (with respect to $\nabla^{\mathcal{S}}$). Similarly, let $\mathcal{V}$ be trivialised over U by parallel transport of an orthonormal frame $\{v_\beta\}$ of $\mathcal{V}_a$ (with respect to $\nabla^{\mathcal{V}}$). We will let $\{e_i(x)\}$ be local orthonormal frame for T^*_xM obtained by parallel transport of a fixed orthonormal frame $e_i(a) = \partial_{i,a}$ along radial geodesics, with respect to the Levi-Civita connection on T^*M. Let $c_i = e_i(a).(-)$ be Clifford multiplication on $\mathcal{E}_a$ by $e_i(a)$.

Then the covariant derivative $\nabla^{\mathcal{E}}$ is given by the formula:

$$\nabla^{\mathcal{E}}_i = \nabla^{\mathcal{E}}_{\partial_i} = \frac{\partial}{\partial x_i} + \frac{1}{4}\sum_{j;\, k<l} x_j R_{ijkl}(0)c_k c_l + \sum_{k<l} f_{ikl}(x)c_k c_l + g_i(x)$$

where:

$$\begin{aligned} R_{ijkl} &= \langle R(\partial_i, \partial_j)e_k, e_l\rangle = \text{Riemann curvature tensor of } M \\ f_{ikl}(x) &\in C^\infty(U), \text{ with } f_{ikl}(x) = O(|\ x\ |^2) \\ g_i(x) &\in C^\infty(U, \mathrm{End}_{\mathbb{C}}(\mathcal{V})) = C^\infty(U, \mathrm{End}_{\mathbb{C}l(M)}(\mathcal{E})) \quad \text{with } g_i = O(|\ x\ |) \end{aligned}$$

(Here $|\ x\ |^2 := \sum_{i=1}^{2m} x_i^2$ is the Euclidean norm of x.)

Proof: Using the geodesic (exponential) coordinate system above, we have $a = (0, .., 0)$, so we will write 0 for a.

Define orthonormal framings $\{s_\alpha\}$ of $\mathcal{S}$ and $\{v_\beta\}$ of $\mathcal{V}$ on a geodesically convex neighbourhood U of a as stated above (and in the Lemma 16.3.1). By the fact that $\nabla^{\mathcal{E}}$ is the tensor product connection of $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{V}}$, it follows that the framing $\{s_\alpha \otimes v_\beta\}$ is a orthonormal framing of $\mathcal{E}$ on U, which is parallel along radial geodesics. Likewise, for $T^*M_{|U}$, by the orthonormal frame field $\{e_i(x)\}$, with the further provision that $e_i(0) = \partial_{i,0}$ (The derivative of the exponential map $\exp_a : T_aM \to M$ is the identity map, so $\{\partial_i\}$ can be taken as the image of an orthonormal basis in $T_a(M)$).

We first claim that with the above trivialisations of $\mathcal{E}$ and T^*M on U, the operation of Clifford multiplication $c(e_i(x))$ by $e_i(x) \in T^*_xM$ on $\mathcal{E}_x$ is the same as $c_i = c(e_i(0))$. That is, the Clifford multiplication $c(e_i(x))$ is a *constant endomorphism* of $\mathrm{End}(V_x) \simeq \mathrm{End}(V_0)$. This follows from the fact that $\nabla^{\mathcal{E}}$ is a Clifford connection, and is seen as follows.

That is, let $\partial_{r,x} := \exp_{a*}(\partial_r)$ denote the radial vector field on U, and $e_{i,r}$ denote $e_i(\exp_a(rx))$. Let s be a vector in $\mathcal{E}_0$, and let $s_r := s(\exp_a(rx)) = P_r s$, where P_r denotes parallel transport from 0 to $\exp_a(rx)$ along the radial ray $r \mapsto rx$, with respect to $\nabla^{\mathcal{E}}$. Since $\nabla^{\mathcal{E}}$ is a Clifford connection, we have:

$$\nabla^{\mathcal{E}}_{\partial_r}(c(e_{i,r})s_r) = c(\nabla_{\partial_r}(e_{i,r}))s_r + c(e_{i,r})\nabla^{\mathcal{E}}_{\partial_r}(s_r) = 0$$

because $e_{i,r}$ is parallel along $\exp_a(rx)$ with respect to the Levi-Civita connection ∇, and s_r are parallel along $\exp_a(rx)$ with respect to the Clifford connection $\nabla^{\mathcal{E}}$ by definition of s_r. Now if we write:

$$c(e_{i,r})s_{\alpha,r} = \sum_{\beta} A_{\beta,\alpha}(r)s_{\beta,r}$$

with respect to the parallel frame $\{s_{\alpha,r}\}$ of $\mathcal{E}_{\exp(rx)}$, then

$$0 = \nabla_{\partial_r}(c(e_{i,r})s_{\alpha,r}) = \sum_{\beta} (\partial_r A_{\beta,\alpha}(r))\, s_{\beta,r}$$

by Leibnitz rule, and since $s_{\beta,r}$ is parallel. Hence $\partial_r A_{\beta,\alpha}(r) = 0$, and hence $A_{\beta\alpha}(r) = A_{\beta\alpha}(0)$. Thus Clifford multiplication $c(e_{i,r})$ is a constant operator along the geodesic rays, with the trivialisation above. Hence the claim.

Let us denote the Cartan connection 1-form on U for $\nabla^{\mathcal{S}}$ by $\omega^{\mathcal{S}}$. Similarly denote the Cartan connection 1-form for $\nabla^{\mathcal{V}}$ as $\omega^{\mathcal{V}}$. By definition,

$$\omega^{\mathcal{E}} = \omega^{\mathcal{S}} + \omega^{\mathcal{V}} \tag{53}$$

Now, because of the trivialisations we have chosen, we may appeal to the Lemma 16.3.1, we may write:

$$\omega^{\mathcal{S}}(x) = -\frac{1}{2}\sum_{i,j} F^{\mathcal{S}}_{ij}(0)x_j dx_i + f(x) \tag{54}$$

where $f(x) \in C^\infty(U, \Lambda^1 T^*M \otimes \mathrm{End}(\mathcal{S})$ is $O(|\, x\,|)^2$. We have already seen in the proof of Weitzenbock's formula in 15.4.3 for the spin bundle $\mathcal{S}$ that:

$$F^{\mathcal{S}}_{ij}(x) = \Omega^{\mathcal{S}}_{ij} = -\frac{1}{2}\sum_{k,l} R_{ijkl}(x)c(e_k(x))c(e_l(x))$$

so that:

$$F^{\mathcal{S}}_{ij}(0) = -\frac{1}{2}\sum_{k<l} R_{ijkl}(0)c_k c_l$$

where R is the Riemann curvature tensor of M. Finally, since by the constancy of Clifford multiplication on U proved above, we have

$$\Lambda^1(U) \otimes \mathrm{End}(S_0) \simeq C^\infty(U, \Lambda^1 T^*M \otimes \mathrm{End}(S_x))$$

via the isomorphism $\omega(x) \otimes c_k c_l \mapsto \omega(x) \otimes c(e_k(x))c(e_l(x))$. Thus we may write

$$f(x) = \sum_{k<l} f_{kl}(x)c_k c_l = \sum_{i;k<l} f_{ikl}(x)c_k c_l dx_i$$

where $f_{ikl} \in C^\infty(U)$ and $f_{ikl} = O(|\, x\,|^2)$. Substituting in equation (54) we find that:

$$\omega^{\mathcal{S}}(x) = \sum_i \left[\frac{1}{4}\sum_{j;k<l} R_{ijkl}(0)x_j c_k c_l + \sum_{k<l} f_{ikl}(x)c_k c_l\right] dx_i \quad \text{where } \ f_{ikl} \in C^\infty(U) \ \text{ is } \ O(|\, x\,|^2) \tag{55}$$

Now, applying the Lemma 16.3.1 to the twisting connection $\nabla^{\mathcal{V}}$, we again find that with the framing and coordinate system we have used:

$$\omega^{\mathcal{V}}(x) = -\frac{1}{2}\sum_{i,j} F^{\mathcal{V}}_{ij}(0)x_j dx_i + h$$

where $h \in C^\infty(U, \Lambda^1 T^*M \otimes \mathrm{End}(\mathcal{V}))$ is $O(|\ x\ |^2)$. Thus

$$\omega^{\mathcal{V}}(x) = \sum_i g_i(x) dx_i \tag{56}$$

where $g_i(x) := -\frac{1}{2}\sum_j F^{\mathcal{V}}_{ij}(0) x_j dx_i + h_i \in C^\infty(U, \mathrm{End}(\mathcal{V}))$ and also $g_i = O(|\ x\ |)$.

Plugging the equations (55) and (56) into (53), we find that:

$$\omega^{\mathcal{E}} = \sum_i \left[\frac{1}{4} \sum_{j;k<l} R_{ijkl}(0) x_j c_k c_l + \sum_{k<l} f_{ikl}(x) c_k c_l + g_i(x) \right] dx_i$$

where $f_{ikl}(x) \in C^\infty(U)$ is $O(|\ x\ |^2)$ and $g_i(x) \in C^\infty(U, \mathrm{End}(\mathcal{V}))$ is $O(|\ x\ |)$. Since $\nabla^{\mathcal{E}}_i s = \partial_i s + \omega^{\mathcal{E}}(\partial_i)s$, the lemma follows. □

16.4. ***u*-scaling.** We will let U be a neighbourhood of 0 in $\mathbb{R}^{2m}$ which maps diffeomorphically onto a geodesically convex neighbourhood of a fixed point $a \in M$, where M is a spin manifold of dimension $2m$. This U is to be thought of as the same U encountered in all the Lemmas of the last subsection. Then, for $x \in U$, we will have $u^{1/2}x \in U$ for all $u \in (0, 1]$.

