PROBLEMS IN COMPLEX ANALYSIS

These problems are not in any particular order. I have collected them from a number of text books. I have provided hints and solutions wherever I considered them necessary. These are problems are meant to be used in a first course on Complex Analysis. Use of measure theory has been minimized.

Updated in September 2012. Thanks to Sourav Ghosh for pointing out several errors in previous version.

Notation: $U = \{z : |z| < 1\}$ and $T = \{z : |z| = 1\}$.Def: f is analytic or holomorphic on an open set if it is differentiable at each point. $H(\Omega)$ is the class of all holomorphic functions on Ω. $\overset{uc}{\to}$ stands for uniform convergence on compact sets.

1. Find a sequence of complex numbers $\{z_n\}$ such that $\sin z_n$ is real for all n and $\to \infty$ as $n \to \infty$?

2. At what points is $f(z) = |z|$ differentiable? At what points is $f(z) = |z|^2$ differentiable?

3. If f is a differentiable function from a region Ω in $\mathbb{C}$ into $\mathbb{R}$ prove that f is necessarily a constant.

4. Find all entire functions f such that $f^n(z) = z$ for all z, n being a given positive integer.

5. If f and $\bar{f}$ are both analytic in a region Ω show that they are constants on Ω.

6. If f^2 and $(\bar{f})^5$ are analytic in a region show that f is a constant on that region.

7. If f is analytic in a region Ω and if $|f|$ is a constant on Ω show that f is a constant on Ω.

8. Define $Log(z) = \log|z| + i\theta$ where $-\pi < \theta \leq \pi$ and $z = |z|e^{i\theta}$ $(z \neq 0)$. Prove that Log is not continuous on $\mathbb{C}\backslash\{0\}$.

9. Prove that the function Log defined in above problem is differentiable on $\mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$. Find its derivative and prove that there is no power series $\sum\limits_{n=0}^{\infty} a_n(z-c)^n$ convergent in $\mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ whose sum is Log.

10. Let p be a non-constant polynomial, $c > 0$ and $\Omega = \{z : |p(z)| < c\}$. Prove that $\partial\Omega = \{z : |p(z)| = c\}$ and that each connected component of Ω contains a zero of p.

11. Prove that there is no differentiable function f on $\mathbb{C}\backslash\{0\}$ such that $e^{f(z)} = z$ for all $z \in \mathbb{C}\backslash\{0\}$.

12. Let γ be a piecewise continuously differentiable map : $[0, 1] \to \mathbb{C}$ and $h : \gamma^* \to \mathbb{C}$ be continuous(γ^* is the range of γ). Show that $f(z) = \int\limits_{\gamma} \frac{h(\zeta)}{\zeta - z} d\zeta$ defines a holomorphic function on $\mathbb{C}\backslash\gamma^*$.

13. If γ is as in above problem show that the total variation of γ is $\int\limits_0^1 |\gamma'(t)|\, dt$.

14. If p is a polynomial and if the maximum of $|p|$ on a region Ω is attained at an interior point show, without using The Maximum Modulus Principle, that p is a constant.

15. If $f(x + iy) = \sqrt{|xy|}$ show that f is not differntiable at 0 even though Cauchy-Riemann equations are satisfied.

16. Show that $\log\sqrt{x^2 + y^2}$ is a harmonic function on $\mathbb{C}\backslash\{0\}$ which is not the real part of any holomorphic function.

17. If f is holomorphic on Ω and e^f is constant on Ω show that f is constant on Ω.

18. If f is an entire function and $\operatorname{Re} f$ (or $\operatorname{Im} f$) is bounded above or below show that f is constant.

19. Prove that $\frac{|a-b|}{|1-\bar{a}b|} \geq \frac{|a|-|b|}{1-|ab|}$ if either $|a|$ and $|b|$ are both less than 1 or both greater than 1.

20. If $f : U \to U$ is holomorphic show that $\frac{|f(\beta)-f(\alpha)|}{|1-\bar{f(\beta)}f(\alpha)|} \leq \frac{|\beta-\alpha|}{|1-\bar{\beta}\alpha|}$ for all $\alpha, \beta \in U$.

21. Prove that a holomorphic function from U into itself has atmost one fixed point unless it is the identity map.

22. If f is a bijective bi-holomorphic map of U show that f maps open balls in U onto open balls.

23. Let Ω be a region, $f \in C(\Omega)$ and let f^n be holomorphic in Ω for some positive integer n. Show that f is holomorphic in Ω.

24. If f is an entire function such that $|f(z)| \leq 1+\sqrt{|z|}$ for all $z \in \mathbb{C}$ show that f is a constant.

If f is an entire function such that $|f(z)| \leq M\,|z|^N$ for $|z|$ sufficiently large (where M is a positive cosnatnt) show that f is a polynomial.

25. Find the largest open set on which $\int\limits_0^1 \frac{1}{1+tz} dt$ is analytic. Do the same for $\int\limits_0^\infty \frac{e^{tz}}{1+t^2} dt$.

26. If f and g are holomorphic functions on a region Ω with no zeros such that $\{z : \frac{f'}{f}(z) = \frac{g'}{g}(z)\}$ has a limit point in Ω find a simple relation between f and g.

27. If f is a holomorphic function on a region Ω which is not identically zero show that the zeros of the function form an atmost countable set.

28. Is Mean Value Theorem valid in the complex case? (i.e., if f is analytic in a convex region and z_1, z_2 are two points in the region can we always find a point ζ on the line segment from z_1 to z_2 such that $f(z_2) - f(z_1) = f'(\zeta)(z_2 - z_1)$?)

29. Let f be holomorphic on a region Ω with no zeros. If there is a holomorphic function h such that $h' = \frac{f'}{f}$ show that f has a holomorphic logarithm on Ω (i.e. there is a holomorphic function H such that $e^H = f$. Show that h need not exist and give sufficient a condition on Ω that ensures existence of h.

30. Prove that a bounded harmonic function on $\mathbb{R}^2$ is constant.

31. If f is a non-constant entire function such that $|f(z)| \geq M\,|z|^n$ for $|z| \geq R$ for some $n \in \mathbb{N}$ and some M and R in $(0, \infty)$ show that f is a polynomial whose degree is atleast n.

