PROBLEMS IN COMPLEX ANALYSIS

These problems are not in any particular order. I have collected them from a number of text books. I have provided hints and solutions wherever I considered them necessary. These are problems are meant to be used in a first course on Complex Analysis. Use of measure theory has been minimized.

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Notation: $U = \{z : |z| < 1\}$ and $T = \{z : |z| = 1\}$.Def: f is analytic or holomorphic on an open set if it is differentiable at each point. $H(\Omega)$ is the class of all holomorphic functions on Ω. $\stackrel{uc}{\to}$ stands for uniform convergence on compact sets.

1. Find a sequence of complex numbers $\{z_n\}$ such that $\sin z_n$ is real for all n and $\to \infty$ as $n \to \infty$?

2. At what points is $f(z) = |z|$ differentiable? At what points is $f(z) = |z|^2$ differentiable?

3. If f is a differentiable function from a region Ω in $\mathbb{C}$ into $\mathbb{R}$ prove that f is necessarily a constant.

4. Find all entire functions f such that $f^n(z) = z$ for all z, n being a given positive integer.

5. If f and $\bar{f}$ are both analytic in a region Ω show that they are constants on Ω.

6. If f^2 and $(\bar{f})^5$ are analytic in a region show that f is a constant on that region.

7. If f is analytic in a region Ω and if $|f|$ is a constant on Ω show that f is a constant on Ω.

8. Define $Log(z) = \log|z| + i\theta$ where $-\pi < \theta \leq \pi$ and $z = |z| e^{i\theta}$ $(z \neq 0)$. Prove that Log is not continuous on $\mathbb{C}\backslash\{0\}$.

Consider the sequences $\{-1 + i/n\}$ and $\{-1 - i/n\}$.

9. Prove that the function Log defined in above problem is differentiable on $\mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$. Find its derivative and prove that there is no power series $\sum\limits_{n=0}^{\infty} a_n(z-c)^n$ convergent in $\mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ whose sum is Log.

The main part is to verify continuity of Log. Differntiability is automatic since its inverse is differentiable. The last part is follows from previous problem and basic facts about power series.

10. Let p be a non-constant polynomial, $c > 0$ and $\Omega = \{z : |p(z)| < c\}$. Prove that $\partial\Omega = \{z : |p(z)| = c\}$ and that each connected component of Ω contains a zero of p.

If $|p(z)| = c$ and there is no sequence $\{z_n\}$ converging to z with $|p(z_n)| < c$ $\forall n$ then Maximum Modulus Principle is violated. This proves the first assertion. Let C be any component of Ω. If p has no zero in Ω then, since $\partial C \subset \partial\Omega$ we have $|p(z)| \leq c$ and $\left|\frac{1}{p(z)}\right| \leq c$ by Maximum Modulus Principle applied to the region C. Hence p is a constant.

11. Prove that there is no differentiable function f on $\mathbb{C}\backslash\{0\}$ such that $e^{f(z)} = z$ for all $z \in \mathbb{C}\backslash\{0\}$.

If it exists, compare it with Log.

12. Let γ be a piecewise continuously differentiable map : $[0,1] \to \mathbb{C}$ and $h : \gamma^* \to \mathbb{C}$ be continuous(γ^* is the range of γ). Show that $f(z) = \int\limits_{\gamma} \frac{h(\zeta)}{\zeta - z} d\zeta$ defines a holomorphic function on $\mathbb{C}\backslash\gamma^*$.

13. If γ is as in above problem show that the total variation of γ is $\int\limits_0^1 |\gamma'(t)|\, dt$.

14. If p is a polynomial and if the maximum of $|p|$ on a region Ω is attained at an interior point show, without using The Maximum Modulus Principle, that p is a constant.

Compute the integral of $\frac{p(z)}{z-a}$ over a circle with centre a contained in Ω.

15. If $f(x+iy) = \sqrt{|xy|}$ show that f is not differntiable at 0 even though Cauchy-Riemann equations are satisfied.

16. Show that $\log\sqrt{x^2+y^2}$ is a harmonic function on $\mathbb{C}\backslash\{0\}$ which is not the real part of any holomorphic function.

17. If f is holomorphic on Ω and e^f is constant on Ω show that f is constant on Ω.

18. If f is an entire function and $\operatorname{Re} f$ (or $\operatorname{Im} f$) is bounded above or below show that f is constant.

19. Prove that $\frac{|a-b|}{|1-\bar{a}b|} \geq \frac{|a|-|b|}{1-|ab|}$ if either $|a|$ and $|b|$ are both less than 1 or both greater than 1.

20. If $f : U \to U$ is holomorphic show that $\frac{|f(\beta)-f(\alpha)|}{\left|1-\bar{f(\beta)}f(\alpha)\right|} \leq \frac{|\beta-\alpha|}{\left|1-\bar{\beta}\alpha\right|}$ for all $\alpha, \beta \in U$.

Let $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. Apply Schwartz Lemma to $\phi_{f(\beta)} \circ f \circ \phi_{-\beta}$.

21. Prove that a holomorphic function from U into itself has atmost one fixed point unless it is the identity map.

Apply Schwartz Lemma to $\phi_a^{-1} \circ f \circ \phi_a$ where a is a fixed point.

22. If f is a bijective bi-holomorphic map of U show that f maps open balls in U onto open balls.

The only bijective bi-holomorphic maps of U are $e^{i\theta}\phi_a$ and these map are compositions of inversions, translations and dilations.[ϕ_a is defined in Problem 20)].

23. Let Ω be a region, $f \in C(\Omega)$ and let f^n be holomorphic in Ω for some positive integer n. Show that f is holomorphic in Ω.

Use definition.

24. If f is an entire function such that $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$ show that f is a constant.

If f is an entire function such that $|f(z)| \leq M\,|z|^N$ for $|z|$ sufficiently large (where M is a positive cosnatnt) show that f is a polynomial.

Consider $\frac{f(z)-f(0)}{z}$ for the first part. For the second part use Liouville's Theorem for $N = 0$. Let $g(z) = \frac{f(z)-f(0)}{z}$ for $z \neq 0$ and $f'(0)$ for $z = 0$. Show that g satisfies the same hypotheis as f with N replaced by $N - 1$.

25. Find the largest open set on which $\int\limits_0^1 \frac{1}{1+tz}dt$ is analytic. Do the same for $\int\limits_0^\infty \frac{e^{tz}}{1+t^2}dt$.

26. If f and g are holomorphic functions on a region Ω with no zeros such that $\{z : \frac{f'}{f}(z) = \frac{g'}{g}(z)\}$ has a limit point in Ω find a simple relation between f and g.

27. If f is a holomorphic function on a region Ω which is not identically zero show that the zeros of the function form an atmost countable set.

There exist compact sets K_n increasing to Ω : look at distances of points of Ω from $\mathbb{C}\backslash\Omega$.

28. Is Mean Value Theorem valid in the complex case? (i.e., if f is analytic in a convex region and z_1, z_2 are two points in the region can we always find a point ζ on the line segment from z_1 to z_2 such that $f(z_2) - f(z_1) = f'(\zeta)(z_2 - z_1)$?)

29. Let f be holomorphic on a region Ω with no zeros. If there is a holomorphic function h such that $h' = \frac{f'}{f}$ show that f has a holomorphic logarithm on Ω (i.e. there is a holomorphic function H such that $e^H = f$. Show that h need not exist and give sufficient a condition on Ω that ensures existence of h.

30. Prove that a bounded harmonic function on $\mathbb{R}^2$ is constant.

31. If f is a non-constant entire function such that $|f(z)| \geq M|z|^n$ for $|z| \geq R$ for some $n \in \mathbb{N}$ and some M and R in $(0, \infty)$ show that f is a polynomial whose degree is atleast n.

Let $z_1, z_2, ..., z_k$ be the zeros of f in $\{z : |z| \leq R\}$. Let $g(z) = \frac{(z-z_1)(z-z_2)...(z-z_k)}{f(z)}$. Then g is an entire function which satisfies an inequality of the type $|g(z)| \leq A + B|z|^m$ for all z. Conclude that f must be a rational function, hence a polynomial.

32. If f is an entire function which is not a constant prove that $\max\{|f(z)| : |z| = r\}$ is an increasing function of r which $\to \infty$ as $r \to \infty$.

33. If $f \in C(U \cup T) \cap H(U)$ and $f(z) = 0$ on $\{e^{i\theta} : \alpha < \theta < b\}$ for some $a < b$ show that f is identically 0.

Consider $f(z)f(ze^{i\phi_1})f(ze^{i\phi_2})...f(ze^{i\phi_k})$ for suitable $\phi_1, \phi_2, ..., \phi_k$

34. True or false: if f and g are entire functions such that $f(z)g(z) = 1$ for all z then f and g are constants. [What is the answer if f and g are polynomials?]

35. If $f : U \to U$ is holomorphic, $a \in U$ and $f(a) = a$ prove that $|f'(a)| \leq 1$.
Define ϕ_a as in Problem 20 above and apply Schwartz Lemma to $\phi_a \circ f \circ \phi_{-a}$.

36. The result of Problem 35 holds for any region that is conformally equivalent to U. [A conformal equivalence is a bijective biholomorphic map].

37. According to Riemann Mapping Theorem, any simply connected region other than $\mathbb{C}$ is conformally equivalent to U. Hence, above problem applies to any such region. Is the result valid for $\mathbb{C}$?

38. Prove that only entire functions that are one-to-one are of the type $f(z) = az + b$.

[Let $g(z) = f(\frac{1}{z}), z \in \mathbb{C}\backslash\{0\}$. If g has an essential singularity at 0 then $g(\{z : |z| > 1\})$ is a non-empty open set and hence it must intersect the dense set $g(U\backslash\{0\})$. But this contradicts the fact f (and hence g) is 0ne-to-one. If

g has a removable singularity at 0 then f would be a constant and it cannot be injective. Thus g has a pole at 0 and we can write $g(z) = \frac{h(z)}{z^N}$ in $\mathbb{C}\backslash\{0\}$ where h is entire and N is a positive integer. Now $f(z) = z^N h(\frac{1}{z})$, $z \in \mathbb{C}\backslash\{0\}$. This yields $|f(z)| \leq M\,|z^n|$ for $|z|$ sufficiently large and we conclude that f must be a polynomial by Problem 24) above. Since f is one-to-one we see that its derivative is a polynomial with no zeros, hence a constant]

39. Prove that $\{z : 0 < |z| < 1\}$ and $\{z : r < |z| < R\}$ are not conformally equivalent if $r > 0$.

If ϕ is a holomorphic equivalence then $\frac{1}{\phi}$ extends to a holomorphic map g on U and there is a holomorphic map h on U such that $e^h = g$. Use this to show that there is a holomorphic logarithm on $\{z : r < |z| < R\}$ and get a contradiction by comparing with the principal branch of log.

40. Let $0 < r_1 < R_1$ and $0 < r_2 < R_2$. Prove that $\{z : r_1 < |z| < R_1\}$ and $\{z : r_2 < |z| < R_2\}$ are conformally equivalent $\Leftrightarrow \frac{R_1}{r_1} = \frac{R_2}{r_2}$

[This is standard text book material. Note that all simply connected regions other than $\mathbb{C}$ are conformally equivalent to each other, but the result is far from being true for doubly connected regions (like annuli)]

41. Show that if a holomorphic map f maps U into itself it need not have a fixed point in U. Even if it extends to a continuous map of the closure of U onto itself the same conclusion holds.

[Look at ϕ_a of Problem 20]

42. If f is holomorphic on U, continuous on the closure of U and $|f(z)| < 1$ on T prove that f has at least one fixed point in U. Can it have more than one fixed point?

By Rouche's Theorem it has exactly one fixed point.

43. If f is holomorphic $: U \to U$ and $f(0) = 0$ and if $\{f_n\}$ is the sequence of iterates of f $(i.e. f_1 = f, f_{n+1} = f \circ f_n, n \geq 1)$ prove that the sequence $\{f_n\}$ converges uniformly on compact subsets of U to 0 unless f is a rotation.

If f is not a rotation then $|f'(0)| < 1$. Consider $\sup\{\left|\frac{f(z)}{z}\right| : |z| \leq r\}$ where $r = \sup\{|z| : z \in K\}$, K being a given compact subset of U.

44. Let f be a homeomorphism of $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ (with the metric induced by the stereographic projection). Assume that f is differntiable at all points of $\mathbb{C} \cup \{\infty\}$ except $f^{-1}\{\infty\}$. Prove that f is a Mobius Transformation.

This is clear if $f^{-1}\{\infty\} = \infty$. Let $f(a) = \infty$ and $f(\infty) = b$. Let $T(z) = \frac{bz+c}{z-a}$ where $c \neq ab$. Consider $f \circ T^{-1}$. Show that this map is entire. Since it is one-to-one it must be a polynomial of degree 1.

45. Prove that the only conformal equivalences $: U\backslash\{0\} \overset{onto}{\to} U\backslash\{0\}$ are rotations.

Prove that such a map extends to a conformal equivalence of U. Hence it must be $\phi_a e^{i\theta}$ for some a and θ.

46. Prove that $\pi \cot \pi z = \frac{1}{z} + \sum\limits_{n=1}^{\infty} \frac{2z}{z^2-n^2}$ if z is not an integer.

Integrate $\frac{\pi \cot \pi\zeta}{\zeta^2 - z^2}$ over the rectangle with vertices $\pm(n+1/2) \pm ni$.

47. Prove or disprove: $Log(z_1 z_2) = Log(z_1) + Log(z_2)$

48.

a) Discuss convergence of the following infinite products:
$\prod\limits_{n=1}^{\infty} \frac{1}{n^p} (p > 0), \prod\limits_{n=1}^{\infty} (1 + \frac{i}{n}), \prod\limits_{n=1}^{\infty} \left|1 + \frac{i}{n}\right|$.

b) Prove that $\prod\limits_{n=2}^{\infty} (1 - \frac{1}{n^2}) = \frac{1}{2}$ and $\prod\limits_{n=0}^{\infty} (1 + z^{2^n}) = \frac{1}{1-z}$ if $|z| < 1$. [See Problem 51) for $\prod\limits_{n=1}^{\infty} (1 + \frac{i}{n})$].

c) $\prod\limits_{n=1}^{\infty} (1 - \frac{1}{p_n})$ where $p_1, p_2, ...$ is the sequence of primes.

[$\prod\limits_{n=1}^{N} \frac{1}{(1 - \frac{1}{p_n})} = \sum\limits_{j \in A_N} \frac{1}{j}$ where A_N is the set of all positive integers whose prime factorizations do not involve primes greater than P_N. Hence the given product diverges. Also, we can conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{p_N} = \infty$].

49. Let $\text{Re}(a_n) > 0$ for all n. Prove that $\prod\limits_{n=1}^{\infty} [1 + |1 - a_n|]$ converges if and only if $\sum\limits_{n=1}^{\infty} |Log(a_n)| < \infty$.

50. Prove or disprove the following:

$\sum\limits_{n=1}^{\infty} |Log(a_n)| < \infty \Leftrightarrow \sum\limits_{n=1}^{\infty} |1 - a_n| < \infty$ and $\sum\limits_{n=1}^{\infty} Log(a_n)$ is convergent $\Leftrightarrow$ $\sum\limits_{n=1}^{\infty} [1 - a_n]$ is convergent.

First part is true: $Log(1+z)$ behaves like z near 0. If $a_n = 1 + \frac{(-1)^n}{\sqrt{n}} i$ then $\sum\limits_{n=1}^{\infty} [1 - a_n]$ is convergent but $\sum\limits_{n=1}^{\infty} Log(a_n)$ is not convegent. If $a_n = e^{\frac{(-1)^n}{\sqrt{n}} i}$ then

$\sum_{n=1}^{\infty}[1-a_n]$ is not convergent but $\sum_{n=1}^{\infty} Log(a_n)$ is convergent.

51. Prove that $\prod_{n=1}^{\infty} z_n$ converges $\Leftrightarrow \sum Log(z_n)$ converges. Use this to prove that $\prod_{n=1}^{\infty}(1+i/n)$ is not convergent.

For $\Rightarrow$: w.l.o.g take $z_n = e^{i\theta_n}, -\pi < \theta_n \leq \pi$ and assume $e^{i(\theta_1+...+\theta_n)} \to 1$. If $\sum_{k=1}^{N} \theta_n$ is close to $2k_N\pi$ then θ_N is close to $2(k_N - k_{N-1})\pi$ and lies in $(-\pi,\pi]$ so $k_N = k_{N-1}$!

52. Prove that $\sin \pi z = \pi z \prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})$

$\sin \pi z = e^{g(z)} z \prod_{n=1}^{\infty}(1-\frac{z}{n})e^{z/n}$ for some entire function g. Use Problem 46 to find g.

53. Let $B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z}$. Prove that if $0 < |a_n| < 1$ and $\sum[1-|a_n|] < \infty$ then the product conveges uniformly on comapct subsets of U and that $B(z)$ is a holomorphic function on this disk with zeros precisely at the points $a_n, n = 1, 2, ...$ Prove that $\{a_n\}$ can be chosen so that every point of T is a limit point; prove that T is a natural boundary of B in this case (in the sense B cannot be extended to a holomorphic function on any larger open set.

[Standard text book stuff]

54. Say that a function $f : \mathbb{R} \to \mathbb{R}$ is analytic if for each $a \in \mathbb{R}$ there exists $\delta_a > 0$ such that on $(a-\delta_a, a+\delta_a)$, f has a power series expansion. Show that the zeros of an analytic function on $\mathbb{R}$ have no limit points.

The power series expansion in $(a-\delta_a, a+\delta_a)$ yields a holomorphic function in $B(a,\delta_a)$ whose restriction to $(a-\delta_a, a+\delta_a)$ is f. Fix R and use compactness of $[-R,R]$ to show that there is an open rectangle in $\mathbb{C}$ containing $[-R,R]$ and a holomorphic function on that rectangle whose restriction to $[-R,R]$ is f. Thus, f has atmost finitely many zeros in $[-R,R]$.

