A Subsidy-Surplus Model and the Skorokhod Problem in an Orthant

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Abstract

We consider the deterministic Skorokhod problem in an orthant of the form

$$Zw(t) = w(t) + \int_0^t b(u, Yw(u), Zw(u))du + \int_0^t R(u, Yw(u-), Zw(u-))dYw(u)$$

with $(Yw)_i(\cdot)$ nondecreasing, and $(Yw)_i(\cdot)$ not increasing while $(Zw)_i(\cdot) > 0$. This can be viewed upon as a subsidy-surplus model in an interdependent economy. Existence of a unique solution is established under fairly general conditions (viz. with $R(\cdot,\cdot,\cdot)$ satisfying a uniform spectral radius condition). Comparison result for (SP) vis-a-vis the usual partial order on the orthant is studied; we show that the more "inward looking" the reflection vectors and the drift, the larger will be the values of Yw but smaller will be the values of Zw. In addition to showing that the Leontief-type output is a feasible subsidy, connection between (SP) and "minimality" of feasible subsidies is discussed; (consequently it is suggested that (SP) may be taken as a continuous time feedback - form analogue of open Leontief model).

In the stochastic case, $(Y(t), Z(t))$ turns out to be a strong Markov process if $w(\cdot)$ arises from a Levy process. Relevance of comparison result to recurrence/positive recurrence of $Z(\cdot)$ process is pointed out.

Key words and phrases : Leontief inverse, variational distance, maximal function, spectral radius, contraction map, deterministic/stochastic Skorokhod equation, feasibility, minimality, Levy process, Brownian motion, recurrence, positive recurrence.

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1 Introduction

The Skorokhod problem in nonsmooth domains has been an object of intensive study ever since reflected Brownian motion in an orthant with oblique reflection was suggested as a heavy traffic limit for tandem queues; see for example, Harrison and Reiman (1981), Dupuis and Ishii (1991(a)), (1993), Shashiashvili (1994), Mandelbaum and Pats (1998). Prior to these Tanaka (1977) has investigated the Skorokhod problem with normal reflection in convex (not necessarily smooth) domains. The deterministic Skorokhod problem itself attracted attention because, in the case of Brownian motion, one can get the reflected process in a path-by-path fashion once the deterministic problem is solved; in addition to the references given above see Reiman (1984), Bernhard and El Kharroubi (1991), Mandelbaum (1989).

Recently in an interesting paper, Chen and Mandelbaum (1991) have proposed the deterministic Skorokhod problem in an orthant as a continuous time Leontief type input-output model; (this paper is the direct motivation for the present investigation). They assume that the reflection matrix (or input-output/consumption matrix in the language of economics) is constant and that the off diagonal elements are negative. They show in particular that the y-part of the solution of (SP) (that is, the function whose coordinates are nondecreasing) is minimal (or efficient) among all feasible production plans. (They interpret $y_i(t)$ as a feasible cumulative production of commodity i over $[0, t]$). This minimality result is inspired by a similar result of Reiman (1984) in the context of queueing networks. (In queueing theory $y_i(t)$ is interpreted as a measure of cumulative potential output/demand lost over $[0, t]$ due to buffer i being empty).

In the present work we consider the Skorokhod problem in an orthant G given by

$$\begin{aligned} Zw(t) &= w(t) + \int_0^t b(u, Yw(u), Zw(u))du \\ &\quad + \int_0^t R(u, Yw(u-), Zw(u-))d(Yw)(u) \qquad (1.1) \\ (Yw)_i(t) &= \int_0^t I_{\{0\}}((Zw)_i(s))d(Yw)_i(s) \qquad (1.2) \end{aligned}$$

where w is a given r.c.l.l. $\mathbb{R}^d$ valued function, b, R respectively are given $\mathbb{R}^d$ valued, matrix valued functions on $[0, \infty) \times \bar{G} \times \bar{G}$; see Section 2.

We argue that (SP) in an orthant can be considered also as a subsidy-surplus model in an interdependent economy; the y-part is taken as cumulative subsidy and the z-part as the current surplus; so $y(\cdot)$ can be considered as a measure of the subsidy needed to have a surplus (≥ 0) in each sector of the economy. In such a set up, $R(\cdots) = ((r_{ij}(\cdots)))$ can be taken as a policy of allocation of funds/subsidies among various sectors of the economy. It is natural to assume that the policy of allocation depends not only on time and current surplus *but also on the cumulative subsidy.* For example the policy can be that a sector which has received a greater subsidy upto the present is to be allotted less, or should play greater role in mobilising subsidy for other sectors. So r_{ij}'s are functions of y as well in addition to t, z. We also introduce a drift $b(\cdots)$, called innovations; this again can depend on t, y, z. (It may be mentioned that in all the papers cited above the drift and the reflection matrix are either constant or state dependent; to our knowledge, the coefficients depending on y has not been considered before).