Definition 16.4.1 (The scaling operators)**.** Let $u \in (0, 1]$. For a smooth section

$$\alpha \in C^\infty((0,\infty) \times U, \Lambda^i_{\mathbb{C}}(\mathbb{R}^{2m*}) \otimes \mathrm{End}(\mathcal{V}))$$

where $\mathcal{V}$ is a fixed complex vector space, define the operator:

$$\delta_u(\alpha) = u^{-i/2}\alpha(ut, u^{1/2}x)$$

For an operator

$$T : C^\infty((0,\infty) \times U, \Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*}) \otimes \mathrm{End}(\mathcal{V})) \to C^\infty((0,\infty) \times U, \Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*}) \otimes \mathrm{End}(\mathcal{V}))$$

define the operator $\delta_u T \delta_u^{-1}$, which may often be denoted as T^u, by

$$\delta_u T \delta_u^{-1} \alpha = \delta_u(T(\delta_u^{-1}\alpha))$$

Lemma 16.4.2. We have the following identities:

$$\begin{aligned}
\delta_u \phi(x) \delta_u^{-1} &= \phi(u^{1/2}x) \quad \text{for} \ \ \phi \in C^\infty(U) \\
\delta_u \partial_t \delta_u^{-1} &= u^{-1}\partial_t \\
\delta_u \partial_i \delta_u^{-1} &= u^{-1/2}\partial_i \\
\delta_u e(\omega)\delta_u^{-1} := \delta_u(\omega \wedge (-))\delta_u^{-1} &= u^{-1/2}e(\omega) \ \ \text{for} \ \ \omega \in \mathbb{R}^{2m*} \\
\delta_u i(\omega)\delta_u^{-1} := \delta_u(\omega \lrcorner (-))\delta_u^{-1} &= u^{1/2}i(\omega) \ \ \text{for} \ \ \omega \in \mathbb{R}^{2m*}
\end{aligned}$$

Proof: If $\phi(x) \in C^\infty(U)$ is regarded as the operator of multiplication, then for $\alpha \in C^\infty((0,\infty) \times U, \Lambda^i(\mathbb{R}^{2m*}) \otimes \mathrm{End}(\mathcal{V}))$ we have:

$$\begin{aligned}
[\delta_u \phi(x) \delta_u^{-1}(\alpha)](t,x) &= \delta_u(\phi(x)(u^{i/2}\alpha(u^{-1}t, u^{-1/2}x) \\
&= u^{-i/2}\phi(u^{1/2}(x))u^{i/2}\alpha(uu^{-1}t, u^{1/2}u^{-1/2}t) \\
&= \phi(u^{1/2}(x))\alpha(t,x)
\end{aligned}$$

which proves the first identity. The next two are similar. For the fourth one, note:

$$[\delta_u e(\omega)\delta_u^{-1}(\alpha)](t,x) = \delta_u(\omega \wedge u^{i/2}\alpha(u^{-1}t, u^{-1/2}x) = u^{\frac{-i-1}{2}} u^{i/2}\omega \wedge \alpha(t,x) = u^{-1/2}(e(\omega)\alpha)(t,x)$$

Similarly the last identity, since $i(\omega)$ reduces degree in Λ^*. □

Remark 16.4.3. We have defined the scaling $\delta_u(-)\delta_u^{-1}$ on 1-forms ω on U, viewed as endomorphisms $e(\omega)$ or $i(\omega)$ of $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})\otimes\mathcal{V}$. Our aim is to deform the Dirac Laplacian over U into the generalised harmonic oscillator by letting the scaling factor $u\to 0$.

We note here that in the local formula for $\nabla_i^{\mathcal{E}}$ derived in Lemma 16.3.2, the terms involving c_k, c_l signify *Clifford multiplication* by e_k and e_l, regarded as endomorphisms of $\mathcal{S}(M)_a$. That is, as elements of $\mathbb{C}l(M)_a = \mathrm{End}_{\mathbb{C}}(\mathcal{S}(M)_a)$. Thus c_kc_l in *not a nilpotent* endomorphism, and the hope is that after scaling, it will become a nilpotent endomorphism, indeed the element $e_k\wedge e_l$ in $\mathcal{A}=\Lambda^*_{\mathbb{C}}(T^*M_a)$. That this is indeed the case is the content of the next lemma.

Lemma 16.4.4 (The u-scaling on $\mathbb{C}l_{2m}$)**.** Note that the constant section e_i on U corresponds to the endomorphism $e_i.(-)$ in $\mathrm{End}_{\mathbb{C}}(S_{2m})=\mathbb{C}l_{2m}$. If one identifies $\mathbb{C}l_{2m}$ with the full exterior algebra $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$, then e_i maps to $c_i = e(e_i)-i(e_i)$. The scaled Clifford section c_i, by definition in Lemma 16.4.2, is:

$$\delta_u(c_i)\delta_u^{-1} = \delta_u(e(e_i)-i(e_i))\delta_u^{-1} = u^{-1/2}e(e_i)-u^{1/2}i(e_i)$$

Now we may extend this definition all over $\mathbb{C}l_{2m}$ by setting:

$$\delta_u(c_1.c_2)\delta_u^{-1} = (\delta_u c_1\delta_u^{-1})(\delta_u c_2\delta_u^{-1})$$

since Clifford multiplication in $\mathbb{C}l_{2m}$ corresponds to composition of maps in $\mathrm{End}_{\mathbb{C}}(S_{2m})$. Then we have:

(i): If c is a *homogeneous* element in $\mathbb{C}l_{2m}$, that is $c=\sum_{|I|=k}a_Ic_I$, where c_I denotes the Clifford product $c_{i_1}.c_{i_2}...c_{i_k}$ in $\mathbb{C}l_{2m}$, we have:

$$\delta_u c\delta_u^{-1} = u^{-k/2}[e(\sigma(c))+O(u)] = u^{-k/2}[\sigma(c)\wedge(-)+O(u)]$$

where σ is the symbol map.

(ii): For any $c=\sum_{|I|\le k}f_I(x)c_I\in\mathbb{C}l(U)$, where $f_I\in C^\infty(U)$, and with leading homogeneous term of degree k. Then

$$\lim_{u\to 0}u^k\delta_u c\delta_u^{-1} = \sum_{|I|=k}f_I(0)(e_I\wedge(-))$$

Proof: We first prove (i). Let $\{e_i\}$ denote an orthonormal frame for $T^*(\mathbb{R}^{2m})$, and let $c(x)=\sum_{|I|=k}a_I(x)c_I$ be homogeneous of degree k. Then, by definition, Then:

$$\begin{aligned}(\delta_u c\delta_u^{-1})(x) &= \sum_{|I|=k}a_I(u^{-1/2}e(e_{i_1})-u^{1/2}i(e_{i_1})).(u^{-1/2}e(e_{i_2})-u^{1/2}i(e_{i_2}))....(u^{-1/2}e(e_{i_k})-u^{1/2}i(e_{i_k}))\\ &= u^{-k/2}(\sum_I a_I\, e(e_{i_1}\wedge e_{i_2}\wedge ...e_{i_k}) + (\text{terms with } \ u^j \text{ with } \ j\ge -k/2+1)\\ &= u^{-k/2}(e(\sigma(c))+O(u))\end{aligned}$$

Now (ii) follows immediately from (i). □

Proposition 16.4.5. Let M be a spin manifold of dimension $2m$, and $D: C^\infty(M,\mathcal{E})\to C^\infty(M,\mathcal{E})$ be the Dirac operator on the Dirac bundle $\mathcal{E}=\mathcal{S}(M)\otimes\mathcal{V}$. For each $a\in M$, there exists a coordinate chart U around a, and framings of $\mathcal{S}$, $\mathcal{V}$ and T^*M such that:

(i): The rescaled covariant derivative $\nabla^{\mathcal{E},u} := \delta_u\nabla_i^{\mathcal{E}}\delta_u^{-1}$ is given by:

$$\nabla_i^{\mathcal{E},u} = u^{-1/2}\left(\partial_i + \frac{1}{4}\sum_j R_{ij}x_j + \rho(u)\right)$$

where $R_{ij}=\sum_{k<l}R_{ijkl}(a)e_k\wedge e_l$ is the curvature 2-form (as an endomorphism of $\Lambda^*_{\mathbb{C}}(T^*_aM)$), R_{ijkl} being the Riemann curvature tensor of M at a, and $\rho(u)\in\mathrm{End}(\mathcal{V}_a)$ is $O(u^{1/2})$.

(ii): The rescaled Dirac Laplacian is given by:

$$u(D^u)^2 := u(\delta_u D \delta_u^{-1})^2 = -\sum_{i=1}^{2m}\left(\partial_i + \frac{1}{4}\sum_{j=1}^{2m} R_{ij}x_j\right)^2 + F + g(u)$$

where $F = \Omega^{\mathcal{V}}(a)$ is the curvature 2-form of $\mathcal{V}$ at a, and $g(u) \in \mathrm{End}(\mathcal{V}_a) = O(u^{1/2})$.