55. If $f : \mathbb{C} \to \mathbb{C}$ has power series expansion around each point then it has a single power series expansion valid on all of $\mathbb{C}$. Is it true that if $f : \mathbb{R} \to \mathbb{R}$ has power series expansion around each point then it has a single power series expansion valid on all of $\mathbb{R}$?

$\frac{1}{1+x^2}$

56. Does there exist an entire function f such that $|f(z)| = |z|^2 e^{\mathrm{Im}(z)}$ for all z? If so, find all such functions. Do the same for $|f(z)| = |z| e^{\mathrm{Im}(z)\,\mathrm{Re}(z)}$.

57. Does there exist a holomorphic function f on U such that $\{f(\frac{1}{n})\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, ...\}$, i.e. $f(\frac{1}{n}) = \frac{1}{n}$ if n is even and $f(\frac{1}{n}) = \frac{1}{n+1}$ if n is odd?

58. If the radius of convegence of $\sum\limits_{n=0}^{\infty} a_{n,k}(z-a)^n$ exceeds R for each k and $\sum\limits_{n=0}^{\infty} a_{n,k}(z-a)^n \to 0$ uniformly on $\{z : |z - z_0| = r\}$ then it converges uniformly on $\{z : |z - z_0| \le r\}$ provided $R > r + |z_0 - a|$.

59. Let f be continuous and bounded on $\{z : |z| \le 1\} \backslash F$ where F is a finite subset of T. If f is holomorphic on U and $|f(z)| \le M$ on $\partial U \backslash F$ show that $|f(z)| \le M$ on U.

Consider $\prod\limits_{j=1}^{k} a_j^2 e^{\epsilon Log(1-\frac{z}{a_j})} f(z)$ where $F = \{a_1, a_2, ..., a_k\}$.

60. Let $\Omega = \{z : \mathrm{Re}(z) > 0\}$. If f is continuous on the closure of Ω, holomorphic on Ω and if $|f(z)| \le 1$ on $\partial\Omega$ does it follow that the same inequality holds on Ω?.

61. Let $\Omega = \{z : a < \mathrm{Im}(z) < b\}, f \in H(\Omega)$ and f be bounded and continuous on the closure of Ω. Prove that if $|f(z)| \le 1$ on $\partial\Omega$ then the same inequality holds on Ω.

Compose the maps $z \to \pi\frac{z-a}{b-a}, z \to e^z$ and $z \to \frac{z-i}{z+i}$. Apply the result of problem 59. [See also problem #85 below].

Second proof: consider $\frac{1}{i+\epsilon(z-ia)} f(z)$ and apply Maximum Modulus Theorem for the rectangle $\{z : a < \mathrm{Im}(z) < b,\} - R < \mathrm{Re}\, z < R\}$ with R sufficiently large.

62. Prove that $f(z) = \frac{z}{(1-z)^2}$ is one-to-one on U and find the image of U.

$f(z) = \frac{1}{(1-z)^2} - \frac{1}{1-z}$. First find $\{\frac{1}{1-z} : z \in U\}$. Answer: $\mathbb{C} \backslash (-\infty, -\frac{1}{4}]$.

63. If p and q are polynomials with $\deg(q) > \deg(p) + 1$ prove that the sum of the residues of $\frac{p}{q}$ at all its poles is 0.

Integrate over a large circle.

64. Evaluate $\int\limits_\gamma \frac{1}{(z-2)(2z+1)^2(3z-1)^3} dz$ and $\int\limits_\gamma \frac{1}{(z-10)(z-\frac{1}{2})^{100}} dz$ where $\gamma(t) = e^{2\pi it} (0 \le t \le 1)$

Use problem 63.

65. Find the number of zeros of $z^7 + 4z^4 + z^3 + 1$ in U and the annulus $\{1 < |z| < 2\}$.

Apply Rouche's Theorem to $z^7 + 4z^4$ and the given function.

66. Let $p(z) = z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0$ and $R = \sqrt{1 + |c_0|^2 + |c_1|^2 + ... + |c_{n-1}|^2}$. Prove that all the zeros of p are in $\{z : |z| < R\}$.

Compare with $q(z) = z^n$ (Apply Cauchy-Schwartz).

67. Let $1 < a < \infty$. prove that $z + a - e^z$ has exactly one zero in the left half plane $\{z : \mathrm{Re}(z) < 0\}$.

Let $R > 1 + a$ and let γ be the line segment from $-Ri$ to Ri followed by the semi-circle $|z| = R$, $\frac{\pi}{2} \le \arg(z) \le \frac{3\pi}{2}$. Compare zeros of $z + a - e^z$ with the zeros of $z + a$ inside γ.

68. If $0 < |a| < 1$ show that the equation $(z-1)^n e^z = a$ has exactly n solutions in $\mathrm{Re}\, z > 0$. Prove that all the roots are simple roots. If $|a| \le \frac{1}{2^n}$ prove that all the roots are in $\{z : |z - 1| < \frac{1}{2}\}$.

$|a| < |(z-1)^n e^z|$ if $|z - 1| = |a|^{1/n}$ and $(z-1)^n e^z - a$ has no zeros outside the ball $\{z : |z - 1| < |a|^{1/n}\}$ and inside the right half plane: $|(z-1)^n e^z| > |a|\, e^{\mathrm{Re}\, z} > |a|$; there are no multiple roots because the derivative has no zeros.

69. Prove that $f(z) = 1 + z^2 + z^{2^2} + ... + z^{2^n} + ...$ has U as its natural boundary in the sense it cannot be extended to a holomorphic function on any open which properly contains U.

If θ is a dyadic rational then f is unbounded on the ray $\{re^{i\theta} : 0 < r < 1\}$ since $\left|\sum_{n=m}^{\infty} (re^{2\pi i(k/2^m)})^{2^n}\right| - \left|\sum_{n=0}^{m-1} (re^{2\pi i(k/2^m)})^{2^n}\right| \ge \sum_{n=m}^{\infty} r^{2^n} - m.$

70. If p is a polynomial such that $|p(z)| = p(|z|)$ for all z prove that $p(z) = cz^n$ for some $c \ge 0$ and some $n \in \mathbb{N} \cup \{0\}$.

p has no zeros in $\mathbb{C} \backslash \{0\}$.

71. Prove that above result holds if p is replaced by an entire function.

Compute $\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta$ in terms of the power series expansion around 0.

72. Prove the two dimensional Mean value Property:

the average of a holomorphic function over an open ball is the value at the centre.

73. Construct a conformal equivalence between the first quadrant and the upper half plane. Also, find a conformal equivalence between U and its intersection with the right half plane.

First part: z^2; second part: compose $\frac{1+iz}{i+z}$, z^2 and $\frac{i(i+z)}{i-z}$.

74. Find a conformal equivalence between the sector $\{z \neq 0 : \theta_1 < \arg(z) < \theta_2\}$ with $0 < \theta_1 < \theta_2 < \pi/2$ and U.

Use previous problem and the function z^α.

75. Prove that if γ is a closed path in a region Ω and $f \in H(\Omega)$ then $\operatorname{Re}(\int\limits_\gamma \bar{f(z)} f'(z) dz) = 0$.

Compute $\frac{d}{dt} |f(\gamma(t))|^2$.

76. Prove or disprove: given any sequence $\{a_n\}$ of complex numbers there is a holomorphic function f in some neighbourhood of 0 such that $f^{(n)}(0) = a_n$ for all n.

77. If f is holomorphic on $\Omega \backslash \{a\}$ prove that $e^{f(z)}$ cannot have a pole at a.

If f has an essential singularity at a then so does e^f. Suppose f has a pole of order k at a. If possible, let $\left|e^{f(z)}\right| \to \infty$ as $z \to a$. Let $g(z) = f(z)(z-a)^k$. Then $\operatorname{Re} \frac{g(z)}{(z-a)^k} \to \infty$ as $z \to a$. Choose θ such that $\alpha = g(a)e^{-i\theta k} \in (-\infty, 0)$. If $z_n = a + \frac{1}{n}e^{i\theta}$ then $\operatorname{Re}[n^k \frac{g(z_n)}{g(a)}] \to -\infty$, but $\operatorname{Re}[\frac{g(z_n)}{g(a)}] \to 1$ a contradiction.

78. Prove that $\int\limits_0^{2\pi} \log\left|1 - e^{i\theta}\right| d\theta = 0$.

$\int\limits_0^{2\pi} \log\left|1 - re^{i\theta}\right| d\theta = 0$ for $r \in (0,1)$ by Mean Value Theorem for harmonic functions. Split the integral into integrals over $\{\theta : r < \cos\theta\}$ and $\{\theta : r \geq \cos\theta\}$ and justify interchange of limit (as $r \to 1$) and the integrals. You may need the inequality $\cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$.

79. Use above result to prove Jensen's Formula:

If $f \in H(B(0,R)), f(0) \neq 0, 0 < r < R$ and $\alpha_1, \alpha_2, ..., \alpha_N$ are the zeros of f in $B(0,r)^-$ listed according to multiplicities then $|f(0)| \prod\limits_{n=1}^{N} \frac{r}{|\alpha_n|} = e^{\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta}$. Also prove Jensen's inequality: $\log|f(0)| \leq \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta$.

Let $g(z) = f(z) \prod\limits_{n=1}^{m} \frac{r^2 - \bar{a}_n z}{r(\alpha_n - z)} \prod\limits_{n=m+1}^{N} \frac{\alpha_n}{\alpha_n - z}$. Prove that $\log|g(0)| = \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|g(re^{i\theta})\right| d\theta$.

80. Let Ω be an open set containing 0 and $f \in H(\Omega)$. Prove that $\bar{f(z)} = f(\bar{z})$ for all z with $|z|$ sufficiently small $\Leftrightarrow f^{(n)}(0) \in \mathbb{R}$ for all $n \geq 0$.

81. If $f \in H(U), f(0) = 0, f'(0) \neq 0$ prove that there is no $g \in H(U \backslash \{0\})$ such that $g^2 = f$.

82. If f is an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$ prove that $|f(z)| \geq c\,|z|$ for some positive number c for all z with $|z|$ sufficiently large.

Consider $\frac{1}{f(\frac{1}{z})}$

83. Let Ω be a region, $\{f_n\} \subset H(\Omega)$ and assume that $\{f_n\}$ is uniformly bounded on each compact subset of Ω. Let C be the set of points where $\{f_n\}$ is convergent. If this set has a limit point in Ω prove that $\{f_n\}$ converges uniformly on compact subsets of Ω to a holomorphic function.

[The family $\{f_n\}$ is normal. Let $\{f_{n_k}\}$ converge uniformly on compact subsets to f. Then $f \in H(\Omega)$. If g is another subsequential limit of $\{f_n\}$ then $f = g$ at point where $\{f_n(z)\}$ converges. Thus $f = g$ on a set with limit points in Ω]

84. Prove or disprove: If Ω is a region, $\{f_n\} \subset H(\Omega), f_n^{(k)}(z) \to 0$ as $n \to \infty$ for each $z \in \Omega$ and each $k \in \{0, 1, 2, ...\}$ then $\{f_n\}$ converges (to 0) uniformly on compact subsets of Ω

[This is a trivial consequence of problem #83 above if $\{f_n\}$ is uniformly bounded on each compact subset of Ω. What if this assumption is dropped?]

85. Give an example of a function f which is continuous on a closed strip, holomorphic in the interior, bounded on the boundary but not bounded on the strip! [See also problem #61 above].

$\cos(\cos z)$

86. Let $u(z) = \operatorname{Im}\{(\frac{1+z}{1-z})^2\}$. Show that u is harmonic in U and $\lim\limits_{r \to 1} u(re^{i\theta}) = 0$ for all θ. Why doesn't this contradict the Maximum Modulus Principle for harmonic functions?

[Answer to second part: Limit is taken only along radii]

87. If $\phi(|z|)$ is harmonic in the region $\{z : \operatorname{Re}(z) > 0\}$ (ϕ being real valued and "smooth") prove that $\phi(t) \equiv a \log t + b$ for some a and b.

88. Let $f : \bar{U} \to \mathbb{C}$ be a continuous function which is harmonic in U. Prove that f is holomorphic in U if and only if $\int\limits_{-\pi}^{\pi} f(e^{it}) e^{int} dt = 0$ for all positive integers n.

f is the Poisson integral of its values on the boundary. Replace the Poisson kernel $P_r(t)$ by $\sum\limits_{n=-\infty}^{\infty} r^{|n|}e^{int}$ and interchange the sum and the integral. Note that $\sum\limits_{n=0}^{\theta} c_n \bar{z}^n$ is holomorphic if and only if it is a constant.

89. Let $\Omega = \{z : \mathrm{Re}(z) > 0\}$. If f is bounded and continuous on $\partial\Omega$ show that it is the restriction of a continuous function on $\bar{\Omega}$ which is harmonic in Ω.

Let $F(x+iy) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{xf(it)}{x^2+(y-t)^2} dt$. Prove that $\int\limits_{-\Delta}^{\Delta} \frac{f(it)}{x+i(y-t)} dt$ is holomorphic on Ω and it converges to F uniformly on compact subsets of Ω.

90. Prove that the square of a real harmonic function is not harmonic unless it is a constant. When is the product of two real harmonic functions harmonic? Find all holomorphic functions f such that $|f|^2$ is harmonic.

91. If $f : \Omega \to \mathbb{C}$ and f and f^2 are harmonic prove that either f is holomorphic or $\bar{f}$ is holomorphic. Prove the converse.

92. If u is a non-constant harmonic in a region Ω prove that the zeros of the gradient of u in Ω have no limit point.

93. If u is harmonic in a region Ω prove that partial derivatives of u of all orders are harmonic.

94. Let $S = \{x \in \mathbb{R} : a \leq x \leq b\}$. Let Ω be a region containing S. Prove that if $f \in H(\Omega \backslash S) \cap C(\Omega)$ then $f \in H(\Omega)$.

Prove that integral of f over any triangle in Ω is 0.

95. Let $f, f_n (n = 1, 2, ...)$ be holomorphic functions on a region Ω. If $\mathrm{Re}(f_n) \overset{uc}{\to} \mathrm{Re}(f)$ show that $f_n \overset{uc}{\to} f$.

Enough to do this in small balls. Use the formula $\mathrm{Im}[f_n(z)] = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \mathrm{Im}[\frac{e^{it}R+z}{e^{it}R-z}] \,\mathrm{Re}\, f_n(a+e^{it}R) dt$ for $z \in B(a,R)$ if the closure of $B(a,R)$ is contained in Ω. [There is a similar formula for $\mathrm{Im}[f(z)]$].

96. Let $f(z) = \int\limits_{-1}^{1} \frac{1}{t-z} dt, z \in \mathbb{C} \backslash [-1, 1]$. Prove that f is holomorphic, its imaginary part is bounded, but the real part is not. Prove that $\lim\limits_{z\to\infty} zf(z)$

exists and find this limit. Find a bounded non-constant holomorphic function on $\mathbb{C}\backslash[-1,1]$.

97. Give an example of a region Ω such that Ω^c is infinite and every bounded holomorphic function on Ω is a constant.

Take $\Omega = \mathbb{C}\backslash\{1,2,...\}$
Remark: it can be shown that there are non-constant bounded holomorphic functions on $\mathbb{C}\backslash[-1,1]$ but there are no such functions on $\mathbb{C}\backslash K$ if K is a compact subset of $\mathbb{R}$ with Lebesgue measure 0. Thus the complement of the Cantor set gives a region whose complement is uncountable such that every bounded holomorphic function on it is a constant.

98. If Ω is any region contained in $\mathbb{C}\backslash(-\infty,0]$ show that there exists a bounded non-constant holomorphic function on Ω.

More generally if there is a non-constant holomorphic function ϕ on Ω such that $\phi(\Omega)$ is contained in $\mathbb{C}\backslash(-\infty,0]$ the same conclusion holds.
Look at $e^{iLog(\phi(z))}$.

99. If Ω is $\mathbb{C}\backslash(-\infty,0]$ or a horizontal strip or a vertical strip or U^c show that there exist non-constant bounded holomorphic functions on Ω.
$[e^{iLog(z)}, e^{iz}, e^z, \frac{1}{z}]$

100. Prove that there is no holomorphic function f on U^c such that $|f(z)| \to \infty$ as $|z| \to 1$.
First assume that f has no zeros and look at $\frac{1}{f(\frac{1}{z})}$. Use Laurent series expansion of $\frac{1}{f(\frac{1}{z})}$. For the general case use the existence of an entire function whose zeros match the zeros of f.

101. Prove that there is no continuous bijection from $\bar{\Omega}$, where $\Omega = \{z : \text{Re}(z) > 0\}$, onto $\bar{U}$ which maps Ω onto U and is holomorphic in Ω.

Write down all holomorphic bijections from Ω onto U and show that each of them extend to continuous functions on $\bar{\Omega}$ uniquely with range properly contained in $\bar{U}$ [In fact the range misses exactly one point].

102. Let Ω be a bounded region, $f \in C(\bar{\Omega}) \cap H(\Omega)$ and assume that $|f|$ is a non-zero constant on $\partial\Omega$. If f is not a constant on Ω show that f has atleast one zero in Ω.

103. Let f be a non-constant entire function. Prove that the closure of $\{z : |f(z)| < c\}$ coincides with $\{z : |f(z)| \leq c\}$ for all $c > 0$.

104. Prove that if $f \in H(\Omega), [a,b] \subset \Omega$ (where $[a,b]$ is the line segment from a to b) then $|f(b) - f(a)| \leq |b-a| \, |f'(\xi)|$ for some $\xi \in [a,b]$. Also prove that $|f(b) - f(a) - (b-a)f'(a)| \leq \frac{|b-a|^2}{2} \left| f^{'}{}'(\eta) \right|$ for some $\eta \in [a,b]$.

105. Evaluate $\int\limits_{\gamma} \frac{z^2+1}{z(z^2+4)} dz$ where $\gamma(t) = re^{2\pi i t}(0 \leq t \leq 1)$ where $0 < r < 2$. No computation is needed!

Compute the same integral for $r > 2$.

Use partial fractions for second part.

106. Give an example of a bounded holomorphic function f on $\mathbb{C}\backslash\mathbb{R}$ which cannot be extended to any larger open set.