Also $r_{ij}(\cdots)$ being positive or negative, for $i \neq j$, has a natural interpretation in our setup. $\int_0{}^t r_{ij}^+(\cdots)dy_j(\cdot)$ can be taken as the part of the subsidy mobilised for Sector j but spent actually on Sector i over $[0, t]$; such a situation is not meaningless. Next, $\int_0{}^t r_{ij}^-(\cdots)dy_j(\cdot)$ can be taken as the part of the subsidy for Sector j raised from Sector i. Many obvious examples can be given to illustrate this situation.

These considerations make it necessary to distinguish between "the subsidy mobilised for Sector j " and "the subsidy given to Sector j ". Also the difference between "the subsidy mobilised for Sector j " and "subsidy mobilised for Sector j from other sectors" can be taken as" the subsidy mobilised from external sources". Our uniform spectral radius condition (A3) would mean that the subsidy mobilised from external sources is nonzero; so this would be an "open system" in the jargon of input-output analysis. The normalisation $r_{ii} \equiv 1$ just means that nonzero subsidy is actually given to Sector i from the subsidy mobilised for Sector i !

Similar interpretations apply to $b^+(\cdots), b^-(\cdots)$; see Section 2.

We now briefly outline the contents of the paper. In Section 2 we describe the subsidy-surplus model in terms of the deterministic Skorokhod problem. In Section 3 we prove the existence of a unique solution. For this we use a generalization of the contraction mapping argument of Harrison and Reiman (1981); however in our approach we simultaneously approximate $y, z-$ parts, using the variation norm for the y-part and the sup norm for the z-part. Besides an estimate on the variational distance of maximal functions due to Shashiashvili (1994), the key to our approach is an a priori estimate for the standard subsidy (that is, y-part of the solution of (SP)) in terms of a Leontief-type output function, and the fact that the set of subsidies bounded by the Leontief type output is invariant under the map T. The maps T and S are defined by (3.15) - (3.17).

In Section 4 we establish some comparison results, a key assumption being $r_{ij}(\cdots), i \neq j$ are nonpositive. This involves a detailed bare-hands analysis of the map (T, S). The results are intuitive in the sense that the more "inward looking" the reflection vectors (that is, the greater the internal mobilization of subsidies) the larger will be the values of y but smaller will be the values of z. As far as we are aware there has been no previous work on comparison result for Skorokhod problem even when the coefficients have no y-dependence.

The next section deals with feasibility and minimality. As can be expected, a subsidy y is called feasible if the solution of the corresponding vector integral equation (see (5.1)) is nonnegative. It is shown that the Leontief-type output function is a feasible subsidy. Various natural notions of minimality are defined. Among other things, we prove a generalization of the minimality result of Reiman (1984), Chen and Mandelbaum (1991) alluded to above.

Section 6 deals briefly with the stochastic case when the exogeneous flow $w(\cdot)$ is driven by a Levy process. It is shown that under our assumptions the solution $\{(Y(t), Z(t)) : t \geq 0\}$ of the stochastic (SP) is a strong Markov process. $Z(\cdot)$ itself will be Markov if $b(\cdots), R(\cdots)$ are independent of y. When the exogeneous flow is Brownian, results of Section 4 enable us to say something about recurrence, positive recurrence of the $Z(\cdot)$-processes corresponding to different reflection fields - viz. a more inward looking process is recurrent/positive recurrent if a less inward looking one is. In the last section we mention a few open problems.

To sum up, the subsidy-surplus model outlined in Section 2 can be taken as the continuous-time, feedback form analogue of the open Leontief model. In the continuous time case, when $R(\cdots)$ is not constant, it is not clear in general if optimal solutions (or efficient plans) exist under the usual type of minimality criteria (of the sort considered in Section 5). In this context note that the condition (2.1) of the Skorokhod problem, viz. standard subsidy for Sector i can be mobilised only when Sector i has no surplus, is essentially a minimality condition. And this minimality condition (which is also quite natural, but perhaps less obvious than other minimality conditions) proves to be mathematically more tractable. See paragraph 3 in Section 7.