32. If f is an entire function which is not a constant prove that $\max\{|f(z)| : |z| = r\}$ is an increasing function of r which $\to \infty$ as $r \to \infty$.

33. If $f \in C(U \cup T) \cap H(U)$ and $f(z) = 0$ on $\{e^{i\theta} : \alpha < \theta < b\}$ for some $a < b$ show that f is identically 0.

34. True or false: if f and g are entire functions such that $f(z)g(z) = 1$ for all z then f and g are constants. [What is the answer if f and g are polynomials?]

35. If $f : U \to U$ is holomorphic, $a \in U$ and $f(a) = a$ prove that $|f'(a)| \leq 1$.

36. The result of Problem 35 holds for any region that is conformally equivalent to U. [A conformal equivalence is a bijective biholomorphic map].

37. According to Riemann Mapping Theorem, any simply connected region other than $\mathbb{C}$ is conformally equivalent to U. Hence, above problem applies to any such region. Is the result valid for $\mathbb{C}$?

38. Prove that only entire functions that are one-to-one are of the type $f(z) = az + b$.

39. Prove that $\{z : 0 < |z| < 1\}$ and $\{z : r < |z| < R\}$ are not conformally equivalent if $r > 0$.

40. Let $0 < r_1 < R_1$ and $0 < r_2 < R_2$. Prove that $\{z : r_1 < |z| < R_1\}$ and $\{z : r_2 < |z| < R_2\}$ are conformally equivalent $\Leftrightarrow \frac{R_1}{r_1} = \frac{R_2}{r_2}$

41. Show that if a holomorphic map f maps U into itself it need not have a fixed point in U. Even if it extends to a continuous map of the closure of U onto itself the same conclusion holds.

42. If f is holomorphic on U, continuous on the closure of U and $|f(z)| < 1$ on T prove that f has at least one fixed point in U. Can it have more than one fixed point?

43. If f is holomorphic : $U \to U$ and $f(0) = 0$ and if $\{f_n\}$ is the sequence of iterates of f ($i.e. f_1 = f, f_{n+1} = f \circ f_n, n \geq 1$) prove that the sequence $\{f_n\}$ converges uniformly on compact subsets of U to 0 unless f is a rotation.

44. Let f be a homeomorphism of $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ (with the metric induced by the stereographic projection). Assume that f is differntiable at all points of $\mathbb{C} \cup \{\infty\}$ except $f^{-1}\{\infty\}$. Prove that f is a Mobius Transformation.

45. Prove that the only conformal equivalences : $U \backslash \{0\} \overset{onto}{\to} U \backslash \{0\}$ are rotations.

46. Prove that $\pi \cot \pi z = \frac{1}{z} + \sum\limits_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ if z is not an integer.

47. Prove or disprove: $Log(z_1 z_2) = Log(z_1) + Log(z_2)$

48.
a) Discuss convergence of the following infinite products:
$\prod\limits_{n=1}^{\infty} \frac{1}{n^p} (p > 0), \prod\limits_{n=1}^{\infty} (1 + \frac{i}{n}), \prod\limits_{n=1}^{\infty} \left|1 + \frac{i}{n}\right|$.

b) Prove that $\prod_{n=2}^{\infty}(1-\frac{1}{n^2}) = \frac{1}{2}$ and $\prod_{n=0}^{\infty}(1+z^{2^n}) = \frac{1}{1-z}$ if $|z| < 1$. [See Problem 51) for $\prod_{n=1}^{\infty}(1+\frac{i}{n})$].

c) $\prod_{n=1}^{\infty}(1-\frac{1}{p_n})$ where $p_1, p_2, ...$ is the sequence of primes.

49. Let $\text{Re}(a_n) > 0$ for all n. Prove that $\prod_{n=1}^{\infty}[1+|1-a_n|]$ converges if and only if $\sum_{n=1}^{\infty}|Log(a_n)| < \infty$.

50. Prove or disprove the following:
$\sum_{n=1}^{\infty}|Log(a_n)| < \infty \Leftrightarrow \sum_{n=1}^{\infty}|1-a_n| < \infty$ and $\sum_{n=1}^{\infty}Log(a_n)$ is convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty}[1-a_n]$ is convergent.

51. Prove that $\prod_{n=1}^{\infty} z_n$ converges $\Leftrightarrow \sum Log(z_n)$ converges. Use this to prove that $\prod_{n=1}^{\infty}(1+i/n)$ is not convergent.

52. Prove that $\sin \pi z = \pi z \prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})$

53. Let $B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z}$. Prove that if $0 < |a_n| < 1$ and $\sum[1-|a_n|] < \infty$ then the product conveges uniformly on comapct subsets of U and that $B(z)$ is a holomorphic function on this disk with zeros precisely at the points $a_n, n = 1, 2, ...$ Prove that $\{a_n\}$ can be chosen so that every point of T is a limit point; prove that T is a natural boundary of B in this case (in the sense B cannot be extended to a holomorphic function on any larger open set.

54. Say that a function $f : \mathbb{R} \to \mathbb{R}$ is analytic if for each $a \in \mathbb{R}$ there exists $\delta_a > 0$ such that on $(a-\delta_a, a+\delta_a)$, f has a power series expansion. Show that the zeros of an analytic function on $\mathbb{R}$ have no limit points.

55. If $f : \mathbb{C} \to \mathbb{C}$ has power series expansion around each point then it has a single power series expansion valid on all of $\mathbb{C}$. Is it true that if $f : \mathbb{R} \to \mathbb{R}$

has power series expansion around each point then it has a single power series expansion valid on all of $\mathbb{R}$?

56. Does there exist an entire function f such that $|f(z)| = |z|^2 e^{\mathrm{Im}(z)}$ for all z? If so, find all such functions. Do the same for $|f(z)| = |z| e^{\mathrm{Im}(z)\,\mathrm{Re}(z)}$.

57. Does there exist a holomorphic function f on U such that $\{f(\frac{1}{n})\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, ...\}$, i.e. $f(\frac{1}{n}) = \frac{1}{n}$ if n is even and $f(\frac{1}{n}) = \frac{1}{n+1}$ if n is odd?