(iii): The limit as $u \to 0$ of $u(D^u)^2)$ is given by:

$$\lim_{u\to 0} u(D^u)^2 = -\sum_{i=1}^{2m}\left(\partial_i + \frac{1}{4}\sum_{j=1}^{2m} R_{ij}x_j\right)^2 + F$$

the right hand side being exactly the generalised harmonic oscillator introduced in Proposition 16.1.7

Proof: We note that by the Lemma 16.3.2, and with the geodesically convex neighbourhood U of a and synchronous framings of T^*M and $\mathcal{V}$ constructed there, we have $a = (0,..,0)$ and:

$$\nabla_i^{\mathcal{E}} = \nabla_{\partial_i}^{\mathcal{E}} = \frac{\partial}{\partial x_i} + \frac{1}{4}\sum_{j;\,k<l} x_j R_{ijkl}(0)c_k c_l + \sum_{k<l} f_{ikl}(x)c_k c_l + g_i(x)$$

where:

$$\begin{array}{rcl} R_{ijkl} & = & \langle R(\partial_i, \partial_j)e_k, e_l\rangle = \text{Riemann curvature tensor of } M \\ f_{ikl}(x) & \in & C^\infty(U), \quad \text{with} \quad f_{ikl}(x) = O(|\, x\,|^2) \\ g_i(x) & \in & C^\infty(U, \mathrm{End}_{\mathbb{C}}(\mathcal{V})) = C^\infty(U, \mathrm{End}_{\mathbb{C}l(M)}(\mathcal{E})) \quad \text{with} \quad g_i = O(|\, x\,|) \end{array}$$

Now, by the previous Lemma 16.4.2, $\delta_u \partial_i \delta_u^{-1} = u^{-1/2}\partial_i$.

Next, since $R_{ij} := \sum_{k<l} R_{ijkl}(0)c_k c_l \in \mathrm{End}(\mathcal{E}_a) = \mathrm{End}(S_{2m}) \otimes \mathrm{End}(\mathcal{V}_a) \simeq \mathbb{C}l(M)_a \otimes \mathrm{End}(\mathcal{V}_a) = \Lambda^* T_a^* M) \otimes \mathrm{End}(\mathcal{V}_a)$, so we have

$$\delta_u c_k c_l \delta_u^{-1} = u^{-1}(e_k \wedge e_l + O(u))$$

by (i) of the Lemma 16.4.4. On the other hand $\delta_u x_j \delta_u^{-1} = u^{1/2}x_j$ by the same Lemma. Thus :

$$\delta_u(\sum_j R_{ij}x_j)\delta_u^{-1} = \delta_u\left(\sum_{j;\,k<l} R_{ijkl}(0)x_j c_k c_l\right)\delta_u^{-1} = u^{-1/2}(\sum_j R_{ij}x_j + h(u))$$

where $h(u) = O(u)$ and $R_{ij} := \sum_{k<l} R_{ijkl}(0)e_k \wedge e_l$.

Since $f_{ikl}(x) \in C^\infty(U)$ and $O(|\, x\,|^2)$, we have $\delta_u f_{ikl}\delta_u^{-1} = f_{ikl}(u^{1/2}x) = O(u)$. On the other hand we observed above that $\delta_u(c_k c_l)\delta_u^{-1} = u^{-1}(e_k \wedge e_l + O(u))$. Thus

$$\delta_u f_{ikl}(x)c_k c_l \delta_u^{-1} = O(1)$$

Finally, since $g_i(x) \in \mathrm{End}(\mathcal{V})_x$ is $O(|\, x\,|)$, we have:

$$\delta_u g_i \delta_u^{-1} = g_i(u^{1/2}x) = O(u^{1/2})$$

Thus we write

$$\rho(u) := u^{1/2}\delta_u\left(\sum_{k<l} f_{ikl}c_k c_l + g_i\right)\delta_u^{-1} + h(u)$$

and by the foregoing, we have $\rho(u) = O(u^{1/2})$, and

$$\delta_u \nabla_i^{\mathcal{E},u}\delta_u^{-1} = u^{-1/2}\left(\partial_i + \frac{1}{4}\sum_j R_{ij}x_j) + \rho(u)\right)$$

with $\rho(u) = O(u^{1/2})$. This proves (i).

To see (ii), we use the formula for the Dirac Laplacian derived in Corollary 15.4.6, viz.

$$D^2 = \nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} + \frac{1}{2}\Omega^{\mathcal{E}} = \nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} + \frac{1}{4}k + \Omega^{\mathcal{V}}$$

where $\Omega^{\mathcal{V}}$ is the curvature operator of $\mathcal{V}$. Also, in the proof of the Corollary 15.4.8 and Lemma 12.2.4, we have seen that the first term is:

$$\nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} = \Delta^{\mathcal{E}} = -\sum_{i,j} g^{ij}(\nabla^{\mathcal{E}}_i\nabla^{\mathcal{E}}_j - \nabla^{\mathcal{E}}_{\nabla_{\partial_i}\partial_j}) = -\sum_{i,j} g^{ij}(\nabla^{\mathcal{E}}_i\nabla^{\mathcal{E}}_j - \sum_k \Gamma^k_{ij}\nabla^{\mathcal{E}}_k)$$

where Γ^k_{ij} are the Christoffel symbols of the Riemannian metric on M with respect to the basis ∂_i. Since $g^{ij}(a) = \delta^{ij}$, we have $g^{ij}(x) = \delta_{ij} + h_{ij}(x)$, where $h_{ij}(x) = O(|\, x\,|)$.

Similarly, since $\nabla_{\partial_i}\partial_j(a) = \nabla_{e_i}e_j = 0$ for all i, j, we have $\Gamma^k_{ij}(x) = O(|\, x\,|)$ by choice of synchronous framing of T^*M on U. Thus we may write:

$$\nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} \;=\; -\sum_i \nabla^{\mathcal{E}}_i\nabla^{\mathcal{E}}_i + (\sum_{i,j} h_{ij}(x)\nabla^{\mathcal{E}}_i\nabla^{\mathcal{E}}_j + g_{ij}\Gamma^k_{ij}\nabla^{\mathcal{E}}_k)$$

Thus, using (i) above, we have:

$$\begin{aligned}
u\delta_u(\nabla^{\mathcal{E}*}\nabla^{\mathcal{E}})\delta_u^{-1} &= -\sum_i (u^{1/2}\delta_u\nabla^{\mathcal{E}}_i\delta_u^{-1})^2 + \sum_{i,j} h_{ij}(u^{1/2}x)(u^{1/2}\delta_u\nabla^{\mathcal{E}}_i\delta_u^{-1})(u^{1/2}\delta_u\nabla^{\mathcal{E}}_j\delta_u^{-1}) \\
&+ u^{1/2}g_{ij}(u^{1/2}x)\Gamma^k_{ij}(u^{1/2}x)(u^{1/2}\delta_u\nabla^{\mathcal{E}}_k\delta_u^{-1}) \\
&= -\sum_i \left(\partial_i + \frac{1}{4}\sum_j R_{ij}x_j + \rho_i(u)\right)^2 \\
&+ \sum_{i,j} h_{ij}(u^{1/2}x)\left(\partial_i + \frac{1}{4}\sum_l R_{il}x_l + \rho_i(u)\right)\left(\partial_j + \frac{1}{4}\sum_l R_{jl}x_l + \rho_j(u)\right) \\
&+ g_{ij}(u^{1/2}x)\Gamma^k_{ij}(u^{1/2}x)\left(\partial_k + \frac{1}{4}\sum_l R_{kl}x_l + \rho_k(u)\right)
\end{aligned}$$

Since $\rho_i(u) = O(u^{1/2})$, $h_{ij}(u^{1/2}x) = O(u^{1/2})$ and $\Gamma^k_{ij} = O(u^{1/2})$, we have finally:

$$u\delta_u(\nabla^{\mathcal{E}*}\nabla^{\mathcal{E}})\delta_u^{-1} = -\sum_i \left(\partial_i + \frac{1}{4}\sum_j R_{ij}x_j\right)^2 + \delta(u) \tag{57}$$

where $\delta(u) = O(u^{1/2})$.

Note that

$$u\delta_u k_M \delta_u^{-1} = uk(u^{1/2}x) = \epsilon(u) = O(u^{1/2}) \tag{58}$$

Finally, reverting to the Corollary 15.4.6, we have that

$$\Omega^{\mathcal{V}}(s\otimes\sigma) = R^{\mathcal{V}}(s\otimes\sigma) = \sum_{i<j} c_ic_js\otimes\Omega^{\mathcal{V}}_{ij}\sigma$$

where $\Omega^{\mathcal{V}}_{ij}(x) = \Omega^{\mathcal{V}}(e_i, e_j)(x)$ is the curvature endomorphism of $\mathcal{V}_x$. Thus again appealing to (i) of Lemma 16.4.4:

$$\begin{aligned}
u\delta_u\Omega^{\mathcal{V}}\delta_u^{-1} &= u\sum_{i<j}(\delta_u c_ic_j\delta_u^{-1}\Omega^{\mathcal{V}}_{ij}(u^{1/2}x) \\
&= u\sum_{i<j}(u^{-1}(e_i\wedge e_j + O(u))(\Omega^{\mathcal{V}}_{ij}(0) + O(u)) \\
&= \sum_{i<j}\Omega^{\mathcal{V}}(0)(e_i\wedge e_j) + \gamma(u) = F + \gamma(u)
\end{aligned} \tag{59}$$

where $\gamma(u) = O(u^{1/2})$, and $F = \sum_{i<j} \Omega^{\mathcal{V}}_{ij}(0) e_i \wedge e_j$ is the curvature endomorphism of $\mathcal{V}$ at $a = 0$.

Adding together the equations (57), (58) and (59), we arrive at (ii), with $g(u) := \delta(u) + \epsilon(u) + \gamma(u)$ being $O(u^{1/2})$.