Take $f(z) = \begin{cases} \frac{1+iz}{1-iz} & \text{if } \operatorname{Im} z > 0 \\ \frac{1-iz}{1+iz} & \text{if } \operatorname{Im} z < 0 \end{cases}$ and note that $\lim\limits_{\operatorname{Im} z \to 0} f(z)$ exists only for $\operatorname{Re} z = 0$.

107. If $f \in H(0 < |z| < R)$ and $\int\limits_{0<x^2+y^2<R} |f(x+iy)| \, dxdy < \infty$ prove that f has either a removable singularity or a pole of order one at 0.

The coeffcients $\{a_n\}$ in the Laurent series expansion satisfy $\int\limits_0^R r^{n+1} dr \, |a_n| < \infty$.

108. In the previous problem if $\int\limits_{0<x^2+y^2<R} |f(x+iy))|^2 \, dxdy < \infty$ prove that f has a removable singularity at 0.

109. Show that there is no function $f \in H(U) \cap C(\bar{U})$ such that $f(z) = \frac{1}{z} \forall z \in \partial U$.

$[zf(z) - 1 \in H(U) \cap C(\bar{U})$ and vanishes on $\partial U]$.

110. If $f \in C(U), f_n \in H(U)$ and $f_n \to f$ in $L^1(U)$ then $f \in H(U)$.

$[\int\limits_0^1 \int\limits_{-\pi}^{\pi} \left| f_n(re^{i\theta}) - f(re^{i\theta}) \right| r dr d\theta \to 0$ and hence $\int\limits_{-\pi}^{\pi} \left| f_{n_k}(re^{i\theta}) - f(re^{i\theta}) \right| d\theta \to 0$ for almost all r for some subsequence $\{n_k\}$ of $\{1, 2, ...\}$. We can find a sequence $r_j \uparrow 1$ such that $\int\limits_{-\pi}^{\pi} \left| f_{n_k}(re^{i\theta}) - f(re^{i\theta}) \right| d\theta \to 0$ for $r = r_1, r_2, ...$ By Cauchy's Integral Formula we have $f_n(z) - f_m(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f_n(\zeta) - f_m(\zeta)}{\zeta - z} d\zeta \forall z \in B(0, \alpha/2)$ where $\gamma(t) = \alpha e^{2\pi i t}, 0 \leq t \leq 1$. It follows easily from this that $\{f_{n_k}\}$ is uniformly

Cauchy on $B(0,\alpha)$. This proves (by Morera's Theorem) that $f \in H(B(0,\alpha))$ and $\alpha \in (0,1)$ is arbitrary.

111. Any conformal equivalence of $\mathbb{C}\backslash\{0\}$ is of the form cz or of the form $\frac{c}{z}$ where c is a constant.

[This requites the Big Picard's Theorem. Consider the Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$. By Big Picard's Theorem and the fact that f is injective neither $f(z)$ nor $f(\frac{1}{z})$ has an essential singularity at 0. This forces $\sum_{n=-\infty}^{\infty} c_n z^n$ to be a finite sum. Thus, f is a rational function . Since f is holomorphic on $\mathbb{C}\backslash\{0\}$, we can write $f(z) = \frac{p(z)}{z^j}$ for some $j \in \{0,1,2,...\}$ and some polynomial p with $p(0) \neq 0$. It follows that its derivative has no zeros in $\mathbb{C}\backslash\{0\}, i.e. z^j p'(z) - jz^{j-1}p(z)$ is a polynomial with no zeros in $\mathbb{C}\backslash\{0\}$. This implies that $z^j p'(z) - jz^{j-1}p(z) = cz^n$ for some $n \in \{0,1,2,...\}$ and $f'(z) = \frac{cz^n}{z^{2j}} = cz^{n-2j}$. Thus, $f(z) = cz^{k+1}/(k+1)$ where $k = n-2j$. [Note that there is holomorphic function on $\mathbb{C}\backslash\{0\}$ whose derivative is $\frac{1}{z}$. Thus, $k \neq -1$]. The fact that f is injective shows that $k+1 = \pm 1$].

112. If $x_1 > x_2 > x_3 > ..., \{x_n\} \to 0$ and $f \in H(U)$ with $f(x_n) \in \mathbb{R} \forall n$ then $f^{(k)}(0) \in \mathbb{R} \forall k$.

[Clearly $f(0)$ and $f'(0)$ are real. Now, $f^{(k+1)}(0) = ((k+1)!)(\lim_{t\to 0} \frac{f(t)-[c_0+c_1t+c_2t^2+...+c_kt^k]}{t^{k+1}})$ where $c_j = \frac{f^{(j)}(0)}{j!}$. Taking limit along the sequence $\{x_n\}$ we see that $f^{(k+1)}(0) \in \mathbb{R}$ if $f^{(l)}(0) \in \mathbb{R}$ for $l \leq k$].

113. Let $\{f_n\} \subset H(D)$ where D is an open disc. Assume that $f_n(D) \subset D\backslash\{0\} \forall n$ and that $\lim_{n\to\infty} f_n(a) = 0$ where is the center of D. Then $\lim_{n\to\infty} f_n(z) = 0$ uniformly on compact subsets of D.

[$\{f_n\}$ is normal. If a subsequence converges uniformly on comapct subsets then either the limit has no zeros or it is identically zero].

114. Let $\{u_n\}$ be a sequence of (strictly) positive harmonic functions on an open set Ω such that $\sum u_n = \infty$ at one point. Then the series diverges at every point. Moreover, if it converges at one point it converges uniformly on compact subsets of Ω.

[Apply problem 113) above to $\{\prod_{n=1}^{N} e^{u_n+iv_n}\}$ where v_n is a harmonic conjugate of u_n. Of course, it suffices to ptove the result in each closed disc contained in Ω, so existence of harmonic conjugate is guaranteed].

115. Find all limit points of the sequence $\{\frac{1}{n}\sum_{k=1}^{n} k^{ia}\}_{n=1,2,...}$ where a is a non-zero real number.

[$\sum_{k=1}^{n}(\frac{k}{n})^{ia}\frac{1}{n} \to \int_0^1 x^{ia}dx = \frac{1}{1+ia}$. We claim that the set of limit points of $\{n^{ia}\}$ is precisely the unit circle $\{|z| = 1\}$ and this would show that the desired set is $\{z : |z| = \frac{1}{\sqrt{1+a^2}}\}$. Given $\alpha \in \mathbb{R}$ and $\epsilon > 0$ we need to show the existance of integers n and m such that $|\alpha - a\log(n) - 2m\pi| < \epsilon$. Equivalently, $\frac{\alpha-2m\pi}{a} - \frac{\epsilon}{|a|} < \log(n) < \frac{\alpha-2m\pi}{a} + \frac{\epsilon}{|a|}$. The interval $(e^{\frac{\alpha-2m\pi}{a}-\frac{\epsilon}{|a|}}, e^{\frac{\alpha-2m\pi}{a}+\frac{\epsilon}{|a|}})$ has length larger than 1 if $-\frac{m}{a}$ is sufficiently large and so it would contain an integer n. Also $e^{\frac{\alpha-2m\pi}{a}-\frac{\epsilon}{|a|}} > 1$ for such m and so n is positive].

116. Let f have an isolated singularity at a point a. Prove that e^f cannot have a pole at a.

[If f has a removable singularity the conclusion holds. Suppose f has an essential singularity at a. We claim that $\{e^{f(z)} : 0 < |z-a| < \delta\}$ is dense in $\mathbb{C}$ for each δ. Of course, these implies that e^f does not have a pole at a. We know that $\{f(z) : 0 < |z-a| < \delta\}$ is dense in $\mathbb{C}$ for each δ. Let $c \in \mathbb{C}\backslash\{0\}$ and $\epsilon > 0$. Let $e^d = c$ and choose z such that $0 < |z-a| < \delta$ and $|f(z) - d| < \epsilon$. Then $\left|e^{f(z)} - e^d\right| < \epsilon[e^{2|d|+\epsilon}]$. This proves the claim. Finally, if f has a pole at a then there is a positive integer m such that $(z-a)^m f(z) = g(z)$ (say) is holomorphic in a neighbourhood of a and $g(a) \neq 0$. Thus, $e^{f(z)} = e^{\frac{g(z)}{(z-a)^m}} = e^{\frac{p(z)}{(z-a)^m}} e^{h(z)}$ near a with h holomorphic near a, p being a polynomial of degree at most m. If $e^{\frac{p(z)}{(z-a)^m}}$ has a removable singulairty or a pole then $e^{\frac{p(z)}{(z-a)^m}}(z-a)^k$ would be bounded near a for some integer $k \geq 0$. Put $z = a + N$ where N is a positive integer and note that $e^{\frac{p(z)}{(z-a)^m}}(z-a)^k \to \infty$ as $N \to \infty$. Thus $e^{\frac{p(z)}{(z-a)^m}}$ must have an essential singularity at a so does $e^{\frac{p(z)}{(z-a)^m}} e^{h(z)}$].

117. Let f be holomorphic on U and assume that for each $r \in (0,1)$, $f(re^{it})$ has a constant argument (i.e. $f(re^{it}) = \left|f(re^{it})\right| e^{ia_r}$ where the real number a_r does not depend on t. Show that f is a constant.

[The set $U\backslash\{z : f(z) \in (-\infty, 0]\}$ is open. On this set $Log(f)$ has a constant imaginary part which implies it is a constant. Thus f is a constant on $U\backslash\{z : f(z) \in (-\infty, 0]\}$. If this open set is non-empty then f is a constant everywhere. If it is empty then $\mathrm{Im}(f) = 0$ on U which implies of course that f is a constant]

118. [based on problem 117)] Let $f \in H(\Omega)$ and suppose $|f|$ is harmonic in Ω. Show that f is a constant.

[f and $|f|$ both have mean value property and this implies that the hypothesis of previous problem is satisfied].

119. Let $f \in H(U), f(U) \subset U, f(0) = 0$ and $f(\frac{1}{2}) = 0$. Show that $|f'(0)| \leq \frac{1}{2}$. Give an example to show that equality may hold.

[Let $g = \frac{f}{h}$ where $h(z) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z} = \frac{2z-1}{2-z}$. Use Maximum Modulus principle to conclude that Schwartz Lemma applies to g. Now

$|f'(0)| = |h(0)|\,|g'(0)| \leq |h(0)| = \frac{1}{2}$. Equality holds when $f = zh(z)$]

120. Let $f \in H(U), f(U) \subset U, f(0) = 0, f'(0) = 0, f''(0) = 0..., f^{(k)}(0) = 0$ where k is a positive integer. Show that $\left|f(\frac{1}{2})\right| \leq \frac{1}{2^k}$ and find a necessary and sufficient condition that $\left|f(\frac{1}{2})\right| = \frac{1}{2^k}$.

[Let $g(z) = \frac{f(z)}{z^k}$. Then $g \in H(U)$ and Maximum Modulus Theorem implies $g(U) \subset U$ (unless g is a constant, in which case $\left|f(\frac{1}{2})\right| \leq \frac{1}{2^k}$ with equality holding when the constant has modulus 1). Hence $\left|f(\frac{1}{2})\right| = \left|(\frac{1}{2})^k g(\frac{1}{2})\right| < \frac{1}{2^k}$

unless $f(z) = cz^k$ with $|c| \leq 1$. Equality holds if and only if $f(z) = cz^k$ with $|c| = 1$].

121. If f and $zf(z)$ are both harmonic then f is analytic.

[C-R equations hold]

122. Prove that $f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} \sin(n\alpha)e^{in\theta}$ is harmonic in U.

[$\sum_{n=0}^{\infty} r^n \sin(n\alpha)e^{in\theta}$ is holomorphic]

123. If $\Omega = \{z : \mathrm{Re}(z) > 0\}$ and f is a bounded holomorphic function on Ω with $f(n) = 0 \forall n \in \mathbb{N}$ show that $f(z) = 0 \forall z \in \Omega$.

[Let $g(z) = f(\frac{1-z}{1+z})$ on U. A well known result (which is an easy consequence of Jansen's Formula) says that the zeros $a_1, a_2, ...$ of a bounded holomorphic function g on U which is not identically 0 satisfies $\sum[1 - |a_n|] < \infty$. Since $\sum[1 - \left|\frac{1-n}{1+n}\right|] = \infty$, g must vanish identically].

124. Show that there is a holomorphic function f on $\{z : \mathrm{Re}(z) > -1\}$ such that $f(z) = \frac{z^2}{2} - \frac{z^3}{(2)(3)} + \frac{z^4}{(3)(4)} - ...$ for $|z| < 1$.

[$f(z) = (1 + z)Log(1 + z) - z$]

125. Consider the series $z+\frac{z^2}{2}+\frac{z^3}{3}+...$ on U and $i\pi-(z-2)+\frac{(z-2)^2}{2}-\frac{(z-2)^3}{3}+ ...$ on $\{z : |z - 2| < 1\}$. (These two regions are disjoint). Show that there is a region Ω and a function $f \in H(\Omega)$ such that Ω contains both U and $\{z : |z - 2| < 1\}$, $f(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + ...$ on U and $f(z) = i\pi - (z - 2) + \frac{(z-2)^2}{2} - \frac{(z-2)^3}{3} + ...$ on $\{z : |z - 2| < 1\}$.

[Let $\Omega = U \cup \{z : |z - 2| < 1\} \cup \{z : \mathrm{Im}\, z > 0\}, f(z) = Log(1 - z)$ on $\Omega \cap \{z : \mathrm{Re}\, z < 1\}, f(z) = \log(1 - z)$ on $\Omega \cap \{z : \mathrm{Re}\, z > 1\}$ where $\log(z) = \log|z| + i\theta$ if $z = |z| e^{i\theta}$ with $0 < \theta < 2\pi, f(z) = Log(1 - z) = \log(1 - z)$ on the ray $\{iy : y > 0\}$].

126. Let $f : U \to U$ be holomorphic with $f(0) = 0 = f(a)$ where $a \in U\backslash\{0\}$. Show that $|f'(0)| \leq |a|$.

[Consider $g(z) = \frac{f(z)(1-\bar{a}z)}{z(z-a)}$]

127. Prove that a complex valued function u on a simply connected region Ω is harmonic if and only if it is of the form $f + \bar{g}$ for some $f, g \in H(\Omega)$.

[If part is obvious. For the converse let $u_1 = \mathrm{Re}(u), u_2 = \mathrm{Im}(u)$ and let u_1+iv_1, u_2+iv_2 be holomorphic. Then $u = f+\bar{g}$ where $f = \frac{u_1+iv_1+iu_2-v_2}{2}, g = \frac{u_1+iv_1-iu_2+v_2}{2}$]

128. Let $f(z) = z + \frac{1}{z}(z \in \mathbb{C}\backslash\{0\})$. Show that $f(\{z : 0 < |z| < 1\}) = f(\{z : |z| > 1\} = \mathbb{C}\backslash[-2, 2]$ and that $f(\{z : |z| = 1\}) = [-2, 2]$. Show also that f is conformal equivalence of both the regions $\{z : 0 < |z| < 1\}$)and $\{z : |z| > 1\}$ with $\mathbb{C}\backslash[-2, 2]$. Prove that $\{z : |z| > 1\}$ is not simply connected. [How many proofs can you think of?]

129. Show that there is no bounded holomorphic function f on the right-hlaf plane which is 0 at the points $1, 2, 3, ...$ and 1 at the point $\sqrt{2}$. What is the answer if 'bounded' is omitted?

[Let $g(z) = f(\frac{1-z}{1+z})$ for $z \in U$ and note that the zeros $\{\alpha_n\}$ of a non-zero bounded function in $H(\Omega)$ must satisfy the condition $\sum[1 - |\alpha_n|] < \infty$ (as a consequence of Jensen's Theorem)].

130. Prove or disprove: if $\{a_n\}$ has no limit points and $\{c_n\} \subset \mathbb{C}$ then there is an entire function f with $f(a_n) = c_n \forall n$.

[This is true and it follows easily from Mittag Lefler's Theorem].

131. Let Ω be a bounded region, $f \in H(\Omega)$ and $\limsup\limits_{z \to a} |f(z)| \leq M$ for every point a on the boundary of Ω.Show that $|f(z)| \leq M$ for every $z \in \Omega$.

[Let $M_1 = \sup\{|f(z)| : z \in \Omega\}$. (This may be ∞). Let $|f(z_n)| \to M_1$ with $\{z_n\} \subset \Omega$. Let $\{z_{n_k}\}$ be a subsequence converging to (say) z. Of course, $z \in \bar{\Omega}$. If $z \in \partial\Omega$ then $\limsup\limits_{k \to \infty} |f(z_{n_k})| \leq M$ by hypothesis and hence $M_1 \leq M$. If $z \in \Omega$ then f is a constant by Maximum Modulus Theorem].

132. Let f be an entire function such that $\frac{f(z)}{z} \to 0$ as $|z| \to \infty$. Show that f is a constant.

133. Let f be an entire function which maps the real axis into itself and the imaginary axis into itself. Show that $f(-z) = -f(z) \forall z \in \mathbb{C}$.

[Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n, z \in \mathbb{C}$. Clearly, $a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{R} \forall n$. [In fact, $f^{(n)}(x) \in \mathbb{R} \forall n \forall x \in \mathbb{R}$]. Now $\sum\limits_{n=0}^{\infty} a_n (iy)^n$ is purely imaginary and hence $\sum\limits_{n=0}^{\infty} a_{2n} (-1)^n y^{2n} = 0 \forall y$. Thus, $a_{2n} = 0 \forall n$]

134. Let f be a continuous function : $\mathbb{C} \to \mathbb{C}$ such that $f(z^2 + 2z - 6)$ is an entire function. Show that f is an entire function.

[Let $a \in \mathbb{C}, a \neq -7$ and $b^2 + 2b - 6 = a$. In a neighbourhood of b the function $p(z) = z^2 + 2z - 6$ is one-to-one (because $2b + 2 \neq 0$)and the image of this neighbourhood is an open set V.Further, p^{-1} is holomorphic on V. Now $f(z) = (f \circ p) \circ p^{-1}(z) \forall z \in V$ and hence f is differntiable at a.Finally, f has a removable singularity at a. Note that $z^2 + 2z - 6$ can be replaced by any ploynomial; in fact we replace it any entire function p such that $\{p(b) : p'(b) = 0\}$ is isolated].