58. If the radius of convegence of $\sum\limits_{n=0}^{\infty} a_{n,k}(z-a)^n$ exceeds R for each k and $\sum\limits_{n=0}^{\infty} a_{n,k}(z-a)^n \to 0$ uniformly on $\{z : |z - z_0| = r\}$ then it converges uniformly on $\{z : |z - z_0| \leq r\}$ provided $R > r + |z_0 - a|$.

59. Let f be continuous and bounded on $\{z : |z| \leq 1\} \backslash F$ where F is a finite subset of T. If f is holomorphic on U and $|f(z)| \leq M$ on $\partial U \backslash F$ show that $|f(z)| \leq M$ on U.

60. Let $\Omega = \{z : \mathrm{Re}(z) > 0\}$. If f is continuous on the closure of Ω, holomorphic on Ω and if $|f(z)| \leq 1$ on $\partial\Omega$ does it follow that the same inequality holds on Ω?.

61. Let $\Omega = \{z : a < \mathrm{Im}(z) < b\}, f \in H(\Omega)$ and f be bounded and continuous on the closure of Ω. Prove that if $|f(z)| \leq 1$ on $\partial\Omega$ then the same inequality holds on Ω.

62. Prove that $f(z) = \frac{z}{(1-z)^2}$ is one-to-one on U and find the image of U.

63. If p and q are polynomials with $\deg(q) > \deg(p) + 1$ prove that the sum of the residues of $\frac{p}{q}$ at all its poles is 0.

64. Evaluate $\int\limits_{\gamma} \frac{1}{(z-2)(2z+1)^2(3z-1)^3} dz$ and $\int\limits_{\gamma} \frac{1}{(z-10)(z-\frac{1}{2})^{100}} dz$ where $\gamma(t) = e^{2\pi i t} (0 \leq t \leq 1)$

65. Find the number of zeros of $z^7 + 4z^4 + z^3 + 1$ in U and the annulus $\{1 < |z| < 2\}$.

66. Let $p(z) = z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0$ and $R = \sqrt{1 + |c_0|^2 + |c_1|^2 + ... + |c_{n-1}|^2}$. Prove that all the zeros of p are in $\{z : |z| < R\}$.

67. Let $1 < a < \infty$. prove that $z + a - e^z$ has exactly one zero in the left half plane $\{z : \mathrm{Re}(z) < 0\}$.

68. If $0 < |a| < 1$ show that the equation $(z-1)^n e^z = a$ has exactly n solutions in $\mathrm{Re}\, z > 0$. Prove that all the roots are simple roots. If $|a| \leq \frac{1}{2^n}$ prove that all the roots are in $\{z : |z-1| < \frac{1}{2}\}$.

69. Prove that $f(z) = 1 + z^2 + z^{2^2} + ... + z^{2^n} + ...$ has U as its natural boundary in the sense it cannot be extended to a holomorphic function on any open which properly contains U.

70. If p is a polynomial such that $|p(z)| = p(|z|)$ for all z prove that $p(z) = cz^n$ for some $c \geq 0$ and some $n \in \mathbb{N} \cup \{0\}$.

71. Prove that above result holds if p is replaced by an entire function.

72. Prove the two dimensional Mean value Property:

the average of a holomorphic function over an open ball is the value at the centre.

73. Construct a conformal equivalence between the first quadrant and the upper half plane. Also, find a conformal equivalence between U and its intersection with the right half plane.

74. Find a conformal equivalence between the sector $\{z \neq 0 : \theta_1 < \arg(z) < \theta_2\}$ with $0 < \theta_1 < \theta_2 < \pi/2$ and U.

75. Prove that if γ is a closed path in a region Ω and $f \in H(\Omega)$ then $\mathrm{Re}(\int_\gamma \bar{f(z)} f'(z) dz) = 0$.

76. Prove or disprove: given any sequence $\{a_n\}$ of complex numbers there is a holomorphic function f in some neighbourhood of 0 such that $f^{(n)}(0) = a_n$ for all n.

77. If f is holomorphic on $\Omega \backslash \{a\}$ prove that $e^{f(z)}$ cannot have a pole at a.

78. Prove that $\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| d\theta = 0$.

79. Use above result to prove Jensen's Formula:

80. Let Ω be an open set containing 0 and $f \in H(\Omega)$. Prove that $\bar{f(z)} = f(\bar{z})$ for all z with $|z|$ sufficiently small $\Leftrightarrow f^{(n)}(0) \in \mathbb{R}$ for all $n \geq 0$.

81. If $f \in H(U), f(0) = 0, f'(0) \neq 0$ prove that there is no $g \in H(U \backslash \{0\})$ such that $g^2 = f$.

82. If f is an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$ prove that $|f(z)| \geq c\,|z|$ for some positive number c for all z with $|z|$ sufficiently large.

83. Let Ω be a region, $\{f_n\} \subset H(\Omega)$ and assume that $\{f_n\}$ is uniformly bounded on each compact subset of Ω. Let C be the set of points where $\{f_n\}$ is convergent. If this set has a limit point in Ω prove that $\{f_n\}$ converges uniformly on compact subsets of Ω to a holomorphic function.

84. Prove or disprove: If Ω is a region, $\{f_n\} \subset H(\Omega), f_n^{(k)}(z) \to 0$ as $n \to \infty$ for each $z \in \Omega$ and each $k \in \{0, 1, 2, ...\}$ then $\{f_n\}$ converges (to 0) uniformly on compact subsets of Ω

85. Give an example of a function f which is continuous on a closed strip, holomorphic in the interior, bounded on the boundary but not bounded on the strip! [See also problem #61 above].

86. Let $u(z) = \operatorname{Im}\{(\frac{1+z}{1-z})^2\}$. Show that u is harmonic in U and $\lim\limits_{r \to 1} u(re^{i\theta}) = 0$ for all θ. Why doesn't this contradict the Maximum Modulus Principle for harmonic functions?

87. If $\phi(|z|)$ is harmonic in the region $\{z : \operatorname{Re}(z) > 0\}$ (ϕ being real valued and "smooth") prove that $\phi(t) \equiv a \log t + b$ for some a and b.

88. Let $f : \bar{U} \to \mathbb{C}$ be a continuous function which is harmonic in U. Prove that f is holomorphic in U if and only if $\int\limits_{-\pi}^{\pi} f(e^{it})e^{int}dt = 0$ for all positive integers n.