(iii) is immediate from (ii). The proposition follows. □

Now we need to construct the u-scaled heat kernel for the u-scaled heat operator $e^{-tuD^{u\,2}}$. To this end, we have the following:

Proposition 16.4.6. Let $\mathcal{E} \to M$ be a Dirac bundle on the spin manifold M of dimension $2m$, and D the associated Dirac operator. Let $k(t,x)$ be the fundamental solution for the heat operator of the Dirac laplacian:

$$e^{-tD^2} : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$$

with pole at $(a, I_{\mathcal{E}})$ (as discussed in Proposition 16.2.3). Note that for each t, $k(t,x)$ is a smooth section of $\mathrm{End}(\mathcal{E}) = \Lambda^*_{\mathbb{C}} T^*M \otimes \mathrm{End}_{\mathbb{C}} \mathcal{V}$. For $x \in U$, where U is the neighbourhood of a defined in Lemma 16.3.2 and its sequel, set

$$r(u,t,x) := u^m (\delta_u k)(t,x)$$

Then

(i):

$$(\partial_t + u(D^u)^2) r(u,t,x) = 0 \quad \text{for} \;\; t \in (0,\infty) \;\; x \in U$$

(ii): Denoting the identity map $Id : \mathcal{E}_a \to \mathcal{E}_a$ by I_a, we have:

$$\lim_{t\to 0} r(u,t,x) = \delta_a I_a$$

That is, $r(u,t,x)$ is the fundamental solution for the heat equation of the (scaled) elliptic operator $uD^{u\,2}$ on the neighbourhood U, with pole at (a, I_a).

Proof: We first check (i). Note that by Lemma 16.4.2 and that $\partial_t k = -D^2 k$, we have:

$$\begin{aligned}
\partial_t(r(u,t,x)) &= \partial_t(u^m(\delta_u k)(t,x)) = u^{m+1}[(u^{-1}\partial_t)\delta_u k](t,x) \\
&= u^{m+1}[(\delta_u \partial_t \delta_u^{-1})(\delta_u k)](x,t) \\
&= u^{m+1}[\delta_u \partial_t k](t,x) = u^{m+1}[\delta_u(-D^2 k)](t,x) \\
&= -u^{m+1}[(\delta_u D^2 \delta_u^{-1})(\delta_u k)](t,x) = -[u(D^u)^2(u^m \delta_u k)](t,x) = -u(D^u)^2 r(u,t,x)
\end{aligned}$$

This proves (i).

To see (ii), we note that by our asymptotic expansion for $k(t,x)$, we have, denoting $\delta(x,a) = \mid x \mid$:

$$k(t,x) \sim (4\pi t)^{-m} \exp(- \mid x \mid^2 /4t)[k_0(x) + \sum_{i\geq 1} k_i(x) t^i]$$

where $k_0(a) = I_a$. Then note that:

$$\begin{aligned}
r(u,t,x) &= u^m[\delta_u k](t,x) = u^m (4\pi t u)^{-m} \exp(- \mid u^{1/2} x \mid^2 /4ut)[k_0(u^{1/2}x) + \sum_{i\geq 1} (ut)^i \delta_u k_i] \\
&= (4\pi t)^{-m} \exp(- \mid x \mid^2 /4t)[k_0(u^{1/2}x) + \sum_{i\geq 1} (ut)^i \delta_u k_i]
\end{aligned}$$

Now as $t \to 0$, the euclidean heat kernel $(4\pi t)^{-m} \exp(- \mid x \mid^2 /4t) \to \delta_a$. So the first term of the series on the right has the limit we want, viz.

$$\lim_{t\to 0} (4\pi t)^{-m} \exp(- \mid x \mid^2 /4t) k_0(u^{1/2}x) = \delta_a k_0(a) = \delta_0 I_a$$

The other terms of course tend to 0 because they involve strictly positive powers of t. This proves (ii), and the proposition. □

The relation of the fundamental solution $r(u,t,x)$ to $k(t,x)$ naturally implies a relation between their asymptotic expansions. More precisely:

Proposition 16.4.7. Let U be a neighbourhood of $a=0$ as in the last proposition. Let us denote the fibres $\mathcal{E}_0 =: E$, $\mathcal{V}_0 =: V$, and the smooth function in $C^\infty((0,\infty)\times U)$

$$q_t(x) := (4\pi t)^{-m}\exp(-\delta(x,a)^2/4t)$$

There exist $\Lambda^*(\mathbb{R}^{2m*})\otimes \mathrm{End}(V)$-valued polynomials $\gamma_i(t,x)$ on $(0,\infty)\times U$ such that:

$$r(u,t,x)\sim q_t(x)\sum_{i=-2m}^{\infty} u^{i/2}\gamma_i(t,x)$$

satisfying:

$$\left\|\partial_t^j\partial_x^\alpha(r(u,t,x)-\sum_{i=-2m}^{N} u^{i/2}\gamma_i(t,x)\right\|_\infty \le C(N,j,\alpha)u^N$$

where $\|-\|_\infty$ is the supremum norm on U. Furthermore, $\gamma_i(0,0)=0$ for $i\neq 0$, and $\gamma_0(0,0)=I_E$.

Proof: By (iii) of the Proposition 12.3.5, we have the asymptotic expansion:

$$k(t,x)\sim q_t(x)\sum_{i=0}^{\infty} t^i k_i(x)$$

where each $k_i(x)\in C^\infty(U,\mathrm{End}_{\mathbb{C}}(E) = C^\infty(U,\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*})\otimes \mathrm{End}_{\mathbb{C}}(V))$, and $k_0(0)=I_V$. The symbol "$\sim$" means that for the partial sum of the series on the right upto $i=l$ we have a sup-norm estimate:

$$\left\|k(t,x)-q_t(x)\sum_{i=0}^{l} k_i(x)t^i\right\|_\infty \le C_l t^N \quad \text{for all } \ l>N+2m,\ \ t\in(0,T]$$

where the sup-norm is over U.

First we want to replace the *smooth* $k_i(x)$ by *polynomials* $\psi_i(x)$ with coefficients in $\Lambda^*(\mathbb{R}^{2m*})\otimes\mathrm{End}_{\mathbb{C}}(V)$. Note that :

$$r^k e^{-r^2/4t}\le C_k t^{k/2} \quad \text{for all } \ r\in[0,\infty)$$

which implies that:

$$\|x\|^k e^{-\|x\|^2/4t}\le C_k t^{k/2} \quad \text{for all } \ \ x\in\mathbb{R}^{2m}$$

Thus, if we define the polynomials $\psi_i(x)$ to be the Taylor polynomial of $k_i(x)$ of order $2(N+m-i)$, then by Taylor's theorem

$$|\ \ k_i(x)-\psi_i(x)\ |\le A_i\|x\|^{2N+2m-2i} \quad \text{for } \ x\in U$$

so that for some constants B_i independent of x and t:

$$\left|q_t(x)\sum_{i=0}^{l} t^i(k_i(x)-\psi_i(x))\right| \le \sum_{i=0}^{l} A_i q_t(x)\|x\|^{2N+2m-2i} = (4\pi)^{-m}\sum_{i=0}^{l} B_i t^{-m}t^i e^{-\|x\|^2/4t}\|x\|^{2N+2m-2i}\le Ct^N$$

Thus

$$\left\|k(t,x)-q_t(x)\sum_{i=0}^{l} t^i\psi_i(x)\right\|_\infty \le C_l t^N \quad \text{for all } \ l>N+2m,\ \ t\in(0,T]$$

and now $\psi_i(x)$ are polynomials in x with coefficients in $\mathrm{End}_{\mathbb{C}}(E)=\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*})\otimes\mathrm{End}_{\mathbb{C}}(V)$.

For an element of $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*})\otimes\mathrm{End}_{\mathbb{C}}(V)$, let us denote the by $\alpha_{[p]}$ the component of α in the summand $\Lambda^p(\mathbb{R}^{2m*})\otimes\mathrm{End}_{\mathbb{C}}(V)$.

Then the last sup-norm inequality above implies sup-norm inequalities for each p-component, viz.

$$\left\|k(t,x)_{[p]}-q_t(x)\sum_{i=0}^{l} t^i(\psi_i(x))_{[p]}\right\|_\infty \le C_l t^N \quad \text{for all } \ l>N+2m,\ \ t\in(0,T]$$

which implies, on multiplying both sides by $u^{m-p/2}$, and resetting $x \mapsto u^{1/2}x$ (which maps U to itself) and $t \mapsto ut$ that:

$$\left\| u^m u^{-p/2} k(ut, u^{1/2}x)_{[p]} - u^m u^{-p/2} q_{ut}(u^{1/2}x) \sum_{i=0}^{l} (ut)^i (\psi_i(u^{1/2}x))_{[p]} \right\|_\infty \leq u^{m-p/2} C_l (ut)^N$$
$$\text{for all } l > N + 2m, \ \ t \in (0, T],$$

Since $u^m u^{-p/2} k(ut, u^{1/2}x) = u^m \delta_u k(t,x)_{[p]} = r(u,t,x)_{[p]}$, and $u^m q_{ut}(u^{1/2}x) = q_t(x)$, the last inequality can be rewritten as:

$$\left\| r(u,t,x)_{[p]} - u^{-p/2} q_t(x) \sum_{i=0}^{l} (ut)^i (\psi_i(u^{1/2}x))_{[p]} \right\|_\infty \leq u^{m-p/2+N} C_l t^N \quad \text{for all } \ l > N + 2m, \ \ t \in (0, T]$$

We need to let $l \to \infty$, and arrange the sum in the norm signs on the left hand side in powers of u. Note that since ψ_i are polynomials, they will contribute non-negative powers of $u^{1/2}$, so that $u^{j/2}$ will be contributed only by terms from $i = 0$ to $i = j/2 + p/2$, and $j/2$ will run from $-p/2$ onwards.