We now fix some notations.

$$G = \{x \in I\!R^d : x_i > 0, i = 1, 2, \cdots, d\};$$

$D([0,\infty) : I\!R^d)$ denotes the space of all $I\!R^d$-valued right continuous functions on $[0,\infty)$ having left limits at every $t > 0$; for a function w on $[0,\infty), w(t-) := \lim_{s\uparrow t} w(s)$ denotes the left limit at t.

$$D([0,\infty) : \bar{G}) = \{w \in D([0,\infty) : I\!R^d) : w(t) \in \bar{G} \quad \forall t\};$$

$$D_{\uparrow}([0,\infty) : \bar{G}) = \{y \in D([0,\infty) : \bar{G}) : y_i(\cdot) \text{ is nondecreasing function for each } i\};$$

$$D_{\uparrow 0}([0,\infty) : \bar{G}) = \{y \in D_{\uparrow}([0,\infty) : \bar{G}) : y(0) = 0\}.$$

Similarly $D([0,t] : I\!R^d), D([0,t] : \bar{G}), D_{\uparrow}([0,t] : \bar{G}), D_{\uparrow 0}([0,t] : \bar{G})$ can be defined. Whenever the context is clear we shall write $D_{\uparrow} \times D$ for $D_{\uparrow}([0,\infty) : \bar{G}) \times D([0,\infty) : \bar{G})$.

2 The model

Consider a class of d interdependent sectors in an economic system. The system is known to evolve in a certain fashion if no action is imposed on it; (may be called "exogeneous" evolution). It is proposed to introduce a set of "innovations" and a "policy of allocation of funds/subsidies" on the system. The situation may be described by the model below.

Let $\bar{G} = \{x \in I\!R^d : x_i \geq 0, 1 \leq i \leq d\}$ denote the nonnegative d dimensional orthant. We have $w \in D([0,\infty) : I\!R^d), b : [0,\infty) \times \bar{G} \times \bar{G} \to I\!R^d, R : [0,\infty) \times \bar{G} \times \bar{G} \to I\!M_d(I\!R)$; here $I\!M_d(I\!R)$ denotes the space of $d \times d$ matrices with real entries; (we often denote $w(t) = (w_1(t), \cdots, w_d(t)), b(u,y,z) = (b_1(u,y,z), \cdots, b_d(u,y,z)), R(u,y,z) = ((r_{ij}(u,y,z))_{1 \leq i,j \leq d})$ We seek Yw, Zw such that the following hold:

(i) $Zw \in D([0,\infty) : \bar{G})$;

(ii) $Yw \in D_{\uparrow}([0,\infty) : \bar{G})$;

(iii) for $1 \leq i \leq d, t \geq 0$,

$$\int_0^t I_{(0,\infty)}((Zw)_i(s))d(Yw)_i(s) = 0 \tag{2.1}$$

(iv) for $1 \leq i \leq d, t \geq 0$,

$$\begin{aligned}(Zw)_i(t) &= w_i(t) + \int_0^t b_i(u, Yw(u-), Zw(u-))du \\ &\quad + (Yw)_i(t) + \sum_{j \neq i} \int_0^t r_{ij}(u, Yw(u-), Zw(u-))d(Yw)_j(u)\end{aligned} \tag{2.2}$$

We have taken $Yw(0-) = 0 = Zw(0-)$, and suitably normalised so that $r_{ii}(\cdot) \equiv 1$ for all i. Observe that there are two unknowns; (2.2) is called the *Skorokhod equation.* Any (Yw, Zw) satisfying the above is said to solve SP(w, b, R) or the *Skorokhod problem* ((SP) for short) corresponding to (w, b, R).

The following interpretation can be given. We assume that the (effects of) innovations, subsidies, surplus etc. have been quantified in terms of some currency in the discussion.

$w_i(t)$ = net of cumulative *exogeneous* amount of income less expenditure in Sector i over the period $[0, t]$; so $w(\cdot)$ represents the "net evolution" of the system when no action is imposed.