89. Let $\Omega = \{z : \operatorname{Re}(z) > 0\}$. If f is bounded and continuous on $\partial\Omega$ show that it is the restriction of a continuous function on $\bar{\Omega}$ which is harmonic in Ω.

90. Prove that the square of a real harmonic function is not harmonic unless it is a constant. When is the product of two real harmonic functions harmonic? Find all holomorphic functions f such that $|f|^2$ is harmonic.

91. If $f : \Omega \to \mathbb{C}$ and f and f^2 are harmonic prove that either f is holomorphic or $\bar{f}$ is holomorphic. Prove the converse.

92. If u is a non-constant harmonic in a region Ω prove that the zeros of the gradient of u in Ω have no limit point.

93. If u is harmonic in a region Ω prove that partial derivatives of u of all orders are harmonic.

94. Let $S = \{x \in \mathbb{R} : a \leq x \leq b\}$. Let Ω be a region containing S. Prove that if $f \in H(\Omega \backslash S) \cap C(\Omega)$ then $f \in H(\Omega)$.

95. Let $f, f_n (n = 1, 2, ...)$ be holomorphic functions on a region Ω. If $\text{Re}(f_n) \overset{uc}{\to} \text{Re}(f)$ show that $f_n \overset{uc}{\to} f$.

96. Let $f(z) = \int\limits_{-1}^{1} \frac{1}{t-z} dt, z \in \mathbb{C} \backslash [-1, 1]$. Prove that f is holomorphic, its imaginary part is bounded, but the real part is not. Prove that $\lim\limits_{z \to \infty} z f(z)$ exists and find this limit. Find a bounded non-constant holomorphic function on $\mathbb{C} \backslash [-1, 1]$.

97. Give an example of a region Ω such that Ω^c is infinite and every bounded holomorphic function on Ω is a constant.

Remark: it can be shown that there are non-constant bounded holomorphic functions on $\mathbb{C} \backslash [-1, 1]$ but there are no such functions on $\mathbb{C} \backslash K$ if K is a compact subset of $\mathbb{R}$ with Lebesgue measure 0. Thus the complement of the Cantor set gives a region whose complement is uncountable such that every bounded holomorphic function on it is a constant.

98. If Ω is any region contained in $\mathbb{C} \backslash (-\infty, 0]$ show that there exists a bounded non-constant holomorphic function on Ω.

More generally if there is a non-constant holomorphic function ϕ on Ω such that $\phi(\Omega)$ is contained in $\mathbb{C} \backslash (-\infty, 0]$ the same conclusion holds.

99. If Ω is $\mathbb{C} \backslash (-\infty, 0]$ or a horizontal strip or a vertical strip or U^c show that there exist non-constant bounded holomorphic functions on Ω.

100. Prove that there is no holomorphic function f on U^c such that $|f(z)| \to \infty$ as $|z| \to 1$.

101. Prove that there is no continuous bijection from $\bar{\Omega}$, where $\Omega = \{z : \text{Re}(z) > 0\}$, onto $\bar{U}$ which maps Ω onto U and is holomorphic in Ω.

102. Let Ω be a bounded region, $f \in C(\bar{\Omega}) \cap H(\Omega)$ and assume that $|f|$ is a non-zero constant on $\partial\Omega$. If f is not a constant on Ω show that f has atleast one zero in Ω.

103. Let f be a non-constant entire function. Prove that the closure of $\{z : |f(z)| < c\}$ coincides with $\{z : |f(z)| \leq c\}$ for all $c > 0$.

104. Prove that if $f \in H(\Omega), [a,b] \subset \Omega$ (where $[a,b]$ is the line segment from a to b) then $|f(b) - f(a)| \leq |b-a| \, |f'(\xi)|$ for some $\xi \in [a,b]$. Also prove that $|f(b) - f(a) - (b-a)f'(a)| \leq \frac{|b-a|^2}{2} \left| f'{}'(\eta) \right|$ for some $\eta \in [a,b]$.

105. Evaluate $\int\limits_{\gamma} \frac{z^2+1}{z(z^2+4)} dz$ where $\gamma(t) = re^{2\pi i t} (0 \leq t \leq 1)$ where $0 < r < 2$. No computation is needed!

Compute the same integral for $r > 2$.

106. Give an example of a bounded holomorphic function f on $\mathbb{C} \backslash \mathbb{R}$ which cannot be extended to any larger open set.

107. If $f \in H(0 < |z| < R)$ and $\int\limits_{0 < x^2 + y^2 < R} |f(x+iy)| \, dxdy < \infty$ prove that f has either a removable singularity or a pole of order one at 0.

108. In the previous problem if $\int\limits_{0 < x^2 + y^2 < R} |f(x+iy))|^2 \, dxdy < \infty$ prove that f has a removable singularity at 0.

109. Show that there is no function $f \in H(U) \cap C(\bar{U})$ such that $f(z) = \frac{1}{z} \forall z \in \partial U$.

110. If $f \in C(U), f_n \in H(U)$ and $f_n \to f$ in $L^1(U)$ then $f \in H(U)$.

111. Any conformal equivalence of $\mathbb{C} \backslash \{0\}$ is of the form cz or of the form $\frac{c}{z}$ where c is a constant.

112. If $x_1 > x_2 > x_3 > ..., \{x_n\} \to 0$ and $f \in H(U)$ with $f(x_n) \in \mathbb{R} \forall n$ then $f^{(k)}(0) \in \mathbb{R} \forall k$.

113. Let $\{f_n\} \subset H(D)$ where D is an open disc. Assume that $f_n(D) \subset D \backslash \{0\} \forall n$ and that $\lim\limits_{n \to \infty} f_n(a) = 0$ where is the center of D. Then $\lim\limits_{n \to \infty} f_n(z) = 0$ uniformly on compact subsets of D.

114. Let $\{u_n\}$ be a sequence of (strictly) positive harmonic functions on an open set Ω such that $\sum u_n = \infty$ at one point. Then the series diverges at every point. Moreover, if it converges at one point it converges uniformly on compact subsets of Ω.