So define:

$$(\gamma_j)_{[p]}(x,t) := \text{coefficient of } u^{j/2} \text{ in } \ u^{-p/2} \sum_{i=0}^{j/2+p/2} (ut)^i \psi_i(u^{1/2}x)_{[p]}$$

and rewrite the last inequality above for the particular value $l = (j+p)/2$ as:

$$\left\| r(u,t,x)_{[p]} - q_t(x) \sum_{i=-p}^{j+p} u^{i/2} \gamma_i(t,x)_{[p]} \right\|_\infty \leq u^{m-p/2+N} C_j t^N \quad \text{for all } \ (j+p)/2 > N + 2m, \ \ t \in (0, T], \ u \in (0, 1]$$

Set $\gamma_i(t,x) := \sum_{p=0}^{2m} \gamma_i(t,x)_{[p]}$, and note that $p/2 \leq m$ for all p, we have $(j+p)/2 > N + 2m \Leftrightarrow j/2 - m > N + m - p/2 = N'$ will be satisfied if we choose $j > 2N' + 2m$. Replacing $N' = N + m - p/2$ by N, and noting that t^N is bounded on $(0, T]$, we then have the inequality:

$$\left\| r(u,t,x) - q_t(x) \sum_{i=-m}^{2N+2m} u^{i/2} \gamma_i(t,x) \right\|_\infty \leq C u^N \ \text{ for all } \ N, \ \ t \in (0, T] \tag{60}$$

A similar argument maybe given for the derivatives $\partial_i^\alpha \partial_t^\beta r(u,t,x)$, which is omitted.

Now for the final statement about $\gamma_i(0,0)$. Since by definition

$$\begin{aligned} \sum_{j=-m}^{\infty} u^{j/2} \gamma_j(x,t) &= \sum_{j=-m}^{\infty} \sum_{p=0}^{2m} u^{j/2} (\gamma_j)_{[p]}(x,t) \\ &= \sum_{p=0}^{2m} \sum_{i=0}^{\infty} u^{-p/2} (ut)^i \psi_i(u^{1/2}x)_{[p]} \\ &= \sum_{p=0}^{2m} (\delta_u \psi_{[p]})(t,x) = \delta_u \psi(t,x) \end{aligned}$$

where we define:

$$\psi(t,x) := \sum_{i=0}^{\infty} t^i \psi_i(x)$$

Now note that by the above definition,

$$(\delta_u \psi)(0,0) = \sum_{i=0}^{\infty} (u.0)^i \delta_u \psi_i(0) = \psi_0(0) = I_0$$

Thus

$$\sum_{j=-m}^{\infty} u^{j/2} \gamma_j(0,0) = \delta_u \psi(0,0) = I_0$$

which shows that $\gamma_j(0,0) = 0$ for $j \neq 0$, and $\gamma_0(0,0) = I_0$. This proves the proposition. □

Now we can combine the Propositions 16.1.7, 16.4.5, 16.4.6 16.4.7 to deduce the following corollary.

Corollary 16.4.8. In the u-expansion

$$r(u,t,x) \sim q_t(x) \sum_{j=-m}^{\infty} u^{j/2}\gamma_j(t,x)$$

deduced in the last proposition, we have $\gamma_j(t,x) \equiv 0$ for $j < 0$. That is, the Laurent expansion in $u^{1/2}$ of $r(u,t,x)$ about 0 has no poles. Secondly, $q_t(x)\gamma_0(t,x)$ is a formal fundamental solution to the heat equation for the generalised harmonic oscillator H in Mehler's formula of 16.1.7 with pole at $(0, I_0)$. That is,

$$q_t(x)\gamma_0(t,x) = (4\pi t)^{-m} j(tR)^{-1/2} \exp\left(\frac{-1}{4t}\langle x|(tR/2)\coth(tR/2)|x\rangle\right)\exp(-tF)$$

where R is the nilpotent matrix $\sum_{i<j} R_{ij} e_i \wedge e_j \in \Lambda^2(\mathbb{R}^{2m*})$.

Proof: By the Proposition 16.4.6 we have:

$$(\partial_t + u(D^u)^2)r(u,t,x) = 0 \quad \text{for } \ t > 0, \ \ x \in U$$

Let $\gamma_{-s}(t,x)$ be the first term in the series $r(u,t,x) = q_t(x)\sum_{j=-m} u^{j/2}\gamma_j(t,x)$ which is not identically zero. Since all space and time derivatives of $r(u,t,x)$ are uniformly approximated on U upto an arbitrarily large power of u by the space and time derivatives of some partial sum of the asymptotic series above (by Proposition 16.4.7 above), and the $\gamma_i(t,x)$ are polynomials in t and x, it follows that the asymptotic series is a *formal power series* solution to the scaled heat equation $(\partial_t + u(D^u)^2$, that is:

$$(\partial_t + u(D^u)^2)(q_t(x)\sum_{j=-s}^{\infty}(u^{j/2}\gamma_j(t,x)) = 0$$

Denoting by $H := -\sum_{i=1}^{2m}(\partial_i + 1/4\sum_j R_{ij}x_j)^2 + F$ the generalised harmonic oscillator introduced earlier, and noting that by (ii) of Proposition 16.4.5 we have:

$$u(D^u)^2 = H + O(u^{1/2})$$

we have:

$$(\partial_t + H + O(u^{1/2}))(q_t(x)\sum_{j=-s}^{\infty}(u^{j/2}\gamma_j(t,x)) = 0$$

as an identity or formal power series in u. Since the lowest power of u occurring on the right is from the first non-vanishing term of the formal series, we have

$$(\partial_t + H)(u^{-s/2}q_t(x)\gamma_{-s}(t,x)) = 0$$

as an identity in (t,x). It follows that

$$(\partial_t + H)(q_t(x)\gamma_{-s}(t,x)) = 0$$

is a solution to the heat equation. That is, $q_t(x)\gamma_{-s}(t,x)$ is a formal solution to the heat equation for the generalised harmonic oscillator. From the Proposition 16.1.7, it follows that this solution is determined by its initial value at $(t,x) = (0,0)$. But, from the Proposition 16.4.7, we have seen that $\gamma_s(0,0) = 0$ for $s \neq 0$. It follows that $\gamma_{-s}(t,x) \equiv 0$ for all $s > 0$. The first assertion follows.

For the second assertion, the above reasoning shows that we have

$$(\partial_t + H)(q_t(x)\gamma_0(t,x)) = 0$$

with $\gamma_0(0,0) = I_0$ by the last statement of Proposition 16.4.7. Since the fundamental solution for this harmonic oscillator is unique, and

$$p_t(x) = (4\pi t)^{-m} j(tR)^{-1/2} \exp\left(\frac{-1}{4t}\langle x|(tR/2)\coth(tR/2)|x\rangle\right)\exp(-tF)$$

satisfies the same equation, with $p_0(0) = I_0$, we have the second assertion. The proposition follows. □

16.5. **The Index Theorem.**

Theorem 16.5.1 (Atiyah-Singer)**.** Let M be a compact spin manifold of dimension $2m$, and let $\mathcal{E} = \mathcal{S}(M) \otimes \mathcal{V}$ be a Dirac bundle on it, where $\mathcal{S}(M) \to M$ is the spin bundle on M, with its unitary spin connection $\nabla^{\mathcal{S}}$), $\mathcal{V}$ a twisting complex vector bundle with a unitary connection $\nabla^{\mathcal{V}}$ on it, and the Clifford connection $\nabla^{\mathcal{E}}$ the tensor product connection of $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{V}}$. Then the index of the Dirac operator $D^+ : C^\infty(M, \mathcal{E}^+) \to C^\infty(M, \mathcal{E}^-)$ is given by the formula:

$$\mathrm{ind}(D^+) = \int_M \widehat{A}(M) \wedge ch(\mathcal{V})$$

Proof: By the Proposition 16.2.3 we have:

$$\mathrm{ind}\, D^+ = \int_M \mathrm{str}_{\mathcal{E}} k_t(a,a) dV(a) = (4\pi)^{-m} \int_M \mathrm{str}_{\mathcal{E}} k_m(a) dV(a)$$

where we have expanded asymptotically:

$$k_t(x,a) \sim (4\pi t)^{-m} \exp(\delta(x,a)^2/4t) \sum_{i=0}^{\infty} t^i k_i(x) \tag{61}$$

By the Proposition 16.4.7 we have a neighbourhood U of $a = 0$ such that on U

$$r(u,t,x) = u^m \delta_u k_t(x,0) = u^m \sum_{p=0}^{2m} u^{-p/2} k(ut, u^{1/2}x)_{[p]} = \sum_{p=0}^{2m} u^{m-p/2} k(ut, u^{1/2}x)_{[p]} \tag{62}$$

Denote $\mathcal{E}_a = \mathcal{E}_0 = E$, $\mathcal{S}_a = S$, and $\mathcal{V}_a = V$.