135. If f and g are entire functions with no common zeros and if h is an entire function show that $h = fF + gG$ for some entire functions F and G.

[Let $\phi = \frac{h}{g}$ on $\mathbb{C} \backslash g^{-1}\{0\}$. Let $a_1, a_2, ...$ be the zeros of f. Let $c_n = \phi(a_n), n \geq 1$. We can find an entire function G such that $G(a_n) = c_n, n \geq 1$ and such that $\phi - c_n$ and $G - c_n$ have zeros of the same order at a_n for each n. It follows that $F = \frac{h - Gg}{f}$ is entire].

136. Show that the series $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$ converges if $|z| \leq 1$ and $z \neq 1$.

[This is a standard result in Fourier series; we will show that $\sum\limits_{n=1}^{\infty} \frac{\cos(nt)}{n}$ and

$\sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$ both converge if $t \neq 0$.

Let $-\pi \leq t \leq \pi, t \neq 0$. Let $a_n = \cos t + \cos(2t) + \ldots + \cos(nt), n \geq 1, a_0 = 0$. Then $a_n = \mathrm{Re}[e^{it} + e^{2it} + \ldots + e^{int}] = \mathrm{Re}\, \frac{e^{i(n+1)t} - e^{it}}{e^{it}-1}$. Thus, $a_n = \frac{\mathrm{Re}[(e^{i(n+1)t} - e^{it})(e^{-it}-1)]}{|e^{it}-1|^2} = \frac{\cos(nt) - \cos((n+1)t) - 1 + \cos(t)}{|e^{it}-1|^2}$ proving that $\{a_n\}$ is bounded. Now $\sum_{n=N_1}^{N_2} \frac{\cos(nt)}{n} = \sum_{n=N_1}^{N_2} \frac{a_n - a_{n-1}}{n}$. This gives $\sum_{n=N_1}^{N_2} \frac{\cos(nt)}{n} = -\frac{a_{N_1-1}}{N_1} + \sum_{j=N_1}^{N_2-1} a_j(\frac{1}{j} - \frac{1}{j+1}) + \frac{a_{N_2}}{N_2}$. This clearly implies convergence of $\sum_{n=1}^{\infty} \frac{\cos(nt)}{n}$. The proof of convegence of $\sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$ uses the same argument with a_n replaced by $\sin(t) + \sin(2t) + \ldots + \sin(nt)$].

137. Show that the convergence of $\sum_{n=1}^{\infty} \frac{\sin(nz)}{n}$ implies that $z \in \mathbb{R}$.

[$\frac{\sin(nx)\cosh(ny)}{n} \to 0$ and $\frac{\cos(nx)\sinh(ny)}{n} \to 0$. If $y \neq 0$ then $\frac{\cosh(ny)}{n}$ and $\left|\frac{\sinh(ny)}{n}\right| \to \infty$].

138. If $f \in C(\overline{U}) \cap H(U)$ and f is real valued on $T = \partial U$ then f is a constant.

[Maximum modulus principle to e^{if} and e^{-if}]

139. Let $\Omega = \{z : \mathrm{Im}(z) > 0\}$ and $f \in H(\Omega) \cap C(\overline{\Omega})$. If $f(x) = x^4 - 2x^2$ for $0 < x < 1$ find $f(i)$.

[One solution is to use Schwartz Reflection Principle. We can extend f to a holomorphic function on $\Omega \cup \Omega_1$ where $\Omega_1 = \{z : 0 < \mathrm{Re}\, z < 1\}$. It the follows that f and $z^4 - 2z^2$ coincide on a set with limit points and hence $f(z) = z^4 - 2z^2$ on Ω].

140. Let Ω be a region and m denote Lebesgue measure on Ω. If $\{f_n\} \subset H(\Omega) \cap L^2(\Omega)$ and if $\{f_n\}$ converges in $L^2(\Omega)$ to f show that $f \in H(\Omega)$.

[Let $B(a, 2r) \subset \Omega$. Consider $\frac{1}{r_2 - r_1} \int\limits_{r_1 \leq |\zeta - a| \leq r_2} f_n(\zeta) \frac{\zeta - a}{|\zeta - a|(\zeta - z)} dm(\zeta)$ where $z \in B(a, r)$ and $0 < r_1 < r_2 < r$. We can write this as $\frac{1}{r_2 - r_1} \int_{r_1}^{r_2} \int_{-\pi}^{\pi} f_n(a + \rho e^{it}) \frac{\rho e^{it}}{\rho(a + \rho e^{it} - z)} \rho\, d\rho\, dt$. Now $\int_{-\pi}^{\pi} f_n(a + \rho e^{it}) \frac{\rho e^{it}}{(a + \rho e^{it} - z)} dt = (-i) \int_{\gamma} \frac{f_n(\zeta)}{\zeta - z} d\zeta$ where $\gamma(t) = a + \rho e^{it}$. By Cauchy's Integral Formula we now see that if $z \in B(a, r_1/2)$ then $\frac{1}{r_2 - r_1} \int\limits_{r_1 \leq |\zeta - a| \leq r_2} f_n(\zeta) \frac{\zeta - a}{|\zeta - a|(\zeta - z)} dm(\zeta) = \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} (-i) f_n(z) d\rho = (-i) f_n(z)$.

Let $h_z(\zeta) = \frac{i}{r_2 - r_1} \frac{\zeta - a}{|\zeta - a|(\zeta - z)} I_{r_1 \leq |\zeta - a| \leq r_2}$. We have $f_n(z) = \int f_n(\zeta) h_z(\zeta) dm(\zeta)$. Since $h_z \in L^2(\Omega)$ we get $f_n(z) \to \int f(\zeta) h_z(\zeta) dm(\zeta)$ and hence $f(z) = \int f(\zeta) h_z(\zeta) dm(\zeta)$ a.e. $[m]$. It suffices therefore to show that $g(z) = \int f(\zeta) h_z(\zeta) dm(\zeta)$ defines a holomorphic function on $B(a, r_1/2)$. But $g(z) = \int \frac{1}{\zeta - z} d\mu(\zeta)$ where $\frac{d\mu}{dm}(\zeta) = f(\zeta) \frac{1}{r_2 - r_1} \frac{\zeta - a}{|\zeta - a|}$ and g has a power series expansion in $B(a, r_1/2)$ by a standard argument].

141. Let Ω be a region containing $\bar{U}$ and $f \in H(\Omega)$. If $|f(z)| = 1$ whenever $|z| = 1$ show that $U \subset f(\Omega)$.

[If f has no zeros then (using Maximum Modulus Theorem to f and $\frac{1}{f}$ we see that f is a constant. Thus $0 \in f(\Omega)$. Now we apply Rouche's Theorem; if $a \in U$ then $|f(z) - (f(z) - a)| = |a| < 1 = |f(z)|$ whenever $|z| = 1$ and hence f and $f - a$ have the same number of zeros in U. Since f has a zero, so does $f - a$].

142. Let Ω be a bounded region, $f, g : \bar{\Omega} \to \mathbb{C}$ be continuous and holomorphic in Ω. If $|f(z) - g(z)| < |f(z)| + |g(z)|$ on $\partial\Omega$ show that f and g have the same number of zeros in Ω.

[This is a well known generalization of Rouche's Theorem. See e.g., "An Introduction To Classical Complex Analysis" by Robert Burckel, Vol. 1, Theorem 8.18, p.265]

143. Let Ω be a bounded region $f : \bar{\Omega} \to \bar{U}$ be continuous and $f \in H(\Omega)$, f not a constant. If $|f(z)| = 1$ whenever $z \in \partial\Omega$ show that $U = f(\Omega)$.

[This is proved by the same argument as the one used in problem 141) above, with Rouche's Theorem replaced by problem 142)].

Problem 148) below says that any continuous function on $\mathbb{R}$ can be approximated uniformly by an entire function [A result of Carleman]. The next 4 problems are required to solve that problem.

144. Given any continuous fucntion $f : \mathbb{R} \to \mathbb{C}$ there is an entire function g such that g has no zeros and $g(x) > |f(x)|$ $\forall x \in \mathbb{R}$.

Consider a series of the type $a + \sum_{n=1}^{\infty} [\frac{z^2}{n+1}]^{k_n}$. This series converges unifrmly on $\{z : |z| \leq N\}$ if $[\frac{N^2}{n+1}]^{k_n} \leq [\frac{1}{2}]^n$ for $n \geq 2N^2$. This is true if $k_n \geq n$. Thus $h(z) = a + \sum_{n=1}^{\infty} [\frac{z^2}{n+1}]^{k_n}$ defines an entire function provided $k_n \geq n$ $\forall n$. Now, for x real $h(x) > [\frac{x^2}{j+1}]^{k_n} \geq [\frac{j^2}{j+1}]^{k_n} \geq \max\{|f(y)| : j \leq |y| \leq j+1\}$ for $j \leq |x| \leq j+1$ provided k_n is sufficiently large and $a > \max\{|f(y)| : 0 \leq |y| \leq 1\}$. Take $g = e^h$].

145. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then we can write f as $\sum_{n=-\infty}^{\infty} f_n(x - n)$ where each f_n is continuous and $f_n(x) = 0$ if $|x| \geq 1$.

[Let $f_n(x) = \frac{g(x)f(x+n)}{G(x+n)}$ where $G(x) = \sum\limits_{n=-\infty}^{\infty} g(x-n)$ and $g(x) = 1$ for $|x| \leq \frac{1}{2}, g(x) = 0$ for $|x| \geq 1$ and g is piece-wise linear. If $n - \frac{1}{2} \leq x \leq n + \frac{1}{2}$ then $G(x) \geq g(x-n) = 1$].

146. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and $f(x) = 0$ for $|x| \geq 1$. Let $S = \{z : |\mathrm{Re}(z)| > 3 \text{ and } |\mathrm{Re}(z)| > 2\,|\mathrm{Im}(z)|\,\}$. Given $\epsilon > 0$ we can find an entire function g such that $|f(x) - g(x)| < \epsilon\ \forall x \in \mathbb{R}$ and $|g(z)| < \epsilon\ \forall z \in S$.

[Let $f_n(z) = \frac{n}{\sqrt{2\pi}} \int\limits_{-1}^{1} e^{-n^2(z-t)^2} f(t)dt$. It is easily seen that f_n is entire for each n. Also, $f_n \to f$ uniformly on $[-\frac{3}{2}, \frac{3}{2}]$ and $f_n \to 0$ uniformly for $\mathbb{R}\backslash[-\frac{3}{2}, \frac{3}{2}]$. [See problem 149 below]. Hence $f_n \to f$ uniformly on $\mathbb{R}$. If $z \in S$ and $|t| \leq 1$ then $\mathrm{Re}[n^2(z-t)^2] = n^2[(x-t)^2 - y^2]$

$= n^2x^2[1 - \frac{2t}{x} + \frac{t^2}{x^2} - \frac{y^2}{x^2}] \geq n^2x^2[1 - \frac{2}{|x|} - \left|\frac{y}{x}\right|^2] \geq n^2x^2[1 - \frac{2}{3} - (\frac{1}{2})^2] > \frac{3n^2}{4}$.

Hence $|f_n(z)| \leq \frac{n}{\sqrt{2\pi}} \int\limits_{-1}^{1} e^{-\frac{3n^2}{4}} |f(t)|\, dt \leq \frac{4}{3n\sqrt{2\pi}} \int\limits_{-1}^{1} |f(t)|\, dt$]

147. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then there is an entire fucntion g such that $|f(x) - g(x)| < 1\ \forall x \in \mathbb{R}$.

[Write f as $\sum\limits_{n=-\infty}^{\infty} f_n(x-n)$ where each f_n is continuous and $f_n(x) = 0$ if $|x| \geq 1$.For each n there is an entire function g_n such that $|f_n(x) - g_n(x)| < 2^{-2-|n|}$ $\forall x \in \mathbb{R}$ and $|g_n(z)| < 2^{-|n|}\ \forall z \in S$. If $|z| \leq N$ and $|n| > 3N + 3$ then $z - n \in S$ and hence $|g_n(z-n)| < 2^{-|n|}$. This implies that $\sum\limits_{n=-\infty}^{\infty} g_n(x-n)$ converges uniformly on compact subsets of $\mathbb{C}$. Let $g(z) = \sum\limits_{n=-\infty}^{\infty} g_n(x-n)$. g is entire. Also $|f(x) - g(x)| \leq \sum\limits_{n=-\infty}^{\infty} |g_n(x-n) - f_n(x-n)| < \sum\limits_{n=-\infty}^{\infty} 2^{-2-|n|} = \frac{3}{4}$.

148. Let $f : \mathbb{R} \to \mathbb{C}$ and $\eta : \mathbb{R} \to (0, \infty)$ be continuous. Then there is an entire function g such that $|f(x) - g(x)| < \eta(x)\ \forall x \in \mathbb{R}$.

[There is an entire function ϕ with no zeros such that $\phi(x) > \frac{1}{\eta(x)}\ \forall x \in \mathbb{R}$. There is an entire function g such that $|f(x)\phi(x) - g(x)| < 1 \forall x \in \mathbb{R}$].

149. [Used in problem 146) above]

Let $a < b$ and $f : [a, b] \to \mathbb{C}$ be continuous. Let $f_n(x) = \frac{n}{\sqrt{2\pi}} \int\limits_{a}^{b} e^{-n^2(x-t)^2} f(t)dt$.

Then $f_n(x) \to f(x)$ uniformly on $[a + \delta, b - \delta]$ and $f_n(x) \to 0$ uniformly on $\mathbb{R}\backslash[a - \delta, b + \delta]$ for each $\delta > 0$.

[Let f be 0 on $\{b+1, \infty)$ and $(-\infty, a-1]$ and linear in $[a-1, a]$ and $[b, b+1]$.

Note that the second part is trivial. Write $f_n(x)-f(x)$ as $\frac{1}{\sqrt{2\pi}}\int\limits_{\sqrt{n}(a-x)}^{\sqrt{n}(b-x)} e^{-u^2}[f(x+\frac{u}{n})-f(x)]du+\frac{1}{\sqrt{2\pi}}\int\limits_{n(a-x)}^{\sqrt{n}(a-x)} e^{-u^2}[f(x+\frac{u}{n})-f(x)]du$

$+\frac{1}{\sqrt{2\pi}}\int\limits_{\sqrt{n}(b-x)}^{n(b-x)} e^{-u^2}[f(x+\frac{u}{n})-f(x)]du+f(x)[\frac{1}{\sqrt{2\pi}}\int\limits_{n(a-x)}^{n(b-x)} e^{-u^2}du-1]$. $\left|f(x+\frac{u}{n})-f(x)\right| \leq \delta/2$ for $u \in [\sqrt{n}(b-x),]$ and $a \leq x \leq b$ if $n \geq$ some n_δ. We may also choose n_δ such that $\left|\frac{1}{\sqrt{2\pi}}\int\limits_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} e^{-u^2}du-1\right| < \frac{\delta}{8M}$ where M is an upper bound for $|f|$].

150. Show that the family of all analytic maps $f: U \to \{z : \mathrm{Re}(z) > 0\}$ with $|f(0)| \leq 1$ is normal.

[Let $g(z) = \frac{f(z)-f(0)}{f(z)+\bar{f(0)}}$. Then $g(U) \subset U$ and Schwartz Lemma gives $|g(z)| \leq |z|$ which gives $|f(z)| \leq \frac{1+|z|}{1-|z|}$].

151. Let $f \in H(\Omega)$ and f be injective. If $\{z : |z-a| \leq r\} \subset \Omega$ show that

$f^{-1}(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\zeta f'(\zeta)}{f(\zeta)-z}d\zeta \ \forall z \in f(B(a,r))$, where $\gamma(t) = a + re^{2it}, 0 \leq t \leq 1$.

[Let $B(a, r+\epsilon) \subset \Omega$. Then $\frac{1}{2\pi i}\int\limits_{\gamma} \frac{\zeta f'(\zeta)}{f(\zeta)-z}d\zeta$ equals the residue of the integrand at the sole pole $z_0 = f^{-1}(z)$].

152. If $f \in C(\bar{U}) \cap H(U)$ show that $f(z) = i\,\mathrm{Im}(f(0)) + \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,\mathrm{Re}\, f(e^{it})dt$ $\forall z \in U$.

[Just observe that $\mathrm{Re}\int\limits_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,\mathrm{Re}\, f(e^{it})dt = \mathrm{Re}\int\limits_{-\pi}^{\pi} \{\mathrm{Re}\,\frac{e^{it}+z}{e^{it}-z}\} f(e^{it})dt$].

153. If Ω is simply connected show that for any real harmonic function u on Ω, a harmonic conjugate v of u is given by $v(z) = \mathrm{Im}[u(a) + \int\limits_{\gamma}(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})dz]$ where a is a fixed point of Ω and γ is any path from a to z in Ω.

[Since Ω is simply connected u indeed has a harmonic conjugate. Let $g \in H(\Omega)$ with $\mathrm{Re}\, g = u$. We may assume that $g(a) = u(a)$. Now $g(z) = g(a) + \int\limits_{\gamma} g'(\zeta)d\zeta$ and $g'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (from definition of derivative and Cauchy-Riemann equations)].

154. Let Ω be a region and $f, g \in H(\Omega)$. If $|f| + |g|$ attains its maximum on Ω at some point a of Ω then f and g are both constants.

[$|f(a)| + |g(a)| \geq |f(z)| + |g(z)|\ \forall z \in \Omega$. Replace f by $e^{is} f$ and g by $e^{it} g$ where s and t are chosen such that $e^{is} f(a)$ and $e^{it} g(a)$ both belong to $[0, \infty)$. This reduces the proof to the case when $f(a)$ and $g(a)$ both belong to $[0, \infty)$. We now have $f(a)+g(a) \geq |f(z)|+|g(z)| \geq \operatorname{Re} f(z)+\operatorname{Re} g(z) = \operatorname{Re}(f(z)+g(z))$. Maximum Modulus principle applied to $f + g$ shows that $f + g$ is a constant. Now $f(a) + g(a) \geq |f(z)| + |g(z)| \geq \operatorname{Re} f(z) + \operatorname{Re} g(z) = \operatorname{Re}(f(z) + g(z)) = \operatorname{Re}(f(a) + g(a))$ which implies that equality holds throughout. In particular $|f(z)| = \operatorname{Re}(f(z))$ and $|g(z)| = \operatorname{Re}(g(z))\ \forall z$].