115. Find all limit points of the sequence $\{\frac{1}{n} \sum\limits_{k=1}^{n} k^{ia}\}_{n=1,2,...}$ where a is a non-zero real number.

116. Let f have an isolated singularity at a point a. Prove that e^f cannot have a pole at a.

117. Let f be holomorphic on U and assume that for each $r \in (0,1)$, $f(re^{it})$ has a constant argument (i.e. $f(re^{it}) = \left|f(re^{it})\right| e^{ia_r}$ where the real number a_r does not depend on t. Show that f is a constant.

118. [based on problem 117)] Let $f \in H(\Omega)$ and suppose $|f|$ is harmonic in Ω. Show that f is a constant.

119. Let $f \in H(U), f(U) \subset U, f(0) = 0$ and $f(\frac{1}{2}) = 0$. Show that $|f'(0)| \leq \frac{1}{2}$. Give an example to show that equality may hold.

120. Let $f \in H(U), f(U) \subset U, f(0) = 0, f'(0) = 0, f''(0) = 0..., f^{(k)}(0) = 0$ where k is a positive integer. Show that $\left|f(\frac{1}{2})\right| \leq \frac{1}{2^k}$ and find a necessary and sufficient condition that $\left|f(\frac{1}{2})\right| = \frac{1}{2^k}$.

121. If f and $zf(z)$ are both harmonic then f is analytic.

122. Prove that $f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} \sin(n\alpha) e^{in\theta}$ is harmonic in U.

123. If $\Omega = \{z : \text{Re}(z) > 0\}$ and f is a bounded holomorphic function on Ω with $f(n) = 0 \forall n \in \mathbb{N}$ show that $f(z) = 0 \forall z \in \Omega$.

124. Show that there is a holomorphic function f on $\{z : \text{Re}(z) > -1\}$ such that $f(z) = \frac{z^2}{2} - \frac{z^3}{(2)(3)} + \frac{z^4}{(3)(4)} - ...$ for $|z| < 1$.

125. Consider the series $z + \frac{z^2}{2} + \frac{z^3}{3} + ...$ on U and $i\pi - (z-2) + \frac{(z-2)^2}{2} - \frac{(z-2)^3}{3} + ...$ on $\{z : |z-2| < 1\}$. (These two regions are disjoint). Show that there is a region Ω and a function $f \in H(\Omega)$ such that Ω contains both U and $\{z : |z-2| < 1\}$, $f(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + ...$ on U and $f(z) = i\pi - (z-2) + \frac{(z-2)^2}{2} - \frac{(z-2)^3}{3} + ...$ on $\{z : |z-2| < 1\}$.

126. Let $f : U \to U$ be holomorphic with $f(0) = 0 = f(a)$ where $a \in U \backslash \{0\}$. Show that $|f'(0)| \leq |a|$.

127. Prove that a complex valued function u on a simply connected region Ω is harmonic if and only if it is of the form $f + \bar{g}$ for some $f, g \in H(\Omega)$.

128. Let $f(z) = z + \frac{1}{z}(z \in \mathbb{C}\backslash\{0\})$. Show that $f(\{z : 0 < |z| < 1\}) = f(\{z : |z| > 1\} = \mathbb{C}\backslash[-2,2]$ and that $f(\{z : |z| = 1\}) = [-2,2]$. Show also that f is conformal equivalence of both the regions $\{z : 0 < |z| < 1\}$)and $\{z : |z| > 1\}$ with $\mathbb{C}\backslash[-2,2]$. Prove that $\{z : |z| > 1\}$ is not simply connected. [How many proofs can you think of?]

129. Show that there is no bounded holomorphic function f on the right-hlaf plane which is 0 at the points $1, 2, 3, ...$ and 1 at the point $\sqrt{2}$. What is the answer if 'bounded' is omitted?

130. Prove or disprove: if $\{a_n\}$ has no limit points and $\{c_n\} \subset \mathbb{C}$ then there is an entire function f with $f(a_n) = c_n \forall n$.

131. Let Ω be a bounded region, $f \in H(\Omega)$ and $\limsup_{z \to a} |f(z)| \leq M$ for every point a on the boundary of Ω.Show that $|f(z)| \leq M$ for every $z \in \Omega$.

132. Let f be an entire function such that $\frac{f(z)}{z} \to 0$ as $|z| \to \infty$. Show that f is a constant.

133. Let f be an entire function which maps the real axis into itself and the imaginary axis into itself. Show that $f(-z) = -f(z) \forall z \in \mathbb{C}$.

134. Let f be a continuous function : $\mathbb{C} \to \mathbb{C}$ such that $f(z^2 + 2z - 6)$ is an entire function. Show that f is an entire function.

135. If f and g are entire functions with no common zeros and if h is an entire function show that $h = fF + gG$ for some entire functions F and G.

ntire].

136. Show that the series $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$ converges if $|z| \leq 1$ and $z \neq 1$.

137. Show that the convergence of $\sum\limits_{n=1}^{\infty} \frac{\sin(nz)}{n}$ implies that $z \in \mathbb{R}$.

138. If $f \in C(\bar{U}) \cap H(U)$ and f is real valued on $T = \partial U$ then f is a constant.

139. Let $\Omega = \{z : \mathrm{Im}(z) > 0\}$ and $f \in H(\Omega) \cap C(\bar{\Omega})$. If $f(x) = x^4 - 2x^2$ for $0 < x < 1$ find $f(i)$.

n Ω].

140. Let Ω be a region and m denote Lebesgue measure on Ω. If $\{f_n\} \subset H(\Omega) \cap L^2(\Omega)$ and if $\{f_n\}$ converges in $L^2(\Omega)$ to f show that $f \in H(\Omega)$.

141. Let Ω be a region containing $\bar{U}$ and $f \in H(\Omega)$. If $|f(z)| = 1$ whenever $|z| = 1$ show that $U \subset f(\Omega)$.

142. Let Ω be a bounded region, $f, g : \bar{\Omega} \to \mathbb{C}$ be continuous and holomorphic in Ω. If $|f(z) - g(z)| < |f(z)| + |g(z)x|$ on $\partial\Omega$ show that f and g have the same number of zeros in Ω.

143. Let Ω be a bounded region $f : \bar{\Omega} \to U$ be continuous and $f \in H(\Omega)$. If $|f(z)| = 1$ whenever $z \in \Omega$ show that $U = f(\Omega)$.