By the Lemma 14.5.2, 16.2.3 we have

$$\mathrm{str}_E(\alpha \otimes F) dV(a) = (-2i)^m T(\alpha) \mathrm{tr}_V dV(a) = (-2i)^m \mathrm{tr}_V((\alpha \otimes F)_{[2m]})$$

where $(\alpha \otimes F)$ is to be regarded as a differential form with coefficients in $\mathrm{End}(E) = \Lambda^*(\mathbb{R}^{2m*}) \otimes \mathrm{End}(V)$, and tr_V is applied to these coefficients, and the $2m$-component applies to α. In particular, for any element r in $\mathrm{End}(E) = \Lambda^*(\mathbb{R}^{2m*}) \otimes \mathrm{End}(V)$, we have:

$$\mathrm{str}_E(k) = str_E k_{[2m]}$$

Applying this to the equation (62) above, and using the Corollary 16.4.8 we find that:

$$\mathrm{str}_E k(ut,0) = \mathrm{str}_E k(ut,0)_{[2m]} = \mathrm{str}_E r(u,t,0) = q_t(0) \mathrm{str}_E \left(\sum_{j=0}^{\infty} u^{j/2} \gamma_j(t,0) \right) \tag{63}$$

On the other hand we have from (61) and Proposition 16.2.3 that scaling time by u does not affect the integral over M, since only the time independent term $q_t(a)k_m(a)$ contributes to the integral. That is,

$$\begin{aligned} \int_M \mathrm{str}_E k_t(a,a) dV(a) &= \int_M \mathrm{str}_E(q_t(a) t^m k_m(a)) dV(a)) = \int_M (q_{ut}(a)(ut)^m k_m(a)) dV(a) \\ &= \int_M \mathrm{str}_E k_{ut}(a,a) dV(a) \quad \text{for all } \; u \in (0,1], \;\; t \in (0,T] \end{aligned}$$

In particular, by substituting (63) into this relation, and noting that $k(t,0) = k_t(a,a)$ by definitions, we have:

$$\begin{aligned} \int_M \mathrm{str}_E k_t(a,a) dV(a) &= \lim_{u \to 0} \int_M \mathrm{str}_E k_{ut}(a,a) dV(a) = \lim_{u \to 0} \int_M \mathrm{str}_E k(ut,0) dV(a) \\ &= \lim_{u \to 0} \int_M \mathrm{str}_E r(u,t,0) dV(a) = \int_M \mathrm{str}_E q_t(0) \gamma_0(t,0) dV(a) \end{aligned}$$

Since the left hand side is independent of t, we can evaluate the right hand side at $t=1$. From the Corollary 16.4.8,

$$q_1(0)\gamma(1,0) = (4\pi)^{-m} j(R)^{-1/2}\exp(-F)$$

where $R=\sum_{ij} R_{ij} e_i \wedge e_j$ is the nilpotent curvature form with $R_{ij}=\frac{1}{2}R_{ijkl}(a)c_k c_l$, and $F=\sum_{i<j}\Omega^{\mathcal{V}}_{i<j} e_i\wedge e_j$ is the curvature form, (being regarded as a (dim $V\times$ dim V)-matrix whose entries are 2-forms, i.e. in the nilpotent algebra $\mathcal{A}=\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*})$). By the Lemma 14.5.2, we have therefore:

$$\begin{aligned} \mathrm{str}_E(q_1(0)\gamma(1,0))dV &= (4\pi)^{-m}(str_E(j(R)^{-1/2}\exp(-F)) = (-2i)^m(4\pi)^{-m} tr_V(j(R)^{-1/2}\exp(-F))_{[2m]} \\ &= (2\pi i)^{-m}\left(\mathrm{tr}_V(j(R)^{-1/2}\exp(-F)\right)_{[2m]} \end{aligned}$$

where $R=R_0=R_a$, and $F=F_0=F_a$.

Now $j(R_a)^{-1/2}=\left[\det\left(\frac{R_a/2}{\sinh\, R_a/2}\right)\right]^{1/2}$ is by definition the element $\widehat{A}(M)(a)=\sum_{p=0}^{2m}\widehat{A}(M)_{[p]}(a)\in\Lambda^*_{\mathbb{C}}(M)$. Similarly, the Chern character of $\mathcal{V}$ is defined by

$$ch(\mathcal{V})(a)=\sum_{p=0}^{2m} ch_{[p]}(\mathcal{V}) = (2\pi i)^{-m}\mathrm{tr}_{V_a}(\exp(-F(a)))$$

so that

$$(2\pi i)^{-m}\left(\mathrm{tr}_V(j(R)^{-1/2}\exp(-F)\right)_{[2m]} = \widehat{A}(M)\wedge ch(\mathcal{V})_{[2m]}$$

and so

$$\mathrm{ind}(D^+) = \int_M \widehat{A}(M)ch(\mathcal{V})_{[2m]} =: \int_M \widehat{A}(M)ch(\mathcal{V})$$

and the theorem follows. □

Corollary 16.5.2 (Atiyah-Singer). Let M be a compact spin manifold of dimension $2m$. Then, for the Dirac operator $D^{\mathcal{S}}$ of the spin bundle $\mathcal{S}$ (called the *Atiyah-Singer operator*), we have:

$$\mathrm{ind}(D^{\mathcal{S}}) = \widehat{A}\text{-genus of } M := \int_M \widehat{A}(M)$$

Proof: Set $\mathcal{E}=\mathcal{S}$, and $\mathcal{V}=M\times\mathbb{C}$, the trivial bundle of rank 1, whose chern character is 1, and apply the Theorem 16.5.1 above. □

Corollary 16.5.3 (Lichnerowicz). Let M be a compact spin manifold of everywhere strictly positive scalar curvature. Then the $\widehat{A}$- genus of M is zero.

Proof: By the Corollary 15.4.5, there are no harmonic spinors on M, i.e. $\dim\ker D^{\mathcal{S}}=0$. In particular both D^+ and D^- have vanishing kernels, so $\mathrm{ind}\, D^+=0$. This implies $\widehat{A}(M)=0$ by the Corollary 16.5.2 above. □

17. Some Consequences of the Index Theorem

Definition 17.0.4 (L-class). Let $R=\sum_{i<j}R_{ij}e_i\wedge e_j$ denote the curvature form of an oriented Riemannian manifold M of dimension $4m$. Define the *L-class* of M to be

$$L(M) = (-2\pi)^{-m}\left(\det\left(\frac{R/2}{\tanh\, R/2}\right)\right)^{1/2}$$

We note that the justification for taking the square root of the determinant is identical to the one we had for the $\widehat{A}(M)$ class, see Definition 16.1.6. Its top degree component, viz. $L(M)_{[4m]}$ turns out to be a polynomial in the Pontragin forms of M, called the *Hirzebruch L-polynomial.*

Theorem 17.0.5 (Hirzebruch Signature)**.** Let M be a compact oriented manifold of dimension $4m$. Then the cup product pairing:

$$\cup : H^{2m}(M,\mathbb{R}) \otimes H^{2m}(M,\mathbb{R}) \to \mathbb{R}$$

is symmetric, and its signature is a homotopy invariant of M called the *signature of* M, and denoted $\sigma(M)$. There is the following integral formula:

$$\sigma(M) = \int_M L(M)_{[4m]}$$

Proof: We first note that there is the chirality operator τ_{4m} which acts on $\mathcal{E} := \mathbb{C}l(M) = \Lambda^*_{\mathbb{C}}(T^*M)$, and decomposes it into the ± 1-eigenbundles $\mathcal{E}^{\pm}$. With its Levi-Civita connection, we know by 15.1.8 and 15.2.8 that this gives it the structure of a Dirac bundle, and indeed, in the proof of Bochner's theorem 15.4.7, we saw that the Dirac operator $D = d + \delta$. We just need to (a) show that the index of D is the signature $\sigma(M)$, and (b) identify the integrand which is the supertrace $\mathrm{str}_{\mathcal{E}}(k_t(a,a))dV(a)$.

In (v) of Lemma 14.1.7, we showed that

$$\tau_{4m}\phi = i^{p+k(8m+k-1)}(*\phi) \ \text{ for } \ \phi \in \Lambda^k$$

where $p = \left[\frac{4m+1}{2}\right] = 2m$. Thus

$$\tau_{4m} = \epsilon(k)\,* \quad \text{on} \ \Lambda^k \ \text{ where } \ \epsilon(k) := (-1)^{m+k(k-1)/2}$$

In particular τ_{4m} is a real operator on $\Lambda^*_{\mathbb{C}}(T^*M)$, and for the middle dimension $m + 2m(2m-1)/2 = m + m(2m-1) = 2m^2$, so τ_{4m} agrees with $*$ on Λ^{2m}. Hence we have:

$$\begin{aligned}
\Lambda^+_{\mathbb{C}}(T^*M) &= \oplus_{0\le k\le 2m}(1+\epsilon(k)*)\Lambda^k_{\mathbb{C}}(T^*M) = \oplus_{2m\le k\le 4m}(1+\epsilon(k)*)\Lambda^{4m-k}_{\mathbb{C}}(T^*M)\\
\Lambda^-_{\mathbb{C}}(T^*M) &= \oplus_{0\le k\le 2m}(1-\epsilon(k)*)\Lambda^k_{\mathbb{C}}(T^*M) = \oplus_{2m\le k\le 4m}(1-\epsilon(k)*)\Lambda^{4m-k}_{\mathbb{C}}(T^*M)
\end{aligned}$$

We know that $D(\tau_{4m}\omega) = -\tau_{4m}D\omega$, since $\nabla_X\tau_{4m} = i^{2m}\nabla_X\omega_{4m} = 0$ and τ_{2m} anticommutes with e_i in the Clifford algebra. Hence

$$D \circ * = (\pm 1) * \circ D$$

From this it follows that $\omega \in \Lambda^k(M,\mathbb{C})$ is a form in the kernel of $D^2 = d\delta + \delta d = \Delta$, iff $*\omega \in \Lambda^{4m-k}(M,\mathbb{C})$ is in the kernel of $D^2 = \Delta$ as well. Denoting the harmonic forms in $\Lambda^k(M,\mathbb{C})$ by $\mathcal{H}^k$, the above decompositions imply that for $\Delta^+ = D^-D^+$ and $\Delta^- = D^+D^-$ we have:

$$\begin{aligned}
\ker(\Delta^+) &= \oplus_{0\le k\le 2m}(1+\epsilon(k)*)\mathcal{H}^k\\
\ker(\Delta^-) &= \oplus_{0\le k\le 2m}(1-\epsilon(k)*)\mathcal{H}^k
\end{aligned}$$

Now, for $0 \le k < 2m$, since $*$ maps $\mathcal{H}^k$ isomorphically to the space $\mathcal{H}^{4m-k}$ with $\mathcal{H}^k \cap \mathcal{H}^{4m-k} = \{0\}$, we see that $(1+\epsilon(k)*)\mathcal{H}^k$ and $(1-\epsilon(k)*)\mathcal{H}^k$ are isomorphic for $0 \le k < 2m$.