155. If f and g are entire functions with $f(n) = g(n)\ \forall n \in \mathbb{N}$ and if $\max\{|f(z)|, |g(z)| \leq e^{c|z|}$ for $|z|$ sufficiently large with $0 < c < 1$ show that $f(z) = g(z)\ \forall z \in \mathbb{C}$. Show that this is false for $c = 1$.

[$c = 1$: take $f(z) = \sin(\pi z), g(z) = \sin(2\pi z)$. Now let $0 < c < 1$. If the conclusion does not hold then $\exists\ a \in (0, 1)$ such that $f(a) \neq g(a)$. Let $\phi(z) = f(z + a) - g(z + a)\ \forall z \in \mathbb{C}$. Then $|\phi(z)| \leq c_1 e^{c|z|}$ for $|z|$ sufficiently large. Consider the disk $B(0, N - a)$ where N is an integer > 1. We apply Jensen's Formula to ϕ on this ball. If $\alpha_1, \alpha_2, ..., \alpha_k$ are the zeros of ϕ in the closure of $B(0, N - a)$ then $|\phi(0)| \prod_{j=1}^{k} \frac{N-a}{|\alpha_j|} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\phi(N-a)e^{it}\right| dt} \leq e^{\log c_1 + c|N-a|}$ for N sufficiently large. Since $\frac{N-a}{|\alpha_j|} \geq 1\ \forall j$ we get $|\phi(0)| \prod_{j=1}^{N} \frac{N-a}{|j-a|} \leq c_1 e^{c|N-a|}$. Also, $|j - a| = j - a \leq j$ so $|\phi(0)| \prod_{j=1}^{N} \frac{N-a}{|j-a|} \leq c_1 e^{c|N-a|}$. This gives

$|\phi(0)|^{1/N} \frac{N-a}{(N!)^{1/N}} \leq c_1^{1/N} e^{c|1-a/N|}$. We conclude that $\limsup \log[\frac{N-a}{(N!)^{1/N}}] \leq c$. However, $\frac{(N!)^{1/N}}{e^{-1}N^{1+1/2N}} \to 1$ as $N \to \infty$ (by Stirling's Formula) and we get $\limsup \log[\frac{N-a}{e^{-1}N^{1+1/2N}}] \leq c$ which says $1 \leq c$, a contradiction].

156. Show that there is a function f in $C(\bar{U}) \cap H(U)$ whose power series does not converge uniformly on $\bar{U}$.

[This is a well known result in the theory of Fourier series. In fact, the power series need not even converge at all points of ∂U. See Theorem 1.14, Chapter VIII Trigonometric Series by A. Zygmund].

157. If $\{f_n\} \subset H(\Omega)$ and $\lim_{n\to\infty} f_n(z) = f(z)$ exists $\forall\ z \in \Omega$ show that there is a dense open subset Ω_0 of Ω such that $f \in H(\Omega_0)$.

[Use Baire Category Theorem]

158. Let $L : H(\Omega) \to H(\Omega)$ be linear and mulitplicative, not identically 0. Show that there is a point $c \in \Omega$ such that $L(f) = f(c)\ \forall\ f \in H(\Omega)$.

[Let $f \in H(\Omega)$ and $c = L(z)$ (where z stands for the identity map). If $c \notin \Omega$ then we get the contradiction $1 = L(1) = L((z-c)\frac{1}{z-c}) = L((z-c))((L(\frac{1}{z-c})) =$

$0((L(\frac{1}{z-c}))$. Thus $c \in \Omega$. Let $g(z) = \frac{f(z)-f(c)}{z-c}$ if $z \neq c$ and $f'(c)$ if $z = c$. Apply L to the identity $f(z) - f(c) = (z-c)g(z)$].

159. Let Ω be a region and $f \in H(\Omega)$ with $f(z) \neq 0 \ \forall z \in \Omega$. If f has a holomorphic square root does it follow that it has a holomorphic logarithm? What if it has a holomorphic $k-th$ root for infinitely many positive integers k?

[$\Omega = U \backslash \{0\}, f(z) = z^2$ is a counter-example to the first part. Suppose now that $k_1 < k_2 < ....$ and $f_j \in H(\Omega)$ with $[f_j(z)]^{k_j} = f(z) \ \forall z \in \Omega, \forall j \geq 1$. Then $\frac{f'}{f} = k_j \frac{f_j'}{f_j}$. If γ is any close path in Ω then $\int\limits_\gamma \frac{f'}{f} = k_j \int\limits_\gamma \frac{f_j'}{f_j}$. If $\gamma_j(t) = f_j(\gamma(t))$, then γ_j is a closed path in $\mathbb{C}$ and $Ind_{\gamma_j}(0) = \frac{1}{2\pi i}\int\limits_\gamma \frac{\gamma_j'}{\gamma_j} = \frac{1}{2\pi i}\int\limits_\gamma \frac{f_j'}{f_j} = \frac{1}{2\pi i k_j}\int\limits_\gamma \frac{f'}{f} \to 0$ as $j \to \infty$. This implies that $\{Ind_{\gamma_j}(0)\}$ vanishes eventually and hence that $\frac{1}{2\pi i k_j}\int\limits_\gamma \frac{f'}{f} = 0$ for j sufficiently large. We have proved that $\int\limits_\gamma \frac{f'}{f} = 0$ for every close path γ in Ω. Hence there exists $h \in H(\Omega)$ such that $\frac{f'}{f} = h'$. Now $(e^{-h}f)' = 0, e^{-h}f$ is a (non-zero) constant and hence f has a holomorphic logarithm.

160. $\lim\limits_{z\to a}\frac{f(z)}{g(z)} = \lim\limits_{z\to a}\frac{f'(z)}{g'(z)}$ if f and g are analytic in some neighbourhood of a, $f(a) = g(a) = 0$ and $g'(a) \neq 0$.

161. If f and g are analytic in some neighbourhood of $a, |f(z)| \to \infty$ and $|g(z)| \to \infty$ as $z \to a$ then $\lim\limits_{z\to a}\frac{f(z)}{g(z)} = \lim\limits_{z\to a}\frac{f'(z)}{g'(z)}$ provided $\lim\limits_{z\to a}\frac{f'(z)}{g'(z)}$ exists.

162. Let f be an entire function such that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f(z) \equiv cz^n$ for some non-negative integer n and some constant c with modulus 1.

[If f has no zeros in U we see that f is a constant. If $\alpha_1, \alpha_2, ..., \alpha_N$ are the zeros of f in $U\backslash\{0\}$ and if 0 is a zero of f of order m (m may be 0) let $B(z) = z^m \frac{z-\alpha_1}{1-\bar{\alpha_1}z}\frac{z-\alpha_2}{1-\bar{\alpha_2}z}...\frac{z-\alpha_N}{1-\bar{\alpha_N}z}$ and $g(z) = f(z)/B(z)$. Then $|f(z)| = 1$ whenever $|z| = 1$ and Maximum Modulus Theorem shows g is a constant. Thus $f(z) = cz^m \frac{z-\alpha_1}{1-\bar{\alpha_1}z}\frac{z-\alpha_2}{1-\bar{\alpha_2}z}...\frac{z-\alpha_N}{1-\bar{\alpha_N}z}$ in U. The two sides must coincide on $\mathbb{C}\backslash\{(\bar{\alpha_j})^{-1} : 1 \leq j \leq N\}$ and we get a contradiction to the fact f is bounded in a neighbourhood of $(\bar{\alpha_j})^{-1}$. This shows that there are no zeros of f other than 0].

164. Let Ω be a region (not necessarily bounded) which is not dense in $\mathbb{C}$, $f \in C(\bar{\Omega}) \cap H(\Omega), |f(z)| \leq M \ \forall z \in \partial\Omega$. Suppose f is bounded on Ω. Then $|f(z)| \leq M \ \forall z \in \Omega$.

[First note that the hypothesis that is bounded on Ω is necessary: $\sin(z)$ is bounded by 1 on the boundary of the upper-half plane but but bounded by 1 in the upper-half plane. Also, the conclusion obviously holds for bounded regions since $|f|$ attains its maximum at some point of $\bar{\Omega}$ in this case.

Since Ω is not dense in $\mathbb{C}$ there is an open ball disjoint from Ω. By translation we may assume that $B(0,\delta) \cap \Omega = \emptyset$. Fix $z_0 \in \Omega$. Let $\epsilon > 0$ and n be a positive integer such that $(|z_0|/\delta)^{1/n} < 1+\epsilon$. Let $R > \max\{|z_0|, \delta(\frac{M_1}{M})^n\}$ where M_1 is a bound for f on Ω. Then $z_0 \in C$ for some component C of $\Omega \cap B(0,R)$. We now apply Maximum Modulus Principle to the function $\frac{f^n(z)}{z}$ on C. Since the $\partial C \subset \partial(\Omega \cap B(0,R)) \subset \partial\Omega \cup \partial B(0,R)$ we see that $\left|\frac{f^n(z)}{z}\right| \leq \max\{\frac{M_1^n}{R}, \frac{M^n}{\delta}\}$ on ∂C since $B(0,\delta) \cap \Omega = \emptyset$. Thus, by Maximum Modulus Principle we get $|f(z_0)| \leq |z_0|^{1/n} \max\{\frac{M_1}{R^{1/n}}, \frac{M}{\delta^{1/n}}\} = |z_0|^{1/n} \frac{M}{\delta^{1/n}}$ in view of the fact that $R > \delta(\frac{M_1}{M})^n$. Finally, since $(|z_0|/\delta)^{1/n} < 1+\epsilon$ we get $|f(z_0)| \leq M(1+\epsilon)$. Since $z_0 \in \Omega$ and $\epsilon > 0$ are arbitrary we are done.

165. In above problem the hypothesis that Ω is not dense can be deleted provided $\Omega \neq \mathbb{C}$.

[Note that the result is obviously false for $\Omega = \mathbb{C}$. Now $\partial\Omega \neq \emptyset$. Let $c \in \partial\Omega$ and consider a small ball $B(c,\rho)$ around c. We may suppose $|f(z)| \leq M+\epsilon$ on $\Omega \cap \partial(B(c,\rho))$. Let $\Omega_1 = \Omega\backslash[B(c,\rho)]^-$. The $|f(z)| \leq M+\epsilon$ on $\partial\Omega_1$ and we can apply above result to Ω_1].

166. If f is an entire function such that $|f(z)| = 1$ whenever $|z| = 1$ show that $f(z) = cz^n$ for some $n \geq 0$ and $c \in \mathbb{C}$ with $|c| = 1$.

[Let n be the order of zero 0f f at 0 and let $\alpha_1, \alpha_2, , , , , \alpha_k$ be the remaining zeros of f (if any) in U. Let $g(z) = f(z)/\{z^n \prod_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha_j}z}\}$. Then $|g(z)| = 1$ whenever $|z| = 1$ and g has no zero in $\bar{U}$. Maximum Modulus Principle applied to g and $\frac{1}{g}$ shows that g is a constant. We now have an equation of the type $f(z) = cz^n \prod_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha_j}z}$ on $\mathbb{C}\backslash\{(\bar{\alpha_j})^{-1} : 1 \leq j \leq k\}$ which contradicts the fact that f is bounded near $(\bar{\alpha_j})^{-1}$. This says that $\alpha_1, \alpha_2, , , , , \alpha_k$ 'don't exist' and $f(z) = cz^n$].

167. Let $f \in H(\Omega\backslash\{a, a_1, a_2, ...\})$ where Ω is a region, $a_n \to a$, $a_n's$ are distinct points of Ω and $a \in \Omega$. If f has a pole at each a_n show that $f(B(a,\epsilon)\backslash\{a, a_1, a_2, ...\})$ is dense in $\mathbb{C}$ for every $\epsilon > 0$.

[Note that a is not an *isolated* singularity of f and hence the usual theorems on classifiation of singularities do not apply directly. However, a standard argument applies: suppose $f(B(a,\epsilon)\backslash\{a, a_1, a_2, ...\})$ is not dense in $\mathbb{C}$ for some $\epsilon > 0$. Let $B(w_0,\rho)$ be an open ball disjoint from $f(B(a,\epsilon)\backslash\{a, a_1, a_2, ...\})$. Let $g(z) = \frac{1}{f(z)-w_0}$ on $B(a,\epsilon)\backslash\{a, a_1, a_2, ...\}$. First note that $|g(z)| \leq \frac{1}{\rho}$ so g has a removable singularity at each of the points $a_1, a_2, ...$ After removing these singularities we see that $g \in H(B(a,\epsilon)\backslash\{a\})$ and we can then remove the singularity at a also!. This gives us g in $H(B(a,\epsilon))$ and $g(a_n) = 0$ for all n such that $a_n \in B(a,\epsilon)$ because f has a pole at a_n. But this contradicts the fact that zeros of g are isolated].

168. If f is a rational function such that $|f(z)| = 1$ whenever $|z| = 1$ show that $f(z) = cz^n \{\prod_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}\} / \{\prod_{j=1}^{m} \frac{z-b_j}{1-\bar{b}_j z}\}$ for some $n \in \mathbb{Z}$ and $a_1, a_2, ..., a_N, b_1, b_2, .., b_m \in \mathbb{C} \backslash T, c \in \mathbb{C}$ with $|c| = 1$.

[Assume first that f does not vanish at 0 and that it does not have a pole at 0. Let and let $\alpha_1, \alpha_2, , , , , \alpha_k$ be the zeros of f (if any) and $b_1, b_2, .., b_m$ the poles of f in U. Let $g(z) = f(z) \prod_{j=1}^{m} \frac{z-b_j}{1-\bar{b}_j z} / \{\prod_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}\}$. Then $|g(z)| = 1$ whenever $|z| = 1$ and g has no zero in $\bar{U}$. Maximum Modulus Principle applied to g and $\frac{1}{g}$ shows that g is a constant. We now have an equation of the type $f(z) = c\{\prod_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}\}/ \{\prod_{j=1}^{m} \frac{z-b_j}{1-\bar{b}_j z}\}$on $\mathbb{C} \backslash \{(\bar{\alpha}_j)^{-1} : 1 \leq j \leq k\} \cup \{(\bar{b}_j)^{-1} : 1 \leq j \leq m\}$. Zero or pole of f at 0 is easy to handle].

169. Let f and g be holomorphic on U with g one-to-one and $f(0) = g(0) = 0$, If $f(U) \subset g(U)$ show that $f(B(0,r)) \subset g(B(0,r))$ for any $r \in (0,1]$.

Let $\Omega = g(U)$. If g is a constant then so is f and there is nothing to prove. Otherwise, Ω is a region. $g^{-1} : \Omega \to U$ is hilomorphic and so is $g^{-1} \circ f : U \to U$. Further, $(g^{-1} \circ f)(0) = 0$. By Schwartz Lemma $\left|(g^{-1} \circ f)(z)\right| \leq |z| \; \forall z \in U$. If $|z| < r$ then $f(z) \in f(U) \subset g(U)$ so we can write $f(z)$ as $g(\zeta)$ for some $\zeta \in U$. Now $|\zeta| = \left|(g^{-1} \circ f)(z)\right| \leq |z| < r$].

170. All injective holomorphic maps from U onto itself are of the type $c\frac{z-a}{1-\bar{a}z}$ with $|a| < 1, |c| = 1$. Find all $m - to - 1$ holomorphic maps of U onto itself for a given positive integer m.

They are all of the type $f(z) = c \prod_{j=1}^{m} \frac{z-a_j}{1-\bar{a}_j z}$ with $\{a_1, a_2, .., a_m\} \subset U$ ($a_j' s$ not necessarily distinct) and $|c| = 1$. First note that if f is of this type and $w \in U$ then the equation $f(z) = w$ is a polynomail equation of degree m. It has no root outside U because $|z| \geq 1$ implies $|z - a_j| \geq \left|1 - \bar{a}_j z\right|$. Hence f is indeed a $m - to - 1$ holomorphic map of U onto itself. Now let f be any $m - to - 1$ holomorphic map of U onto itself. We claim that $|f(z)| \to 1$ as $|z| \to \infty$. Once this claim is established we can apply Maximim Modulus principle to f/g and g/f where $g(z) = \prod_{j=1}^{m} \frac{z-a_j}{1-\bar{a}_j z}$, $a_j' s$ being the zeros of f counted according to multiplicities to complete the proof. Suppose the claim is false. Then there exists a sequence $\{z_n\}$ of distinct points in U and $\delta > 0$ such that $|z_n| \to 1$ and $|f(z_n)| \leq 1 - \delta \; \forall n$. We may assume that $f(z_n) \to w$ (say). Since $|w| \leq 1 - \delta$, we see that $w \in U$. Consider the equation $f(z) = w$. This equation has exactly m solutions by hypothesis. Let $c_1, c_2, ..., c_k$ be the distinct points in $f^{-1}\{w\}$ and let $m_1, m_2, ..., m_k$ be the multiplicities of zeros of $f(z) - w$ at $c_1, c_2, ..., c_k$ respectively. By Theorem 10.30 of Rudin's Real And Complex Analysis there are neighbourhoods $V_1, V_2, ..., V_k$ of $c_1, c_2, ..., c_k$ respectively and one-to-one holomorphic functions $\phi_1, \phi_2, ..., \phi_k$ on these neighbourhoods and integers $n_j, 1 \leq j \leq k$ such that $f(z) = w + [\phi(z)]^{n_j}$ on V_j and such that ϕ maps V_j onto an open abll centered at 0. We may assume that $V_1, V_2, ..., V_k$ are disjoint. Also note that in the Theorem referred to above n_j is the order of zero of $f(z) - w$ at c_j. In other words, $n_j = m_j \; \forall j$. We now get a contradiction by showing that if n is large enough then the equation $f(z) = f(z_n)$ has m solutions in V where $V = V_1 \cup V_2 \cup ... \cup V_k$. Since $z = z_n$ is another solution we get a contradiction. Indeed, $\bar{V}$ is a compact subset of U so $z_n \notin V$ if n is large enough. Let $R = \sup\{|z| : z \in \bar{V}\}$ and choose n such that $|z_n| > R, f(z_n) \neq w$ and $f(z_n) \in f(V_j)$ for each j. [Zeros of $f(z) - w$ are precisely $c_1, c_2, ..., c_k$ and z_n is not one of these points for large n!. Note that $w = f(c_j) \in f(V_j)$ and $f(z_n) \to w$ so $f(z_n) \in f(V_j)$ if n is large enough]. The equation $f(z) = f(z_n)$ has exactly m_j solutions in V_j for each j [see the remark after Theorem 10.30 in Rudin's book]. Thus the number of solutions of $f(z) = f(z_n)$ in V is $m_1 + m_2 + ... + m_k = m$ and the proof is complete.