144. Given any continuous fucntion $f : \mathbb{R} \to \mathbb{C}$ there is an entire function g such that g has no zeros and $g(x) > |f(x)|\ \forall x \in \mathbb{R}$.

145. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then we can write f as $\sum\limits_{n=-\infty}^{\infty} f_n(x - n)$ where each f_n is continuous and $f_n(x) = 0$ if $|x| \geq 1$.

146. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and $f(x) = 0$ for $|x| \geq 1$. Let $S = \{z : |\mathrm{Re}(z)| > 3$ and $|\mathrm{Re}(z)| > 2\,|\mathrm{Im}(z)|\,\}$. Given $\epsilon > 0$ we can find an entire function g such that $|f(x) - g(x)| < \epsilon\ \forall x \in \mathbb{R}$ and $|g(z)| < \epsilon\ \forall z \in S$.

147. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then there is an entire fucntion g such that $|f(x) - g(x)| < 1\ \forall x \in \mathbb{R}$.

148. Let $f : \mathbb{R} \to \mathbb{C}$ and $\eta : \mathbb{R} \to (0, \infty)$ be continuous. Then there is an entire function g such that $|f(x) - g(x)| < \eta(x)\ \forall x \in \mathbb{R}$.

149. [Used in problem 146) above]

Let $a < b$ and $f : [a, b] \to \mathbb{C}$ be continuous. Let $f_n(x) = \frac{n}{\sqrt{2\pi}} \int\limits_a^b e^{-n^2(x-t)^2} f(t)dt$.

Then $f_n(x) \to f(x)$ uniformly on $[a + \delta, b - \delta]$ and $f_n(x) \to 0$ uniformly on $\mathbb{R} \backslash [a - \delta, b + \delta]$ for each $\delta > 0$.

150. Show that the family of all analytic maps $f : U \to \{z : \text{Re}(z) > 0\}$ with $|f(0)| \leq 1$ is normal.

151. Let $f \in H(\Omega)$ and f be injective. If $\{z : |z-a| \leq r\} \subset \Omega$ show that $f^{-1}(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\zeta f'(\zeta)}{f(\zeta)-z} d\zeta \ \forall z \in f(B(a,r))$, where $\gamma(t) = a + re^{2it}, 0 \leq t \leq 1$.

152. If $f \in C(\bar{U}) \cap H(U)$ show that $f(z) = i\,\text{Im}(f(0)) + \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,\text{Re}\, f(e^{it})dt$ $\forall z \in U$.

153. If Ω is simply connected show that for any real harmonic function u on Ω, a harmonic conjugate v of u is given by $v(z) = \text{Im}[u(a) + \int\limits_{\gamma}(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})dz]$ where a is a fixed point of Ω and γ is any path from a to z in Ω.

154. Let Ω be a region and $f, g \in H(\Omega)$. If $|f| + |g|$ attains its amximum on Ω at some point a of Ω then f and g are both constants.

155. If f and g are entire functions with $f(n) = g(n) \ \forall n \in \mathbb{N}$ and if $\max\{|f(z)|, |g(z)| \leq e^{c|z|}$ for $|z|$ sufficiently large with $0 < c < 1$ show that $f(z) = g(z) \ \forall z \in \mathbb{C}$. Show that this is false for $c = 1$.

156. Show that there is a function f in $C(\bar{U}) \cap H(U)$ whose power series does not converge uniformly on $\bar{U}$.

157. If $\{f_n\} \subset H(\Omega)$ and $\lim\limits_{n\to\infty} f_n(z) = f(z)$ exists $\forall\ z \in \Omega$ show that there is a dense open subset Ω_0 of Ω such that $f \in H(\Omega_0)$.

158. Let $L : H(\Omega) \to H(\Omega)$ be linear and mulitplicative, not identically 0. Show that there is a point $c \in \Omega$ such that $L(f) = f(c) \ \forall\ f \in H(\Omega)$.

159. Let Ω be a region and $f \in H(\Omega)$ with $f(z) \neq 0 \ \forall z \in \Omega$. If f has a holomorphic square root does it follow that it has a holomorphic logarithm? What if it has a holomorphic $k-th$ root for infinitely many positive integers k?

160. $\lim\limits_{z\to a} \frac{f(z)}{g(z)} = \lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ if f and g are analytic in some neighbourhood of a, $f(a) = g(a) = 0$ and $g'(a) \neq 0$.

161. If f and g are analytic in some neighbourhood of $a, |f(z)| \to \infty$ and $|g(z)| \to \infty$ as $z \to a$ then $\lim\limits_{z\to a} \frac{f(z)}{g(z)} = \lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ provided $\lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ exists.

162. Let f be an entire function such that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f(z) \equiv cz^n$ for some non-negative integer n and some constant c with modulus 1.

163. Let Ω be a region (not necessarily bounded) which is not dense in $\mathbb{C}$, $f \in C(\bar{\Omega}) \cap H(\Omega), |f(z)| \leq M \ \forall z \in \partial\Omega$. Suppose f is bounded on Ω. Then $|f(z)| \leq M \ \forall z \in \Omega$.

164. In above problem the hypothesis that Ω is not dense can be deleted provided $\Omega \neq \mathbb{C}$.

$\frac{f'(z)}{g'(z)}$ provided $\lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ exists.

165. If f is an entire function such that $|f(z)| = 1$ whenever $|z| = 1$ show that $f(z) = cz^n$ for some $n \geq 0$ and $c \in \mathbb{C}$ with $|c| = 1$.

166. Let $f \in H(\Omega \backslash \{a, a_1, a_2, ...\})$ where Ω is a region, $a_n \to a$, $a_n's$ are distinct points of Ω and $a \in \Omega$. If f has a pole at each a_n show that $f(B(a, \epsilon) \backslash \{a, a_1, a_2, ...\})$ is dense in $\mathbb{C}$ for every $\epsilon > 0$.

167. If f is a rational function such that $|f(z)| = 1$ whenever $|z| = 1$ show that $f(z) = cz^n \{\prod_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha_j} z}\} / \{\prod_{j=1}^{m} \frac{z-b_j}{1-\bar{b_j} z}\}$ for some $n \in \mathbb{Z}$ and $a_1, a_2, ..., a_N, b_1, b_2, .., b_m \in \mathbb{C} \backslash T, c \in \mathbb{C}$ with $|c| = 1$.