For $k = 2m$, we have $\epsilon(2m) = 1$, and $(1 \pm \epsilon(2m)*)\mathcal{H}^{2m}$ are precisely the (± 1)-eigenspaces of $* : \mathcal{H}^{2m} \to \mathcal{H}^{2m}$. By the Hodge theorem, these are precisely the (± 1)-eigenspaces of the star operator $*$ on $H^{2m}(M,\mathbb{C})$. Call them $H^{2m}_{\pm}$. Since:

$$\langle \alpha \cup \beta, [M] \rangle = \int_M \alpha \wedge \beta$$

it follows that the cup product pairing is positive definite (resp. negative definite) on the space $H^{2m}(M,\mathbb{R})^+$ which is the real form of H^{2m+} (since $*$ is a real operator) (resp. $H^{2m}(M,\mathbb{R})^-$, the real form of H^{2m-}). Thus

$$\begin{aligned}
\mathrm{ind}(D^+) &= \dim_{\mathbb{C}}(\ker\Delta^+) - \dim_{\mathbb{C}}(\ker\Delta^-) = \dim_{\mathbb{C}} H^{2m+} - \dim_{\mathbb{C}} H^{2m-}\\
&= \dim_{\mathbb{R}} H^{2m}(M,\mathbb{R})^+ - \dim_{\mathbb{R}} H^{2m}(M,\mathbb{R})^- = \sigma(M)
\end{aligned}$$

Now it remains to identify the integrand. Since every manifold is locally spin, say on some coordinate chart U, and so we have the identification:

$$\mathbb{C}l(M)_{|U} = \mathcal{S}(M)_{|U} \otimes \mathcal{S}(M)_{|U}$$

by (i) in Example 15.1.10. We need to apply the Atiyah-Singer theorem to get the local integrand, with $\mathcal{V} = \mathcal{S}$.

We already know $\widehat{A}(M) = \left[\det\left(\frac{R/2}{\sinh(R/2)}\right)\right]^{1/2}$ where R is the curvature operator. We need to compute

$$\mathrm{tr}_{\mathcal{V}}(\exp(-F)) = \mathrm{tr}_{\mathcal{S}}(\exp(-F))$$

where F is the curvature form of $\mathcal{S}$ with respect to the connection $\nabla^{\mathcal{V}} = \nabla^{\mathcal{S}}$, i.e. the spin connection on $\mathcal{S}$. We have already seen that $F = -R$ as elements of $\Lambda^2 \otimes so(2m) \simeq \Lambda^2 \otimes C_2$. So we need a formula for $\mathrm{tr}_{\mathcal{S}}(\exp(R))$. Note that R is a skew-symmetric $2m \times 2m$-matrix of 2-forms since the spin connection is unitary.

Since we are at a point $a \in M$, we replace $\mathcal{S}_a$ by S_{2m}. First assume $R \in \mathrm{End}_{\mathbb{C}}(S_{2m})$ is a skew-symmetric matrix with *real scalar entries*, instead of 2-form entries. If we find a power series representation for $\mathrm{tr}_{S_{2m}}(\exp(R))$ in this case, then we can use the same power series representation when R has entries in $\Lambda^2_{\mathbb{C}}$, since R would then be nilpotent. (The same principle we applied in the proof of the Proposition 16.1.7).

First note that as a $\mathbb{C}l_{2m}$-module by left multiplication, $\mathbb{C}l_{2m}$ breaks up into 2^m identical copies of S_{2m}, by (i) of Proposition 14.4.1. Thus for an endomorphism $R \in \mathrm{End}_{\mathbb{C}}(S_{2m}) = \mathbb{C}l_{2m}$, we have:

$$\mathrm{tr}_S(\exp(R)) = 2^{-m}\mathrm{tr}_{\mathbb{C}l_{2m}}(\exp(R))$$

Suppose $R \in \mathbb{C}l_{2m}$ is of the special block-diagonal form:

$$R = t_1e_1e_2 + t_2e_3e_4 + ... + t_me_{2m-1}e_{2m}$$

Then, since $e_{2j-1}e_{2j}$ commutes with $e_{2k-1}e_{2k}$ for $j \neq k$, we have:

$$\exp(R) = \exp(t_1e_1e_2)\exp(t_2e_3e_4)...\exp(t_me_{2m-1}e_{2m})$$

We have already seen in the proof of (v) in Proposition 13.2.2 that

$$\exp(t_je_{2j-1}e_{2j}) = \cos\, t_j.I + \sin\, t_je_{2j-1}e_{2j}$$

Note that $e_{2j-1}e_{2j}$ acts as a skew symmetric matrix on the plane spanned by e_{2j-1} and e_{2j}, and a skew-symmetric matrix on the span of 1 and $e_{2j-1}e_{2j}$, and off-diagonal on all the rest of $\mathbb{C}l_{2m}$. Thus it contributes nothing to the trace of R. Similar reasoning applies to any product of distinct doublets $e_{2j-1}e_{2j}$. Thus one sees that:

$$\mathrm{tr}_{\mathbb{C}l_{2m}}\exp(t_1e_1e_2)\exp(t_2e_3e_4)...\exp(t_me_{2m-1}e_{2m}) = \mathrm{tr}_{\mathbb{C}l_{2m}}\cos\, t_1\cos\, t_2....\cos\, t_mI = 2^{2m}\prod_{j=1}^{m}\cos\, t_j$$

Now the endomorphism $R = t_1e_1e_2 + t_2e_3e_4 + ...t_me_{2m-1}e_{2m})$ is in $C_2(V) = \mathrm{Lie\,Spin}(2m) \simeq so(2m)$, and is identified with the matrix with 2×2-blocks of the form:

$$\begin{pmatrix} 0 & -2t_j \\ 2t_j & 0 \end{pmatrix}$$

whose eigenvalues are $\pm\sqrt{-1}(2t_j)$. Thus $\cosh\, R/2$ has eigenvalues $\cosh(\pm\sqrt{-1}t_j) = \cos\, t_j$. Hence $\det\cosh\, R/2 = \prod_{j=1}^m \cos^2\, t_j$. As a consequence, we find that for R of the block diagonal form above:

$$\mathrm{tr}_{S_{2m}}(\exp(R)) = 2^{-m}\mathrm{tr}_{\mathbb{C}l_{2m}}(\exp(R)) = 2^{-m}2^{2m}\left(\det\,\cosh\, R/2)\right)^{1/2} = 2^m\left(\det\,\cosh R/2\right)^{1/2}$$

Now we can assert the same formula for *any* skew-symmetric $2m \times 2m$-matrix by choice of suitable orthonormal basis $e_1, ..., e_{2m}$, since both quantities of the equation above are unaffected by such a change.

Thus

$$\widehat{A}(M) \wedge ch(\mathcal{V}) = (2\pi i)^{-2m}2^m\left[\det\left(\frac{R/2}{\sinh R/2}\right)\det\left(\cosh R/2\right)\right]^{1/2} = (-2\pi)^{-m}\left[\det\left(\frac{R/2}{\tanh\, R/2}\right)\right]^{1/2} = L(M)$$

and we have the signature theorem

$$\sigma(M) = \int_M L(M)$$

from the Atiyah-Singer Theorem 16.5.1. □

We recall the definition of the Pfaffian of a $2m \times 2m$ *skew-symmetric* real matrix A. We note that there is a polynomial of degree m in the entries of A called the *Pfaffian of* A, and which satisfies:

$$(Pf(A))^2 = \det\ A$$

The easiest way to write an explicit formula for the Pfaffian in the entries of A is to note that by an orthogonal transfomation we can bring A into a normal block diagonal form with m 2×2-blocks each of the kind:

$$\begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}$$

Then the Pfaffian is just $\prod_{i=1}^m a_i = (\det\ A)^{1/2}$. If we define the 2-form associated with a skew-symmetric matrix A, viz. $\omega_A := \sum_{i<j} A_{ij} e_i \wedge e_j$ then, at least for A of the form above, we see that

$$Pf(A)\omega_{2m} = \frac{1}{m!}\left(\omega_A \wedge \omega_A \wedge .... \wedge \omega_A\right)$$

An easy computation shows that an orthonormal change of basis $e_i \mapsto Pe_i := f_i$ results in transforming $\sum_{i<j} A_{ij} e_i \wedge e_j$ into $\sum_{k<l} (PAP^t)_{k,l} f_k \wedge f_l$, and so the above formula holds good for all skew- symmetric A.