171. Let Ω_1 and Ω_2 be bounded regions. Let $f : \Omega_1 \to \Omega_2$ be a holomorphic map such that there is no sequence $\{z_n\}$ in Ω_1 converging to a point in $\partial\Omega_1$ such that $\{f(z_n)\}$ converges to a point in Ω_2. Then there is a positive integer m such that f is $m - to - 1$ on Ω_1.

Proof: If $w \in \Omega_2$ then $f - w$ can only have a finite number of zeros in Ω_1 : if it had distinct zeros $z_1, z_2, ...$ then some subsequence $\{z_{n_k}\}$ converges to some $z \in \bar{\Omega}_1$. If $z \in \partial\Omega_1$ then we have a contradiction to the hypotheisis since $f(z_{n_k}) = w \; \forall k$. Thus $z \in \Omega_1$ which forces $f - w$ to be a constant and this contradicts the hypothesis again. Let $n(w)$ be the number of zeros of $f - w$ on Ω_1 for each $w \in \Omega_2$. If we show that n is continuous on Ω_2 we can conclude that it is a constant and this completes the proof. Show that $\{w \in \Omega_2 : n(w) = k$ is open for each k.

172. The condition in Problem 169) above that there is no sequence $\{z_n\}$ in Ω_1 converging to a point in $\partial\Omega_1$ such that $\{f(z_n)\}$ converges to a point in Ω_2 is equivalent to the fact that $f^{-1}(K)$ is compact whenever K is a compact subset of Ω_2.

Suppose $f^{-1}(K)$ is compact whenever K is a compact subset of Ω_2. Let $\{z_n\}$ be a sequenec in Ω_1 converging to a point z in $\partial\Omega_1$. If $f(z_n) \to w \in \Omega_2$ then $K = \{w, f(z_1, f(z_2), ...\}$ is a compact subset of Ω_2 and $f^{-1}(K)$ contains the sequence $\{z_n\}$ with no convergent subsequence in Ω_1. Conversely let the hypothesis of Problem 169 hold and let K be compact in Ω_2. No subsequence of a sequence $\{z_n\}$ in $f^{-1}(K)$ can have a limit point on $\partial\Omega_1$ whcih means $f^{-1}(K)$ is a closed (hence compact) subset of Ω_1.

173. Prove that the analogue of Problem 169) when $\Omega_1 = \Omega_2 = \mathbb{C}$ and $\partial\Omega_1$ is interpreted as (the boundary in $\mathbb{C}_\infty$ i.e.) $\{\infty\}$ holds. Give an example to show that Problem 169) fails for a general unbouded region Ω_1.

First part follows from the fact if f is entire and $|f(z)| \to \infty$ as $|z| \to \infty$ then f is a polynomial. For the second part take $\Omega_1 = \Omega_2 = \{z : \mathrm{Im}(z) > 0\}$ and $f(z) = \sin(z)$.

174. Let $f \in H(U), \theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$ and $\left|f(re^{i\theta_1})\right| = |f(0)| = \left|f(re^{i\theta_2})\right|$ for all $r \in (0,1)$. Show that f is a constant if $\frac{\theta_1-\theta_2}{2\pi}$ is irrational.

Let $g(z) = \frac{f(\delta e^{i\theta_2}z)}{|f(0)|}$. Note that of $f(0) = 0$ then there is nothing to prove. Choose $\delta \in (0,1)$ so small that g has no zeros in U. Since U is simply connected we can write g as e^h for some $h \in H(U)$. Now $\left|g(\frac{r}{\delta}z)\right| = \left|\frac{f(re^{i\theta_2}z)}{|f(0)|}\right| = 1\ \forall z \in U$. Also, $\left|g(\frac{r}{\delta}e^{i(\theta_1-\theta_2)}z)\right| = 1\ \forall z \in U$. These two equations give $e^{\mathrm{Re}\, h([\frac{r}{\delta}z])} = 1$ and $e^{\mathrm{Re}\, h([\frac{r}{\delta}e^{i(\theta_1-\theta_2)}z])} = 1$. That is to say $\mathrm{Re}\, h([\frac{r}{\delta}z]) = 0 = \mathrm{Re}\, h([\frac{r}{\delta}e^{i(\theta_1-\theta_2)}z])$ $\forall r \in (0,1)$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of h. From the first equation here we get $\mathrm{Re}(\sum_{n=0}^{\infty} a_n z^n) = 0$ whenever $z \in (0, \frac{\delta}{r})$. In other words $\mathrm{Im}(\sum_{n=0}^{\infty} \frac{a_n}{i} z^n) = 0$ whenever $z \in (0, \frac{\delta}{r})$. This implies that $\frac{a_n}{i} \in \mathbb{R}\ \forall n$. The second realtion above yields the fact that $\mathrm{Im}(\sum_{n=0}^{\infty} \frac{a_n}{i} e^{i(\theta_1-\theta_2)n} z^n) = 0$ whenever $z \in (0, \frac{\delta}{r})$. This gives $\frac{a_n e^{i(\theta_1-\theta_2)n}}{i} \in \mathbb{R}\ \forall n$. Since not all the coefficients a_n are 0 we see that $e^{i(\theta_1-\theta_2)n} \in \mathbb{R}$ for some n. So $sin[(\theta_1-\theta_2)n] = 0\ \forall n$. This imples that $(\theta_1 - \theta_2)$ is a rational multiple of 2π.

175. Suppose $\theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$ and $f \in H(U)$, $\left|f(re^{i\theta_1})\right| = |f(0)| = \left|f(re^{i\theta_2})\right|$ for all $r \in (0,1)$ implies that f is a constant. Show that $\frac{\theta_1-\theta_2}{2\pi}$ is irrational.

[If $\frac{\theta_1-\theta_2}{2\pi}$ is a rational number $\frac{p}{q}$ $(p, q \in \mathbb{Z})$ let $f(z) = e^{i\sin([ze^{-i\theta_1}]^q)}$]

176. A second order differential equation: let Ω be a convex region and $g \in H(\Omega)$. Show that any holomorphic function f satifying the differential equation $f'' + f = g$ in Ω can be expressed as $h(z)\sin(z) + \phi(z)\cos(z)$ for suitable $h, \phi \in H(\Omega)$.

Let $\xi(z) = f(z) - h(z)\sin(z) - \phi(z)\cos(z)$ where $h(z) = c_1 + \int\limits_{[a,z]} g(\zeta)\cos(\zeta)d\zeta$ and $\phi(z) = c_2 - \int\limits_{[a,z]} g(\zeta)\sin(\zeta)d\zeta$ and c_1, c_2 are chosen such that $f(a) = h(a)\sin(a) + \phi(a)\cos(a), f'(a) = h(a)\cos(a) - \phi(a)\sin(a)$. [I fact, $c_1 = f(a)\sin(a) + f'(a)\cos(a), c_2 = f(a)\cos(a) - f'(a)\sin(a)$]. Straightforward computation show that $\xi'' + \xi = 0$ and $\xi(a) = 0, \xi'(a) = 0$. The coefficients in the power seires expansion of ξ around a are all zero and hence $\xi \equiv 0$.

177. Show that $U\backslash\{0\}$ is not conformally equivalent to $\{z : 1 < |z| < 2\}$.

If possible let $\phi : U\backslash\{0\} \to \{z : 1 < |z| < 2\}$ be a bijective (bi-) holomorphic map. Since ϕ is bounded it extends to a holomorphic function g on U and its range is contained in $\{z : 1 \leq |z| \leq 2\}$. Since g has no zeros the Maximum Modulus Principle applied to g and $\frac{1}{g}$ shows that $g(0) \in \{z : 1 < |z| < 2\}$. Let $c = \phi^{-1}(g(0))$. Then $0 = \lim\limits_n \frac{1}{n} = \lim\limits_n \phi^{-1}(\phi(\frac{1}{n})) = \phi^{-1}(g(0))$ because $\phi(\frac{1}{n}) = g(\frac{1}{n}) \to g(0)$ and ϕ^{-1} is continuous on $\{z : 1 < |z| < 2\}$. This contradicts the fact that $\phi^{-1}(\{z : 1 < |z| < 2\}) \subset U\backslash\{0\}$.

178. Let f be continuous on $\{z : |z| \leq R\}$ and holomorphic on $B(0,R)$. Let $M(r) = \sup\{|f(z)| : |z| = r\}$ and $\phi(r) = \sup\{\operatorname{Re} f(z) : |z| = r\}$ for $0 \leq r \leq R$. Show that $\phi(r) \leq \frac{R-r}{R+r}\operatorname{Re} f(0) + \frac{2r}{R+r}\phi(r)$ and $M(r) \leq \frac{R-r}{R+r}|f(0)| + \frac{2r}{R+r}\phi(r)$ for $0 \leq r \leq R$.

We may assume that $\phi(R) > \operatorname{Re} f(0)$ because $\phi(R) \geq \operatorname{Re} f(0)$ and equality holds only when f is a constant (in which case the desired inequalities hold with equality). Let $g(z) = f(0) - \{\phi(R) - \operatorname{Re} f(0)\}\frac{2z}{1-z}$. This is a conformal equivalence from U onto $\{z : \operatorname{Re}(z) < \phi(R)\}$. [Use the facts that $\frac{1+z}{1-z}$ is a conformal equivalence from U onto $\{\operatorname{Re}(z) > 0\}$ and $\frac{2z}{1-z} = \frac{1+z}{1-z} - 1$ is a conformal equivalence from U onto $\{\operatorname{Re}(z) > -1\}$]. Now $f(B(0,R)) \subset \{z : \operatorname{Re}(z) < \phi(R)\}$. Thus $f(B(0,R)) \subset g(U)$. Writing $f_R(z) = f(Rz)$ we get $f_R(U) \subset g(U)$. We now use Problem 167) above to conclude that $f_R(rU) \subset g(rU)$ for $0 \leq r \leq 1$. In other words, $|z| \leq r \Rightarrow f(z) \in g(B(0,\frac{r}{R}), 0 \leq r \leq R$. Hence $M(r) \leq \sup\{|\zeta| : \zeta \in g(B(0,\frac{r}{R}))\} = \sup\{\left|f(0) - \{\phi(R) - \operatorname{Re} f(0)\}\frac{2z}{1-z}\right| : |z| \leq \frac{r}{R}\} \leq |f(0)| + \{\phi(R) - \operatorname{Re} f(0)\}\frac{2r/R}{1-r/R}$ which gives $M(r) \leq \phi(R)\frac{2r}{R-r} + |f(0)|\frac{R-r}{R+r}$. To prove the inequality $\phi(r) \leq \frac{R-r}{R+r}\operatorname{Re} f(0) + \frac{2r}{R+r}\phi(r)$ we write $u(z) = \phi(R) - f(z)$. By Harnack's Inequality we have $\frac{R-|z|}{R+|z|}\operatorname{Re} u(0) \leq \operatorname{Re} u(z)$ for $|z| \leq R$. This completes the proof.

179. If f is an entire function such that $\operatorname{Re} f(z) \leq B|z|^n$ for $|z| \geq R$ then f is a polynomial of degree at most n.

We have $\phi(r) \leq Br^n$ for $r \geq R$ in the notations of Problem 176). By that problem we get $M(r) \leq \frac{2r-r}{2r+r}|f(0)| + \frac{2r}{2r+r}\phi(r) \leq \frac{1}{3}|f(0)| + \frac{2}{3}Br^n$ and $|f(z)| \leq \frac{1}{3}|f(0)| + \frac{2}{3}B|z|^n$ if $|z| \geq R$. This implies that f is a polynomial of degree at most n.

180. Let Ω be a region and A be a subset of Ω with no limit points in Ω. Show that $\Omega\backslash A$ is a region.

Since A has no limit points it is closed in Ω, so $\Omega\backslash A$ is open in $\mathbb{C}$. Now fix z_0 in $\Omega\backslash A$ and let $S = \{z \in \Omega : \exists\gamma : [0,1] \to \Omega$ with $\gamma(t) \in \Omega\backslash A$ for $t < 1, \gamma(0) = z_0$, $\gamma(1) = z$ and γ is continuous$\}$. It is easy to see that S is closed in Ω. To show that it is open in Ω pick $z \in S\backslash\{z_0\}$ and choose a ball $B(z,\delta)$ such that $B(z,\delta)\backslash\{z\} \subset \Omega\backslash A$. Pick any $\zeta \in B(z,\delta)$. Let $\gamma : [0,1] \to \Omega$ be a map with $\gamma(t) \in \Omega\backslash A$ for $t < 1, \gamma(0) = z_0$, $\gamma(1) = z$ and γ continuous. If $z \notin A$ we can combine γ with the segment $[z,\zeta]$ to conclude that $\zeta \in S$. If $z \in A$ then there exists $t_0 \in [0,1)$ such that $\gamma(t_0) \in B(z,\delta)\backslash\{z\}$. [If this is not true then there would be a discontinuity of γ at $\inf\{t : \gamma(t) = z\}$]. Combine γ restricted to $[0,t_0]$ with $[\gamma(t_0),\zeta]$ to see that $\zeta \in S$ if $z \notin [\gamma(t_0),\zeta]$. If $z \in [\gamma(t_0),\zeta]$ let $z_1 = z + \epsilon e^{i(\frac{\pi}{2}+\theta)}$ where θ is the argument of $\zeta - z$ and $0 < \epsilon < \delta$. Note that $z_1 \in B(z,\delta)\backslash\{z\}$ and that the segments $[\gamma(t_0),z_1],[z_1,\zeta]$ are both contained in the convex set $B(z,\delta)$, as well as in $B(z,\delta)\backslash\{z\} \subset (\Omega\backslash A)$. [If z is on on eof these segments it is easy to see that the ratio of $\zeta - z$ to $z_1 - z$ is real. However, the definition of z_1 show that these two are orthogonal (i.e. Re[$(\zeta - z)$ $(z_1 - z)^-] = 0$). We may now combine γ restricted to $[0,t_0]$ with the segments $[\gamma(t_0),z_1]$ and $[z_1,\zeta]$ to see that $\zeta \in S$. Finally we prove that z_0 is an interior point of S : any point of a ball $B(z_0,\delta)$ that is contained in $\Omega\backslash A$ can be joined by a continuous arc to z_0 by a single line segment.

181. Show that $\mathbb{C}\backslash(Q \times Q)$ is connected.

We prove a more general result:
Let $A \subset \mathbb{R}^n$ be countable. Then $\mathbb{R}^n\backslash A$ is path connected.

Let $x_0 \in A$. Consider the sets $\{x_0 + tx : t > 0\}$ where $\|x\| = 1$. These sets are disjoint and hence only countable many of them can intersect A. Similarly $\{y : \|x\| = r\}$ can intersect A for at most countably many positive numbers r. Removing these we get rays and circles disjoint from A and any two points of $\mathbb{R}^n\backslash A$ can be joined by a path consisting of two line segments and an arc of a circle.

182. Prove the formula $\int\limits_{-\infty}^{\infty} e^{itx}e^{-x^2/2}dx = \sqrt{2\pi}e^{-t^2/2}(t \in \mathbb{R})$ in four different ways.

Contour integration: assume that $t > 0$ and integrate $e^{itx}e^{-x^2/2}$ over the rectangle with vertices $-R, R, R + it, -R + it$.

Power series method: justify $\int\limits_{-\infty}^{\infty}\sum\limits_{n=0}^{\infty}\frac{i^n t^n x^n}{n!}e^{-x^2/2}dx=\sum\limits_{n=0}^{\infty}\frac{i^n t^n}{n!}\int\limits_{-\infty}^{\infty}x^n e^{-x^2/2}dx$ and compute the integrals $\int\limits_{-\infty}^{\infty}x^n e^{-x^2/2}dx$ for each n explicitly.

Using the fact that zeros are isolated: let $\phi(z)=\int\limits_{-\infty}^{\infty}e^{izx}e^{-x^2/2}dx$, show that ϕ is entire and compute $\phi(it)$ for real t. This gives the desired identity with t changed to it and that is good enough!

Differential equation method: prove that $\phi'(t)=-t\phi(t)$. This and the fact that $\phi(0)=\sqrt{2\pi}$ give $\phi(t)=\sqrt{2\pi}e^{-t^2/2}$.

[$\int\limits_0^1(\int\limits_0^s e^{iuz}du)ds=\frac{e^{iz}-1--iz}{i^2z^2}$ and hence $\left|e^{iz}-1-iz\right|\leq|z|^2\left|\int\limits_0^1(\int\limits_0^s e^{iuz}du)ds\right|\leq|z|^2e^{|z|}/2$. This inequality is useful in the lsat two methods].

183. Prove that $|e^z-1-z|\leq\frac{|z|^2}{2}e^{|z|}$ $\forall z\in\mathbb{C}$ and $|e^z-1-z|\leq\frac{|z|^2}{2}$ if $\mathrm{Re}(z)=0$. Also show that $\left|e^z-1-z-z^2/2!-...-z^n/n!\right|\leq\frac{|z|^{n+1}}{(n+1)!}e^{|z|}$ $\forall z\in\mathbb{C}$.