168. Let f and g be holomorphic on U with g one-to-one and $f(0) = g(0) = 0$, If $f(U) \subset g(U)$ show that $f(B(0, r)) \subset g(B(0, r))$ for any $r \in (0, 1]$.

169. All injective holomorphic maps from U onto itself are of the type $c\frac{z-a}{1-\bar{a}z}$ with $|a| < 1, |c| = 1$. Find all $m - to - 1$ holomorphic maps of U onto itself for a given positive integer m.

170. Let Ω_1 and Ω_2 be bounded regions. Let $f : \Omega_1 \to \Omega_2$ be a holomorphic map such that there is no sequence $\{z_n\}$ in Ω_1 converging to a point in $\partial\Omega_1$ such that $\{f(z_n)\}$ converges to a point in Ω_2. Then there is a positive integer m such that f is $m - to - 1$ on Ω_1.

171. The condition in Problem 169) above that there is no sequence $\{z_n\}$ in Ω_1 converging to a point in $\partial\Omega_1$ such that $\{f(z_n)\}$ converges to a point in Ω_2 is equivalent to the fact that $f^{-1}(K)$ is compact whenever K is a compact subset of Ω_2.

172. Prove that the analogue of Problem 169) when $\Omega_1 = \Omega_2 = \mathbb{C}$ and $\partial\Omega_1$ is interpreted as (the boundary in $\mathbb{C}_\infty$ i.e.) $\{\infty\}$ holds. Give an example to show that Problem 169) fails for a general unbouded region Ω_1.

173. Let $f \in H(U), \theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$ and $\left|f(re^{i\theta_1})\right| = |f(0)| = \left|f(re^{i\theta_2})\right|$ for all $r \in (0, 1)$. Show that f is a constant if $\frac{\theta_1 - \theta_2}{2\pi}$ is irrational.

174. Suppose $\theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$ and $f \in H(U)$, $\left|f(re^{i\theta_1})\right| = |f(0)| = \left|f(re^{i\theta_2})\right|$ for all $r \in (0, 1)$ implies that f is a constant. Show that $\frac{\theta_1 - \theta_2}{2\pi}$ is irrational.

175. A second order differential equation: let Ω be a convex region and $g \in H(\Omega)$. Show that any holomorphic function f satifying the differential equation $f'' + f = g$ in Ω can be expressed as $h(z)\sin(z) + \phi(z)\cos(z)$ for suitable $h, \phi \in H(\Omega)$.

176. Show that $U \backslash \{0\}$ is not conformally equivalent to $\{z : 1 < |z| < 2\}$.

177. Let f be continuous on $\{z : |z| \leq R\}$ and holomorphic on $B(0, R)$. Let $M(r) = \sup\{|f(z)| : |z| = r\}$ and $\phi(r) = \sup\{\operatorname{Re} f(z) : |z| = r\}$ for $0 \leq r \leq R$.

Show that $\phi(r) \leq \frac{R-r}{R+r} \operatorname{Re} f(0) + \frac{2r}{R+r}\phi(r)$ and $M(r) \leq \frac{R-r}{R+r} |f(0)| + \frac{2r}{R+r}\phi(r)$ for $0 \leq r \leq R$.

178. If f is an entire function such that $\operatorname{Re} f(z) \leq B|z|^n$ for $|z| \geq R$ then f is a polynomial of degree at most n.

179. Let Ω be a region and A be a subset of Ω with no limit points in Ω. Show that $\Omega \backslash A$ is a region.

180. Show that $\mathbb{C}\backslash(Q \times Q)$ is connected.

[As an easy consequence of this we can show that $\mathbb{R}^n \backslash Q^n$ is connected. (We only have to project to two dimensions)].

181. Prove the formula $\int\limits_{-\infty}^{\infty} e^{itx} e^{-x^2/2} dx = \sqrt{2\pi} e^{-t^2/2} (t \in \mathbb{R})$ in four different ways.

Contour integration, Power series method: justify $\int\limits_{-\infty}^{\infty} \sum\limits_{n=0}^{\infty} \frac{i^n t^n x^n}{n!} e^{-x^2/2} dx = \sum\limits_{n=0}^{\infty} \frac{i^n t^n}{n!} \int\limits_{-\infty}^{\infty} x^n e^{-x^2/2} dx$,

using the fact that zeros are isolated: let $\phi(z) = \int\limits_{-\infty}^{\infty} e^{izx} e^{-x^2/2} dx$, show that ϕ is entire and compute $\phi(it)$ for real t, differential equation method: prove that $\phi'(t) = -t\phi(t)$.

182. Prove that $|e^z - 1 - z| \leq \frac{|z|^2}{2} e^{|z|}$ $\forall z \in \mathbb{C}$ and $|e^z - 1 - z| \leq \frac{|z|^2}{2}$ if $\operatorname{Re}(z) = 0$. Also show that $\left|e^z - 1 - z - z^2/2! - ... - z^n/n!\right| \leq \frac{|z|^{n+1}}{(n+1)!} e^{|z|}$ $\forall z \in \mathbb{C}$.

183. Let f be a non-constant entire function. Show without using Picard's Theorem that $\liminf\limits_{|z| \to \infty} |f(z)| \in \{0, \infty\}$.

184. Let Ω be open and $f \in H(\Omega)$ be one-to-one. Let γ be any closed path in Ω and $\Omega_1 = \{z \in \Omega \backslash \gamma^* : Ind_\gamma(z) \neq 0\}$. Show that $f^{-1}(w) Ind_\gamma(f^{-1}(w)) = \frac{1}{2\pi i} \int\limits_\gamma \frac{z f'(z)}{f(z) - w} dz$ $\forall w \in f(\Omega_1)$.

185. Let $f \in H(U \backslash \{0\})$ and assume that f has an essential singularity at 0. Let $f_n(z) = f(\frac{z}{2^n}), n \geq 1, z \in U \backslash \{0\}$. Show that $\{f_n\}$ is not normal in $H(U \backslash \{0\})$.