Expanding the right hand side, we find that

$$Pf(A) = \frac{1}{m\,!}\sum_{\sigma \in S_{2m}} A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} ... A_{\sigma(2m-1)\sigma(2m)}$$

Definition 17.0.6 (Euler form)**.** Let M be an oriented Riemannian manifold of dimension $2m$. Let $R = \sum_{i<j} R_{ij} e_i \wedge e_j$ be its curvature 2-form, where each R_{ij} is the skew-symmetric matrix $\frac{1}{2}\sum_{k<l} R_{ijkl} c_k c_l$. We can then regard R as a $2m \times 2m$-skew-symmetric matrix whose (k,l)-entry is the 2-form $R^{kl} = \frac{1}{2}\sum_{i<j} R_{ijkl} e_i \wedge e_j$. Then define the *Euler form of* M by the formula:

$$e(M) = \frac{1}{(2\pi)^m} Pf(R) = \frac{1}{(2\pi)^m m!} \sum_{\sigma \in S_{2m}} R^{\sigma(1)\sigma(2)} \wedge R^{\sigma(3)\sigma(4)} ... \wedge R^{\sigma(2m-1)\sigma(2m)}$$

which is a $2m$-form.

Theorem 17.0.7 (Gauss-Bonnet-Chern-Allendoerfer)**.** For M an oriented compact Riemannian manifold of dimension $2m$, we have:

$$\chi(M)(\text{=Euler characteristic of } M) := \int_M e(M) = \sum_{i=0}^{2m} (-1)^i \dim_{\mathbb{C}} H^i(M, \mathbb{C})$$

Proof: In this case the Dirac bundle is $\mathcal{E} = \mathbb{C}l(M) = \Lambda^*_{\mathbb{C}}(T^*M))$, and the grading is *not* the chirality grading, but the *parity* grading (which comes from conjugation by $\omega_{2m} \in \mathrm{Spin}(2m)$ when M is a spin manifold). That is, $\mathcal{E}^+ = \mathbb{C}l^0(M) = \Lambda^{ev}_{\mathbb{C}}(T^*M)$, $\mathcal{E}^- = \mathbb{C}l^1(M) = \Lambda^o_{\mathbb{C}} T^*M)$ (see (ii) of Remark 15.1.11). The Dirac operator is of course $d + \delta$, as we saw in the proof of the Bochner theorem 15.4.7. Thus $D^2 = \Delta$, the Laplace-Beltrami operator on M, and by the Hodge-deRham Theorem (Corollary 9.5.3), we have $\ker(D^- D^+) = \oplus_{i=0}^m H^{2i}(M, \mathbb{C})$, and $\ker(D^+ D^-) = \oplus_{i=0}^m H^{2i+1}(M, \mathbb{C})$. Thus

$$\operatorname{ind} D^+ = \dim\ \ker(D^- D^+) - \dim \ker(D^+ D^-) = \sum_{i=0}^{2m} (-1)^i \dim_{\mathbb{C}} H^i(M, \mathbb{C})$$

Again, to identify the integrand, we may use the fact that a coordinate chart U is spin, and decompose $\mathbb{C}l(U) = \mathcal{S}(U) \otimes \mathcal{S}(U)$. However, to compute the supertrace, we have to compute with respect to this *parity* grading.

In the decomposition $\mathcal{E} = \mathcal{S} \otimes \mathcal{V}$ of a Dirac bundle on a spin manifold, we have assumed $\mathcal{S}$ is given the chirality grading and $\mathcal{V}$ is *ungraded.* The integrand of the Atiyah-Singer index theorem (i.e. the index density) has been calculated for this situation by using the fact that if $\alpha \otimes F$ is an endomorphism of a Clifford module $E = S_{2m} \otimes V$, with $\alpha \in \mathrm{End}_{\mathbb{C}}(S_{2m}) = \Lambda^*(\mathbb{R}^{2m*})$ and $F \in \mathrm{End}_{\mathbb{C}}(V)$, then

$$\text{str}_E(\alpha \otimes F) = \text{tr}_E(\tau_{2m} \circ (\alpha \otimes V)) = (-2i)^m (\alpha_{[2m]} \otimes \text{tr}_V F) \tag{64}$$

(See Lemma 14.5.2). In the present situation, $E = S_{2m} \otimes S_{2m}$, which we are regarding as a graded module with the grading operator $\omega_{2m} \otimes \omega_{2m}$ instead of the earlier grading operator $\tau_{2m} \otimes 1$.

But then, if $\alpha \otimes F \in \text{End}_{\mathbb{C}}(S_{2m}) \otimes \text{End}_{\mathbb{C}}(S_{2m}) = \Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m*}) \otimes \mathbb{C}l_{2m}$, we have:

$$\begin{aligned} \text{str}_E(\alpha \otimes F) &:= tr_E\left[(\omega_{2m} \otimes \omega_{2m}) \circ (\alpha \otimes F)\right] \\ &= \text{tr}_E[((i)^{-m}\tau_{2m} \circ \alpha) \otimes (\omega_{2m} \otimes F)] = (-i)^m \text{tr}_{S_{2m}}(\tau_{2m}\alpha)\text{tr}_{S_{2m}}(\omega_{2m} \circ F) \\ &= (-i)^m(-2i)^m(\alpha)_{[2m]}\text{tr}_{S_{2m}}(\omega_{2m} \circ F) \\ &= (-2)^m \alpha_{[2m]}\text{tr}_{S_{2m}}(\omega_{2m} F) \end{aligned}$$

(Note incidentally that $\text{tr}_{S_{2m}}(\omega_{2m} \circ F) = (-i)^m \text{tr}_{S_{2m}}(\tau_{2m} \circ F) = (-i)^m \text{str}_{S_{2m}} F$)

So for a general endomorphism $k \in \text{End}_{\mathbb{C}}(E) = \Lambda^* \otimes \mathbb{C}l$, we must modify the formula (64) by the formula:

$$\text{str}_E k = (-2)^m [\text{tr}_{S_{2m}}(\omega_{2m} k)]_{[2m]} \tag{65}$$

where k is to be regarded as an element of $\Lambda^*_{\mathbb{C}}(\mathbb{R}^{2m})$ with coefficients in $\mathbb{C}l_{2m} = \text{End}(S_{2m})$, and the trace is to be applied to the coefficients after composing with ω_{2m}.

So in the Atiyah-singer theorem, we will have to make the corresponding modification of the integrand to read:

$$\text{ind}\,(d+\delta) = (-2\pi)^{-m} \int_M \left[\widehat{A}(M)\text{tr}_{\mathcal{S}}(\omega_{2m}\exp(R))\right]_{[2m]}$$

Again, by the same reasoning as in the proof of Hirzebruch signature theorem 17.0.5,

$$\text{tr}_{\mathcal{S}}(\omega_{2m}\exp(R)) = 2^{-m}\text{tr}_{\mathbb{C}l}(\omega_{2m}\exp(R))$$

Taking R of the special form $R = \sum_{j=1}^m t_j e_{2j-1}e_{2j}$ we had computed in the proof of 17.0.5 that

$$\exp(R) = \prod_{j=1}^m ((\cos t_j)I + (\sin t_j)e_{2j-1}e_{2j})$$

which implies, since distinct doublets $e_{2j-1}e_{2j}$ and $e_{2k-1}e_{2k}$ commute, that:

$$\omega_{2m}\exp(R) = \prod_{j=1}^m e_{2j-1}e_{2j}((\cos t_j)I + (\sin t_j)e_{2j-1}e_{2j}) = \prod_{j=1}^m ((-\sin t_j)I + (\cos t_j)e_{2j-1}e_{2j})$$

As in the proof of the Hirzebruch theorem again, only the scalar term contributes to the trace, and this trace is

$$\text{tr}_{\mathbb{C}l}(\omega_{2m}\exp(R)) = (-1)^m(2^{2m})\prod_{j=1}^m \sin\, t_j$$

Hence

$$\text{tr}_{\mathcal{S}}(\omega_{2m}\exp(R)) = (-2)^m \prod_{j=1}^m \sin\, t_j$$

On the other hand, R corresponds to the block $(2m \times 2m)$-matrix whose 2×2 blocks are

$$\begin{pmatrix} 0 & -2t_j \\ 2t_j & 0 \end{pmatrix}$$

so that

$$\det\sinh(R/2) = \prod_{j=1}^m (\sinh(it_j))(\sinh(-it_j) = \prod_{j=1}^{2m}(i\sin\, t_j)(-i\sin\, t_j) = \prod_{j=1}^m \sin^2 t_j$$

So we conclude that for R of the special form above,

$$(-2)^m \left(\det(\sinh(R/2)\right)^{1/2} = \text{tr}_{\mathcal{S}}(\omega_{2m}\exp(R))$$

Now once concludes the above formula for all skew-symmetric R as before, by change of orthonormal basis. Hence the index theorem now reads:

$$\begin{aligned}
\operatorname{ind}(d+\delta) &= (-2\pi)^{-m}\int_M \left[\widehat{A}(M)\operatorname{tr}_{\mathcal{S}}(\omega_{2m}\exp(R))\right]_{[2m]} \\
&= (-2\pi)^{-m}\int_M \left[\det\left(\frac{R/2}{\sinh(R/2)}\right)^{1/2}.(-2)^m\left(\det(\sinh(R/2))^{1/2}\right]_{[2m]} \\
&= (\pi)^{-m}\int_M (\det(R/2))^{1/2} = (2\pi)^{-m}\int_M (\det(R))^{1/2} \\
&= \int_M (2\pi)^{-m} Pf(R) = \int_M e(M)
\end{aligned}$$

This proves the proposition. □

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