$\int\limits_0^1(\int\limits_0^s e^{uz}du)ds=\int\limits_0^1\frac{e^{sz}-1}{z}ds=\frac{1}{z}(\frac{e^z-1}{z}-1)=\frac{e^z-1-z}{z^2}$. Hence $|e^z-1-z|\leq|z|^2\left|\int\limits_0^1(\int\limits_0^s e^{uz}du)ds\right|\leq|z|^2\int\limits_0^1(\int\limits_0^s e^{u|z|}du)ds\leq|z|^2e^{|z|}\int\limits_0^1(\int\limits_0^s du)ds$ and this gives the first inequality. If $\mathrm{Re}(z)=0$ the $|e^{uz}|=1$ and we can replace $e^{|z|}$ by 1 in above inequalities. For the last part use induction and the fact that $\int\limits_0^1[e^{tz}-1-tz-t^2z^2/2!-...-t^nz^n/n!]dt=\frac{1}{z}[e^z-1-z-z^2/2!-z^3/3!-...-z^{n+1}/(n+1)!]$.

184. Let f be a non-constant entire function. Show without using Picard's Theorem that $\liminf\limits_{|z|\to\infty}|f(z)|\in\{0,\infty\}$.

If $g(z)=f(\frac{1}{z})$ has an essential singularity at 0 then $\{g(z):0<|z|<1\}$ is dense in $\mathbb{C}$ and this implies $\liminf\limits_{|z|\to\infty}|f(z)|=0$. If it has a pole or an essential singularity then Problem 24) above shows f is a polynomial.

185. Let Ω be open and $f\in H(\Omega)$ be one-to-one. Let γ be any closed path in Ω and $\Omega_1=\{z\in\Omega\backslash\gamma^*:Ind_\gamma(z)\neq0\}$. Show that $f^{-1}(w)Ind_\gamma(f^{-1}(w))=\frac{1}{2\pi i}\int\limits_\gamma\frac{zf'(z)}{f(z)-w}dz$ $\forall w\in f(\Omega_1)$.

This follows imeditely from Residue Theorem. The integrand has a simple pole at $f^{-1}(w)$ with residue $f^{-1}(w)$! Note that if

$Ind_{\gamma}(a) = 0$ or 1 for any $a \in \mathbb{C}\backslash\gamma^*$ then $f^{-1}(w) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{zf'(z)}{f(z)-w}dz$ $\forall w \in f(\Omega_1)$.

186. Let $f \in H(U\backslash\{0\})$ and assume that f has an essential singularity at 0. Let $f_n(z) = f(\frac{z}{2^n}), n \geq 1, z \in U\backslash\{0\}$. Show that $\{f_n\}$ is not normal in $H(U\backslash\{0\})$.

We can find $\{c_n\}$ such that $|c_{n+1}| < |c_n|, |c_n| \to 0, |c_1| < \frac{1}{4}$ and $\lim\limits_{n\to\infty} f(c_n) = 0$. Let $n_k \in \mathbb{N}$ with $n_k < -\frac{\log|c_k|}{\log 2} \leq n_k + 1, k = 1, 2, ...$ Clearly $n_k \leq n_{k+1}$ and $n_k \to \infty$ as $k \to \infty$. Let $z_k = 2^{n_k}c_k$. Then $\frac{1}{4} \leq |z_k| < \frac{1}{2}$. Note that $f_{n_k}(z_k) = f(2^{-n_k}z_k) = f(c_k) \to 0$. If possible, let $\{f_n\}$ be normal. Let $f_{n_{k_j}} \stackrel{ucc}{\to} h$. Let M be an upper bound for $\{f_{n_{k_j}}\}$ on $\{z : \frac{1}{4} \leq |z_k| \leq \frac{1}{2}\}$. If $\zeta \in B(0, \frac{1}{2^{n_{k_1}+1}})\backslash\{0\}$ then there exists j such that $\frac{1}{2^{n_{k_j}+2}} \leq |\zeta| < \frac{1}{2^{n_{k_j}+1}}$. Since $2^{n_{k_j}}\zeta \in \{z : \frac{1}{4} \leq |z_k| \leq \frac{1}{2}\}$ we get $\left|f_{n_{k_j}}(2^{n_{k_j}}\zeta)\right| \leq M$ which means $|f(\zeta)| \leq M$. Thus, f is bounded in a neighbourhood of 0 contradicting the hypothesis that f has an essential singularity at 0.

187. Let Ω be an open set in $\mathbb{C}$ such that $\mathbb{C}_\infty\backslash\Omega$ is connected. Let γ be closed path in Ω. Show that $Ind_\gamma(a) = 0$ $\forall a \in \mathbb{C}\backslash\Omega$.

Remark: some books give a lengthy proof. Here is a simple proof: let $F(\infty) = 0$ and $F(z) = Ind_\gamma(a)$ for $a \in \mathbb{C}\backslash\gamma^*$. Then F is an integer valued continuous function on $\mathbb{C}_\infty\backslash\gamma^*$. Continuity at ∞ follows from the fact that $\left|\int\limits_\gamma \frac{1}{z-a}dz\right| \leq \frac{1}{|a|-M}L(\gamma)$ where $L(\gamma)$ is the length of γ and $M = \sup\{|z| : z \in \gamma^*\}$. If $\mathbb{C}_\infty\backslash\Omega$ is connected then F is a constant on this set. Since it is 0 at ∞ it is 0 on $\mathbb{C}_\infty\backslash\Omega$ as well.

188. If f is an entire function which is not a transaltion show that $f \circ f$ has a fixed point.

Let $g(z) = \frac{f(f(z))-z}{f(z)-z}$. If $f \circ f$ has no fixed point then f also cannot have a fixed point ans g is an entire function with no zeros. Also $g(z) = 1 \Rightarrow f(f(z)) = f(z)$ which implies that $f(z)$ is a fixed point of f and this is a contradiction. Hence, by Picard's Theorem, g is a constant different from both 0 and 1. Let $f(f(z)) - z = c[f(z) - z]$. From this we have to show that f is a translation. We have $f'(f(z))f'(z) - 1 = c[f'(z) - 1]$ which can be written as $f'(z)[f'(f(z)) - c] = 1 - c$. $(*)$ If $f'(f(z)) = 0$ we can replace z by $f(z)$ in $(*)$ to get $c = 1$, a contradiction. Hence, neither $f'(z)$ nor $f'(f(z))$ can be 0 for any

z. Thus, $f' \circ f$ is an entire function whose range misses 0 and c. Using Picard's Theorem again we conclude that $f' \circ f$ is a constant. By $(*)$ f' is also a constant and hence $f(z) = az+b$ for some constants a and b. But $(f \circ f)(z) = a^2z+ab+b$ has a fixed point unless $a^2 = 1$, i.e. unless $a = 1$.

189. Show that there is a sequence of polynomials $\{p_n\}$ such that $\lim\limits_{n\to\infty} p_n(z) =$
$$\begin{cases} 0 \text{ if } \operatorname{Im}(z) = 0 \\ 1 \text{ if } \operatorname{Im}(z) > 0 \\ -1 \text{ if } \operatorname{Im}(z) > 0 \end{cases}$$
Fix n Let $K = \{z : -n \leq \operatorname{Re}(z) \leq n, \frac{1}{n} \leq \operatorname{Im} z \leq n\} \cup \{z : -n \leq \operatorname{Re}(z) \leq n, -n \leq \operatorname{Im} z \leq -\frac{1}{n}\} \cup \{z : -n \leq \operatorname{Re}(z) \leq n, \operatorname{Im}(z) = 0\}$. This is a compact subset of the open set $\Omega = \{z : -n-1 < \operatorname{Re}(z) < n+1, \frac{1}{2n} < \operatorname{Im} z < n+1\} \cup \{z : -n-1 < \operatorname{Re}(z) < n+1, -n-1 < \operatorname{Im} z < -\frac{1}{2n}\} \cup \{z : -n-1 < \operatorname{Re}(z) < n+1, |\operatorname{Im}(z)| < \frac{1}{3n}\}$.

Let f be 1 on $\{z : -n-1 < \operatorname{Re}(z) < n+1, \frac{1}{2n} < \operatorname{Im} z < n+1\}, -1$ on $\{z : -n-1 < \operatorname{Re}(z) < n+1, -n-1 < \operatorname{Im} z < -\frac{1}{2n}\}$ and 0 elsewhere. Since $\mathbb{C}_\infty \backslash K$ is connected and f is holomorphic on Ω we can find a polynomial p_n such that $|f(z) - p_n(z)| < \frac{1}{n}$ on K.

190. Show that there is a sequence of polynomials $\{p_n\}$ such that $\lim\limits_{n\to\infty} p_n(z) = 0$ $\forall z \in \mathbb{C}$ but the convergence is not uniform on at least one compact set.

If $\{p_n\}$ is the sequence in Problem 189) then $\{p_n^2 - p_n^4\}$ is a sequence of polynomials converging to 0 pointwise. If this sequence converges uniformly on compact subsets of $\mathbb{C}$ then it is uniformly bounded on each compact set. Since $\left|p_n^2 - p_n^4\right| \geq |p_n|^2 \left[|p_n|^2 - 1\right]$, the sequence $\{p_n\}$ is also bounded uniformly on compacts. It is therefore a normal sequence and there must be a subsequence that converges ucc to an entire function, a contradiction.

191. If A is bounded in $\mathbb{C}$ then $\mathbb{C}_\infty \backslash A$ is connected if and only if $\mathbb{C} \backslash A$ is connected. If A is unbounded and $\mathbb{C} \backslash A$ is connected does it follow that $\mathbb{C}_\infty \backslash A$ is connected? If $\mathbb{C}_\infty \backslash A$ is connected does it follow that $\mathbb{C} \backslash A$ is connected?

Let $|z| \leq R$ for all $z \in A$. Let $V_R = \{z : |z| > R\}$. If $\mathbb{C} \backslash A$ is connected and $\mathbb{C}_\infty \backslash A = E \cup F$ with E and F disjoint open subsets of $\mathbb{C}_\infty \backslash A$ let $\infty \in E$.

Then $\mathbb{C} \backslash A = (E \backslash \{\infty\}) \cup F$ which implies that either $F = \emptyset$ or $E = \{\infty\}$. Hence $\{\infty\} = V \cap (\mathbb{C}_\infty \backslash A)$ for some open set V in $\mathbb{C}_\infty$. But then all complex numbers z with $|z|$ sufficiently large are in $V \cap (\mathbb{C}_\infty \backslash A) = \{\infty\}$ which is a contradiction. If $\mathbb{C}_\infty \backslash A$ is connected and $\mathbb{C} \backslash A = E \cup F$ with E and F disjoint open subsets of $\mathbb{C} \backslash A$ then $V_R = (V_R \cap E) \cup (V_R \cap F)$ and the connectedness of V_R shows that

either $V_R \cap E = \emptyset$ or $V_R \cap F = \emptyset$. In the first case $V_R \subset F$ which implies that $F \cup \{\infty\}$ is open in $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty \backslash A = E \cup (F \cup \{\infty\})$ we get $E = \emptyset$. Similarly if $V_R \cap F = \emptyset$ we get $F = \emptyset$.

For the counter-examples consider $\mathbb{C}\backslash\{0\}$ and $\{z : 0 < \mathrm{Re}(z) < 1\}$. To see that $\mathbb{C}_\infty\backslash A$ is connected in the second example consider the closures in $\mathbb{C}_\infty$ of $\{z : 1 \leq \mathrm{Re}(z)\}$ and $\{z : \mathrm{Re}(z) \leq 0\}$.

192. Let Ω be a bounded region, $a \in \Omega$ and $f : \Omega \to \Omega$ be a holomorphic map such that $f(a) = a$. Show that $|f'(a)| \leq 1$.

Let $\{z : |z-a| \leq r\} \subset \Omega$. Let $M = \sup\{|\zeta| : \zeta \in \Omega\}$. Let $g(z) = \frac{r}{M}f(z) + a$. Then $|g(z) - a| \leq r$ and Open Mapping Theorem implies that g maps $B(a,r)$ into itself. Also $g(a) = (1 + \frac{r}{M})a$. Applying Schwartz Lemma to $h(z) \equiv \frac{g(a+rz)-a}{r}(z \in U)$ we get $|h'(0)| \leq 1 - |a|^2/M^2$. This gives $|f'(a)| \leq \frac{M}{r}$. Thus $|f'(a)|$ has a bound which depends only on a and Ω and not on f. Now we note that the iterates $f, f\circ f, f\circ f\circ f, \ldots$ satisfyu the same hypothesis as f and hence $\sup\limits_n |f_n'(a)| < \infty$ where f_n denotes the $n-th$ iterate of f. But this means $\sup\limits_n |f'(a)|^n < \infty$ which means $|f'(a)| \leq 1$.

193. Let $f \in H(U\backslash\{0\})$ and $|f(z)| \leq \log\frac{1}{|z|}$ $\forall z \in U\backslash\{0\}$. Show that f vanishes identically.

$zf(z) \in H(U\backslash\{0\})$ and $|zf(z)| \leq -|z|\log(|z|) \to 0$ as $z \to 0$. Hence $zf(z)$ has a removable singularity at 0 and the extended function on U vanishes at 0. This says that f has removable singularity at 0. By Maximum Moduls Principle applied to $\{z : |z| \leq 1-\delta\}$ we get $|f(z)| \leq \log\frac{1}{1-\delta}$ for $|z| \leq 1-\delta$. Let $\delta \to 0$.

194. Let f be an entire function with $|x|\,|f(x+iy)| \leq 1$ $\forall x, y \in \mathbb{R}$ then $f(z) = 0$ $\forall z \in \mathbb{C}$.

If $x^2 + y^2 = R^2$ and $y \geq 0$ then $R - y = \frac{x^2}{R+y} \leq \frac{x^2}{R} \leq |x|$ and hence $|x + i(y-R)|\,|x + i(y+R)|\,|f(z)| \leq 4R$. Changing y to $-y$ we see that the same inequality holds even if $y < 0$. By Maximum Modulus Principle $|z+Ri|\,|z-Ri|\,|f(z)| \leq 4R$ for $|z| \leq R$. For $|z| \leq R/2$ we get $|f(z)| \leq \frac{4R}{(R-R/2)^2} = \frac{16}{R}$. Clearly this implies that f is bounded, hence constant. The hypothesis implies that the constant is necessarily 0.

195. Let $f_n : U \to U$ be holomorphic and suppose $f_n(0) \to 1$. Show that $f_n \stackrel{ucc}{\to} 1$.

Since $\{f_n\}$ is normal there is a subsequence $f_{n_j} \stackrel{ucc}{\to} g$ (say). Note that $g \in H(U)$ and $g(0) = 1$. If g is not a constant then $g - 1$ has no zeros in some deleted neighbourhood of 0. Let $\delta > 0$ be such that g has no zero on $|z| = \delta$. For $|z| = \delta$ and j sufficiently large we have $|(f_{n_j}(z) - 1) - (g(z) - 1)| < \inf\{|g(z) - 1| : |z| = \delta\}$. Hence $f_{n_j}(z) - 1$ has same number of zeros as $g - 1$ in $B(0,\delta)$. However $g(0) = 1$ and $f_{n_j}(z) - 1$ has no zero on $B(0,\delta)$ because $f_{n_j}(U) \subset U$! This proves that $g(z) = 1$ $\forall z$ so

$f_{n_j} \stackrel{ucc}{\to} 1$. Going to subsequences we conclude that $f_n \stackrel{ucc}{\to} g$.

196. If $n \in \{3, 4, ...\}$ show that the equation $z^n = 2z - 1$ has a unique solution in U.

Note that $|1 + z^n| \leq 1 + 1 = |-2z|$ on ∂U. If we had strict inequality we could conclude that $z^n - 2z + 1$ and $-2z$ have the same number of zeros in U and that is what we are aiming at. However strict inequality fails at $z = 1$. We claim that $(1-t)^n < 1 - 2t$ if $t > 0$ is sufficiently small. Indeed, by L'Hopital's Rule $\lim\limits_{t \to 0} \frac{(1-2t)-(1-t)^n}{t} = n - 2 > 0$. We now have $|z|^n = (1-t)^n < 1 - 2t = -1 + |-2z|$ if $|z| = 1 - t$. Hence $|1 + z^n| \leq 1 + |z^n| < |-2z|$ for $|z| = 1 - t$. This shows that $1 + z^n - 2z$ and $-2z$ have the same number of zeros in $|z| < 1 - t$. This holds for all sufficiently small positive numbers t.

197. Show that there are (restrictions to $\mathbb{R}$ of) entire functions which tend to ∞ faster than any given function. More precisely, if $\phi : (0, \infty) \to (0, \infty)$ is any increasing function then there is an entire function f such that $f(x) \geq \phi(x)$ $\forall x \in (0, \infty)$.

Let $f(z) = 1 + \sum\limits_{j=1}^{\infty} (\frac{z}{j})^{m_j}$ where $m_1 < m_2 < ...$, Then f is entire. We choose $m_j's$ with the additional property $1 + j^{m_j} \geq \phi(((j+1)^2)$. Any number $x > 1$ lies between j^2 and $(j+1)^2$ for some $j \in \mathbb{N}$ and $f(x) \geq 1 + (\frac{x}{j})^{m_j} \geq 1 + j^{m_j} \geq \phi(((j+1)^2) \geq \phi(x)$. If $\psi(x) = \begin{cases} \phi(x-1) \text{ if } x > 1 \\ 0 \text{ if } 0 < x \leq 1 \end{cases}$ then ψ is a increasing function : $(0, \infty) \to (0, \infty)$ and there is an entire function g such that $g(x) \geq \psi(x)$ $\forall x > 0$. Let $f(z) = g(z+1)$.