186. Let Ω be an open set in $\mathbb{C}$ such that $\mathbb{C}_\infty \backslash \Omega$ is connected. Let γ be closed path in Ω. Show that $Ind_\gamma(a) = 0\ \forall a \in \mathbb{C}\backslash\Omega$.

187. If f is an entire function which is not a transaltion show that $f \circ f$ has a fixed point.

188. Show that there is a sequence of polynomials $\{p_n\}$ such that $\lim\limits_{n\to\infty} p_n(z) =$
$$\begin{cases} 0 \text{ if } \operatorname{Im}(z) = 0 \\ 1 \text{ if } \operatorname{Im}(z) > 0 \\ -1 \text{ if } \operatorname{Im}(z) > 0 \end{cases}$$

189. Show that there is a sequence of polynomials $\{p_n\}$ such that $\lim\limits_{n\to\infty} p_n(z) = 0\ \forall z \in \mathbb{C}$ but the convergence is not uniform on at least one compact set.

190. If A is bounded in $\mathbb{C}$ then $\mathbb{C}_\infty \backslash A$ is connected if and only if $\mathbb{C}\backslash A$ is connected. If A is unbounded and $\mathbb{C}\backslash A$ is connected does it follow that $\mathbb{C}_\infty\backslash A$ is connected? If $\mathbb{C}_\infty\backslash A$ is connected does it follow that $\mathbb{C}\backslash A$ is connected?

191. Let Ω be a bounded region, $a \in \Omega$ and $f : \Omega \to \Omega$ be a holomorphic map such that $f(a) = a$. Show that $|f'(a)| \leq 1$.

192. Let $f \in H(U\backslash\{0\})$ and $|f(z)| \leq \log \frac{1}{|z|}\ \forall z \in U\backslash\{0\}$. Show that f vanishes identically.

193. Let f be an entire function with $|x|\,|f(x+iy)| \leq 1\ \forall x, y \in \mathbb{R}$ then $f(z) = 0\ \forall z \in \mathbb{C}$.

194. Let $f_n : U \to U$ be holomorphic and suppose $f_n(0) \to 1$. Show that $f_n \overset{ucc}{\to} 1$.

195. If $n \in \{3, 4, ...\}$ show that the equation $z^n = 2z - 1$ has a unique solution in U.

196. Show that there are (restrictions to $\mathbb{R}$ of) entire functions which tend to ∞ faster than any given function. More precisely, if $\phi : (0,\infty) \to (0,\infty)$ is any increasing function then there is an entire function f such that $f(x) \geq \phi(x)$ $\forall x \in (0, \infty)$.

197. Find a necessary and sufficient condition that $A \equiv \{z : \left|az^2 + bz + c\right| < r\}$ is connected.

198. If $z, c_1, c_2, c_3 \in \mathbb{C}$ and $\frac{1}{z-c_1} + \frac{1}{z-c_2} + \frac{1}{z-c_3} = 0$ show that z belongs to the closed triangular region with vertices c_1, c_2, c_3.

199. Prove the following result of Gauss and Lucas: if p is a polynomial then every zero of p' is in the convex hull of the zeros of p.

200. Let $f \in C(\bar{U}) \cap H(U)$. Show that $\int\limits_{-1}^{1} |f(x)|^2 \, dx \leq \int\limits_{-\pi}^{\pi} \left|f(e^{it})\right|^2 dt$.

201. Prove Brouer's Fixed Point Theorem in two dimensions (: every continuous map $\phi : \bar{U} \to \bar{U}$ has a fixed point) by constructiong a homotopy in $\mathbb{C} \backslash \{0\}$ from the unit circle to a constant (under the assumption that ϕ has no fixed point).

202. If $\phi : T \to \mathbb{C} \backslash \{0\}$ is continuous and if $\phi(-z) = -\phi(z) \ \forall z \in T$ show that there is no continuous function g on T such that $g^2 = \phi$.

203. Prove that if K is a non-empty compact convex subset of $\mathbb{C}$ then every continuous map $\phi : K \to K$ has a fixed point.

204. If $f \in H(B(0, \delta)), f(0) = 0$ and $f(z) \neq 0 \ \forall z \in B(0, \delta) \backslash \{0\}$ show that $|f(z)|$ is not harmonic. (Example: $|z|^n$)

205. Prove Rado's Theorem

Let Ω be a region, $f \in C(\Omega)$ and $f \in H(\Omega_0)$ where $\Omega_0 = \Omega \backslash f^{-1}\{0\}$. Then $f \in H(\Omega)$

Remark: this problem requires some measure theory and properties of subharmonic functions.

206. Let $f \in H(\mathbb{C} \backslash \{0\})$ and suppose f does not have an essential singularity at 0. If $f(e^{it}) \in \mathbb{R} \ \forall t \in \mathbb{R}$ show that $f(z) = \frac{p(z)}{z^k}$ for some non-negative integer k and some polynomial p whose degree does not exceed $2k$.

207 Find a necessary and sufficient condition that $az^2 + bz + c$ (with $a \neq 0$) is one-to-one in U.

208 Let $c_1, c_2, ..., c_n$ be distinct complex numbers. Show that $\sum\limits_{k=1}^{n} \prod\limits_{j \neq k} \frac{c_j - c}{c_j - c_k} = 1$ for all $c \in \mathbb{C}$.

209 Let $c_1, c_2, ..., c_n$ be distinct complex numbers. Show that $\sum\limits_{k=1}^{n} \prod\limits_{j \neq k} \frac{c_j - c}{c_j - c_k} = 1$ for all $c \in \mathbb{C}$.

210.

Let μ be a finite positive measure on the Borel subsets of $(0, \infty)$. If $g \in L^\infty(\mu)$ and $\int_0^\infty e^{-x} p(x) g(x) d\mu(x) = 0$ for every polynomial p show that $g = 0$ a.e. $[\mu]$. Conclude that $\{e^{-x}p(x) : p \text{ is a polynomial}\}$ is dense in $L^1(\mu)$.

211.

Let $\Omega = \mathbb{C}\backslash\{0, 1\}$ and $f \in H(\Omega)$. Show that if f is not a constant then it must be one of four specific Mobius transformations. [Proposed and solved by Walter Rudin in Amer. Math. Monthly]