198. Find a necessary and sufficient condition that $A \equiv \{z : \left|az^2 + bz + c\right| < r\}$ is connected.

If $a = 0$ then A is always connected. Assume $a \neq 0$. We claim that A is connected if and only if $\left|b^2 - 4ac\right| < 4r\,|a|$. Note that $A = \{\zeta - \frac{b}{2a} : \left|\zeta^2 - \beta\right| < \frac{r}{|a|}\}$ where $\beta = \frac{b^2}{4a^2} - \frac{c}{a}$. It suffices to show that $B \equiv \{\zeta : \left|\zeta^2 - \beta\right| < \frac{r}{|a|}\}$ is connected if and only if $\left|b^2 - 4ac\right| < 4r\,|a|$ which translates into $|\beta| < \frac{r}{|a|}$. Let $\alpha^2 = \beta$. If $|\beta| \geq \frac{r}{|a|}$ then the relation $B = [B \cap B(\alpha, \sqrt{\frac{r}{|a|}})] \cup [B \cap B(-\alpha, \sqrt{\frac{r}{|a|}})]$ shows that B is not connected. If $|\beta| < \frac{r}{|a|}$ then $tz \in B$ whenever $z \in B$ and $0 \leq t \leq 1$ proving that B is connected.

199. If $z, c_1, c_2, c_3 \in \mathbb{C}$ and $\frac{1}{z-c_1} + \frac{1}{z-c_2} + \frac{1}{z-c_3} = 0$ show that z belongs to the closed triangular region with vertices c_1, c_2, c_3.

We prove a more general result: if $z, c_1, c_2, ..., c_n \in \mathbb{C}$ and $\frac{1}{z-c_1} + \frac{1}{z-c_2} + ... + \frac{1}{z-c_n} = 0$ we show that z belongs to the convex hull of $c_1, c_2, ..., c_n$.

This requires a standard "Seperation Theorem": if C is a closed convex set in $\mathbb{C}$ and z is a complex number in $\mathbb{C} \backslash C$ then there is a complex number a such that

$\mathrm{Re}(\bar{a}\zeta) < \mathrm{Re}(\bar{a}z)$ for each $\zeta \in C$. Let C be the convex hull of $c_{1,}c_2, ..., c_n$.The given equation gives $\frac{1}{\bar{a}z-\bar{a}c_1} + \frac{1}{\bar{a}z-\bar{a}c_2} + ... + \frac{1}{\bar{a}z-\bar{a}c_n} = 0$. If z does not belongs to the convex hull of $c_{1,}c_2, ..., c_n$ we choose a as above, multiply the numerator and the denominator of each term by the conjugate of the denominator and take real parts on both sides to get a contradiction.

200. Prove the following result of Gauss and Lucas: if p is a polynomial then every zero of p' is in the convex hull of the zeros of p.

We may suppose $p(z) = (z - c_1)(z - c_2)...(z - c_n)$. If $p'(z) = 0$ then $0 = \frac{p'(z)}{p(z)} = \frac{1}{z-c_1} + \frac{1}{z-c_2} + ... + \frac{1}{z-c_n}$ and previous problem can be applied.

201. Let $f \in C(\bar{U}) \cap H(U)$. Show that $\int_{-1}^{1} |f(x)|^2 \, dx \leq \int_{0}^{\pi} \left|f(e^{it})\right|^2 dt$.

Let γ consist of the line segment from -1 to $+1$ and the semi-circular arc $\{e^{it} : 0 \leq t \leq \pi\}$. By Cauchy's Theorem $\int_{\gamma} f(z)\bar{f}(\bar{z})dz = 0$. Hence $\int_{-1}^{1} |f(x)|^2 \, dx = -\int_{0}^{\pi} f(e^{it})\bar{f}(e^{-it})ie^{it}dt$. Apply Cauchy-Schwartz inequality.

202. Prove Brouer's Fixed Point Theorem in two dimensions: every continuous map $\phi : \bar{U} \to \bar{U}$ has a fixed point.

Suppose not. Let $H(t,s) = \begin{cases} (e^{2\pi is} - 2t\phi(e^{2\pi is})) \text{ is } t \in [0,1/2) \text{ and } s \in [0,1] \\ ((2-2t)e^{2\pi is} - \phi((2-2t)e^{2\pi is})) \text{ if } t \in [1/2,1] \text{ and } s \in [0,1] \end{cases}$.
This is a continuous function : $[0,1] \times [0,1] \to \mathbb{C}\backslash\{0\}$. Also, $H(0,s) = e^{2\pi is}, 0 \leq s \leq 1$ and $H(1,s) = -\phi(0), 0 \leq s \leq 1$. This shows that the path $\gamma(s) = e^{2\pi is}, 0 \leq s \leq 1$ is homotopic to a constant path in $\mathbb{C}\backslash\{0\}$. This implies that the index of 0 w.r.t. the path $\gamma(s) = e^{2\pi is}, 0 \leq s \leq 1$ is 0, a contradiction.

203. If $\phi : T \to \mathbb{C}\backslash\{0\}$ is continuous and if $\phi(-z) = -\phi(z)$ $\forall z \in T$ show that there is no continuous function g on T such that $g^2 = \phi$.

Consider $h(z) = \frac{g(-z)}{g(z)}$. We have $h^2 = -1$ and h is continuous. This implies $h(z) = i$ $\forall z$ or $h(z) = -i$ $\forall z$. Let us write $h(z) = c$ so the constant c is either i or $-i$. But then $c^2 = h(-z)h(z) = \frac{g(z)}{g(-z)}\frac{g(-z)}{g(z)} = 1$, a contradiction.

204. Prove that if K is a non-empty compact convex subset of $\mathbb{C}$ then every continuous map $\phi : K \to K$ has a fixed point.

Let $H = \frac{1}{R}K$ where $R > 0$ is so large that $H \subset \bar{U}$. For each $z \in \bar{U}$ there is a unique point $g(z) \in H$ such that $|g(z) - z| \leq |\zeta - z|\ \forall \zeta \in H$. The existence is an easy consequence of compactness of H. Uniqueness is proved as follows: if $|\zeta_1 - z| \leq |\zeta - z|\ \forall \zeta \in H$ and $|\zeta_2 - z| \leq |\zeta - z|\ \forall \zeta \in H$ then $\left|\frac{\zeta_1+\zeta_2}{2} - z\right| \leq \frac{|\zeta_1-z|+|\zeta_2-z|}{2} \leq |\zeta - z|\ \forall \zeta \in H$ and this holds, in particular for $\zeta = \frac{\zeta_1+\zeta_2}{2}$ by convexity of H. This implies that $\zeta_1 - z = \lambda(\zeta_2 - z)$ for some $\lambda \geq 0$ and hence ζ_1, ζ_2, z are colinear. The fact that $\left|\frac{\zeta_1+\zeta_2}{2} - z\right| = \frac{|\zeta_1-z|+|\zeta_2-z|}{2}$ forces z to be 'between' ζ_1 and ζ_2 which implies that $z \in H$ by convexity. But then $\zeta_1 = \zeta_2 = z$. We have now proved the existence of a map $g : \bar{U} \to H$ such that $|g(z) - z| \leq |\zeta - z|\ \forall \zeta \in H$. Now define $f : \bar{U} \to \bar{U}$ by $f(z) = \frac{1}{R}\phi(Rg(z))$. Note that g is continuous: if $z_n \to z$ and $g(z_n) \to \zeta_0$ then $|g(z_n) - z_n| \leq |\zeta - z_n|$ $\forall \zeta \in H\ \forall n$ implies $|\zeta_0 - z| \leq |\zeta - z|\ \forall \zeta \in H$. But then $g(z) = \zeta_0$, by definition. It follows that f is a continuous map from $\bar{U}$ into itself. By Problem 202) above there is a point $z \in \bar{U}$ such that $f(z) = z$. But then $\phi(Rg(z)) = Rz$. But $Rg(z) \in RH = K$ so $Rz = \phi(Rg(z)) \in K$ which means $z \in H$. But this implies $g(z) = z$ and we get $\phi(Rz) = Rz$. Since $Rz \in K$ we are done.

205. If $f \in H(B(0,\delta)), f(0) = 0$ and $f(z) \neq 0\ \forall z \in B(0,\delta)\backslash\{0\}$ show that $|f(z)|$ is not harmonic. (Example: $|z|^n$)

MVP fails.

206. Prove Rado's Theorem

Let Ω be a region, $f \in C(\Omega)$ and $f \in H(\Omega_0)$ where $\Omega_0 = \Omega\backslash f^{-1}\{0\}$. Then $f \in H(\Omega)$

Remark: this problem requires some measure theory and properties of subharmonic functions.

We first prove that Ω_0 is dense in Ω.

Let $A = \{z \in \Omega : \int\limits_{B(z,\delta)} \log|f(\zeta)|\, d\zeta > -\infty$ for some $\delta > 0$ with $[B(z,\delta)]^- \subset \Omega\}$ and $B = \{z \in \Omega : f$ vanishes in some neibourhood of $z\}$. Clearly A and B are disjoint subsets of Ω and B is open. If we show that A is also open we can conclude that one of these sets is Ω. If $B = \Omega$ then $f \in H(\Omega)$ and $f^{-1}\{0\}$ is countable. If $A = \Omega$ then the fact that $\Omega\backslash f^{-1}\{0\}$ is dense in Ω is clear from the fact that $\int\limits_{B(z,\delta)} \log|f(\zeta)|\, d\zeta > -\infty \Rightarrow \{\zeta \in B(z,\delta) : f(\zeta) = 0\}$ is a (Lebesgue) null set. [Of course, $\log|f(\zeta)|$ is bounded above on $B(z,\delta)$ if the closure of this ball is contained in Ω].

It remains to show that A is open. Let $z \in A$ and $\delta > 0$ be such that $\int\limits_{B(z,\delta)} \log|f(\zeta)|\,d\zeta > -\infty$ and $[B(z,\delta)]^{-} \subset \Omega$. Let $w \in B(z,\delta)$ and choose $r > 0$ such that $B(w,r) \subset B(z,\delta)$. Then $\int\limits_{B(w,r)} \log|f(\zeta)|\,d\zeta = \int\limits_{B(w,r)\cap\{|f|\leq 1\}} \log|f(\zeta)|\,d\zeta +$ $\int\limits_{B(w,r)\cap\{|f|>1\}} \log|f(\zeta)|\,d\zeta$. The second term here is non-negative, so it suffices to show that $\int\limits_{B(w,r)\cap\{|f|\leq 1\}} \log|f(\zeta)|\,d\zeta > -\infty$. Since $-\log|f(\zeta)| \geq 0$ on $B(w,r) \cap \{|f| \leq 1\}$ it follows that $\int\limits_{B(w,r)\cap\{|f|\leq 1\}} \log|f(\zeta)|\,d\zeta \geq \int\limits_{B(z,\delta)\cap\{|f|\leq 1\}} \log|f(\zeta)|\,d\zeta =$ $\int\limits_{B(z,\delta)} \log|f(\zeta)|\,d\zeta - \int\limits_{B(z,\delta)\cap\{|f|>1\}} \log|f(\zeta)|\,d\zeta > -\infty$ because $\int\limits_{B(z,\delta)} \log|f(\zeta)|\,d\zeta > -\infty$ and $\int\limits_{B(z,\delta)\cap\{|f|>1\}} \log|f(\zeta)|\,d\zeta < \infty$.

Next we prove the following:
Lemma
Let f be continuous on a region containing $\bar{U}$ and suppose $U \backslash f^{-1}\{0\}$ is dense in U. If $f \in H(U \backslash f^{-1}\{0\})$ then $\operatorname{Re} f$ is harmonic in .

Grant this Lemma for the moment. We can change U to any open ball whose closure is contained in Ω. It would follow that $\operatorname{Re} f$ is harmonic in any ball contained in Ω, hence in Ω. Applying the result to if we see that $\operatorname{Im} f$ is also harmonic. The Cauchy-Riemann equations are satisfied on $\Omega \backslash f^{-1}\{0\}$ which is dense in Ω and since the real and imaginary parts of f are C^{∞} functions, the Cauchy_Riemann hold throughout Ω and the proof of Rado's Theorem is complete.
Proof of the lemma:

let u be subharmonic on a region containing $\bar{U}$. Claim: $u(z) \leq \int\limits_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})dt \ \forall z = re^{i\theta} \in U$. For this let $u_n, n \geq 1$ be continuous functions on ∂U *decreasing* to u. Let $v_n(z) = \int\limits_{-\pi}^{\pi} P_r(\theta - t)u_n(e^{it})dt \ \forall z = re^{i\theta} \in U, v_n(z) = u_n(z)$ for $z \in \partial U$. Then $v_n' s$ are harmonic. Since $u - v_n$ is subharmonic and ≤ 0 on

∂U we see that $u - v_n \leq 0$ in U and letting $n \to \infty$ we get $u(z) \leq \int\limits_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})dt \; \forall z = re^{i\theta} \in U$. We apply this result to the subharmonic function $u = \operatorname{Re} f + \epsilon \log |f|$ [Note that the inequality $u(z) \leq \int\limits_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})dt$ holds if $u(z) = -\infty$, i.e. $f(0) = 0$. It holds for r sufficiently small if $f(0) \neq 0$. Hence u is subharmonic]. We get

$\operatorname{Re} f(z) + \epsilon \log |f(z)| \leq \int\limits_{-\pi}^{\pi} P_r(\theta - t)\{\operatorname{Re} f(e^{it}) + \epsilon \log \left|f(e^{it})\right|\}dt \; \forall z = re^{i\theta} \in U$. If $f(z) \neq 0$ we get $\operatorname{Re} f(z) \leq \int\limits_{-\pi}^{\pi} P_r(\theta - t) \operatorname{Re} f(e^{it})dt$ by letting $\epsilon \to 0$. Changing f to $-f$ we get the reverse inequality. By continuity of $\operatorname{Re} f$ we see that $\operatorname{Re} f(z) = \int\limits_{-\pi}^{\pi} P_r(\theta - t) \operatorname{Re} f(e^{it})dt \; \forall z = re^{i\theta} \in U$. This proves the lemma.

207. Let $f \in H(\mathbb{C}\backslash\{0\})$ and suppose f does not have an essential singularity at 0. If $f(e^{it}) \in \mathbb{R} \; \forall t \in \mathbb{R}$ show that $f(z) = \frac{p(z)}{z^k}$ for some non-negative integer k and some polynomial p whose degree does not exceed $2k$.

Since f has a pole or a removable singularity at 0 we can write $z^k f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \; \forall z \in \mathbb{C}$ for some non-negative integer k.. Also, $a_n = \frac{1}{2\pi}\int\limits_{0}^{2\pi} e^{i(k-n)t} f(e^{it})dt$ $\forall n \geq 0$. By hypothesis, $\bar{a_n} = \frac{1}{2\pi}\int\limits_{0}^{2\pi} e^{-i(k-n)t} f(e^{it})dt \; \forall n \geq 0$. Now $\int\limits_{0}^{2\pi} e^{-i(k-n)t} f(e^{it})dt =$ $-i\int\limits_{\gamma} z^{n-k-1} f(z)dz = 0$ for $n \geq 2k+1$ (by Cauch'y Theorem), where $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$. Hence $z^k f(z) = \sum\limits_{n=0}^{2k} a_n z^n$.

208 Find a necessary and sufficient condition that $az^2 + bz + c$ (with $a \neq 0$) is one-to-one in U.

If it is one-to-one then $2az + b$ has no zeros in U which implies $\left|-\frac{b}{2z}\right| \geq 1$ or $|b| \geq 2|a|$. Conversely, if this condition holds then $az^2 + bz + c = aw^2 + bw + c \Rightarrow (z - w)(az + aw + b) = 0$ and this implies that $z = w$ because $|az + aw + b| \geq |b| - |a||z + w| > |b| - 2|a| \geq 0$.

209 Let $c_1, c_2, ..., c_n$ be distinct complex numbers. Show that $\sum_{k=1}^{n} \prod_{j \neq k} \frac{c_j - c}{c_j - c_k} = 1$ for all $c \in \mathbb{C}$.

The left side is a polynomial of degree $(n-1)$ which has the vale 1 at each of the points $c_1, c_2, ..., c_n$.

210.

Let μ be a finite positive measure on the Borel subsets of $(0, \infty)$. If $g \in L^\infty(\mu)$ and $\int_0^\infty e^{-x} p(x) g(x) d\mu(x) = 0$ for every polynomial p show that $g = 0$ a.e. $[\mu]$. Conclude that $\{e^{-x} p(x) : p \text{ is a polynomial}\}$ is dense in $L^1(\mu)$.

The second part follows immediately from the first. For the first part let $\phi(z) = \int_0^\infty e^{-zx} g(x) d\mu(x)$ for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. A straightforward argument shows that ϕ is analytic in $\{z \in \mathbb{C}\colon\ \mathrm{Re}(z) > 0\}$. Further, $\phi^{(n)}(z) = \int_0^\infty (-x)^n e^{-zx} g(x) d\mu(x)$ for $z \in \mathbb{C}$ and $n \geq 0$. By hypothesis this gives $\phi^{(n)}(1) = 0$ $\forall n \geq 0$. It follows that $\phi(z) = 0$ whenever $\mathrm{Re}(Z) > 0$. In particular $\int_0^\infty e^{-tx} g(x) d\mu(x) = 0$ if $t > 0$. The finite positive measures ν_1 and ν_2 defined by $d\nu_1 = g^+ d\mu$ and $d\nu_2 = g^- d\mu$ have the same Laplace transform and hence they are equal. This means $g(x) d\mu(x) = 0$ which is what we wanted to prove.

211.

Let $\Omega = \mathbb{C} \backslash \{0, 1\}$ and $f \in H(\Omega)$. Show that if f is not a constant then it must be one of six specific Mobius transformations. [Proposed and solved by Walter Rudin in Amer. Math. Monthly]

By Picard's Theorem f cannot have an essential singularity at 0 and 1. Also $f(\frac{1}{z})$ cannot have an essential singularity at 0. Thus $p_1(z)p_2(1-z)f(z)$ is an entire function which has a removable singularity or a pole at ∞ for some polynomials p_1 and p_2. It follows that $f = \frac{p}{q}$ for some polynomials p and q with no common zeros. Since f does not take the value 0 it follows that p can have zeros only at 0 and 1. Also, q satisfies the same property. Thus $p(z)$ is $cz, c(1-z)$ or $cz(1-z)$ for some constant c. The same is true of q. It is now a routine matter to write down all possibilities for f.