In expressions of the form $b_i(u, y(u-), z(u-)), r_{ij}(u, y(u-), z(u-))$ we take u as the time parameter, $y(u) = (y_1(u), \cdots, y_d(u))$ as the *cumulative subsidy given* to the various sectors over the period $[0, u]$, and $z(u)$ as the *current surplus* at time $u; b(\cdot, \cdot, \cdot)$ is considered the *innovations* introduced, and $R(\cdot, \cdot, \cdot)$ is interpreted as the *policy of allocation/mobilization of subsidies.* If $y(\cdot), z(\cdot)$ are respectively the cumulative subsidy and current surplus functions then

$\int_s^t b_i^+(u, y(u-), z(u-))du = \int_s^t b_i^+(u, y(u), z(u))du$

= amount of Sector i *produced* over $[s, t]$ due to innovations (but excluding the subsidy for Sector i);

$\int_s^t b_i^-(u, y(u-), z(u-))du = \int_s^t b_i^-(u, y(u), z(u))du$

= amount of Sector i *consumed* over $[s,t]$ due to innovations (but excluding for the purpose of subsidies);

$\int_s{}^t r_{ij}^+(u, y(u-), z(u-))dy_j(u)$

= amount of subsidy mobilised for Sector j which is actually used in Sector i (but not as subsidy in Sector i) over $[s,t]$;

$\int_s{}^t r_{ij}^-(u, y(u-), z(u-))dy_j(u)$

= amount of subsidy for Sector j mobilised from Sector i over $[s,t]$.

It is possible that part of the subsidy for other sectors mobilised from Sector i could be from the subsidy of Sector i. For example such a situation can arise if $z_i(\cdot) = 0, z_j(\cdot) = 0, r_{ij}(\cdot) \leq 0, i \neq j$ over a certain period; see Example 3.8.

If (Yw, Zw) solve $\mathrm{SP}(w,b,R)$, then we call Yw the *standard subsidy* and Zw the *standard surplus* corresponding to the exogeneous evolution w, innovations b, and subsidy allocation policy R. The condition (2.1) means that *the standard subsidy for Sector j can be mobilised only when Sector j is "empty";* (that is *only when Sector j has no surplus*; mathematically $(Yw)_j(\cdot)$ can increase only when $(Zw)_j(\cdot) = 0$). Consequently observe that

$$\begin{aligned}
&\sum_i \int_0^{\ t} r_{ij}^+(u, Yw(u-), Zw(u-)) \ \ d(Yw)_j(u) \\
&= \ (Yw)_j(t) + \sum_{i \neq j} \int_0^{\ t} r_{ij}^+(u, Yw(u-), Zw(u-))d(Yw)_j(u) \\
&= \ \text{cumulative standard subsidy } \textit{mobilised} \\
&\quad \ \text{for Sector } j \text{ over } [0,t]
\end{aligned} \tag{2.3}$$

And of this $(Yw)_j(t)$ is actually given to Sector j; (so our normalisation $r_{jj}(\cdot) \equiv 1$ basically means that the fraction of the subsidy given to Sector j from that mobilised for Sector j is bounded away from zero under the continuity assumption (A2) of next section; this is quite a natural assumption) Also observe that

$$\sum_i \int_0^{\ t} r_{ij}^-(u, Yw(u-), Zw(u-)) \ \ d(Yw)_j(u)$$

$$
\begin{aligned}
&= \sum_{i \neq j} \int_0^t r_{ij}^-(u, Yw(u-), Zw(u-)) d(Yw)_j(u) \\
&= \text{cumulative standard subsidy mobilised for Sector } j \\
&\quad \text{from other sectors of the system over } [0, t] \qquad (2.4)
\end{aligned}
$$

Therefore (2.3) minus (2.4) gives the cumulative (standard) subsidy mobilised for Sector j from *external sources* over $[0, t]$.

Remarks

1. The coefficients b_i, r_{ij} being functions of y is perhaps new, but quite reasonable. For example, increased food subsidy in times of drought (when food surplus is zero) can have positive effects on other sectors. May be, as extreme examples one can consider public health, education, research and development, government, etc.; such welfare sectors can in principle be "no surplus" sectors.

2. The reason for taking $R(u, Yw(u-), Zw(u-))$ as the infinitesimal element, rather than the more natural $R(u, Yw(u), Zw(u))$ is technical; in the latter case there is no uniqueness even in certain simple situations; see Shashiashvili (1994). In the context of the stochastic problem in Section 6, this would ensure predictability of the processes.

3. While considering queueing networks and reflected diffusions, it is customary to take $w(0) \in \bar{G}$ and hence $Yw(0) = 0$. However, in our discussion such a restriction may not be warranted. For example a particular sector may be in the red to start with and would need a subsidy right away. Also $Yw(0-) = 0 = Zw(0-)$ is a very natural convention to make.

Remark 2.1 : Suppose $b \equiv 0, r_{ij}$ are nonpositive constants for $i \neq j, r_{ii} \equiv 1, w_i(\cdot) =$ negative constant for all i. This situation is formally the classical *Leontief input-output system.* One can write $R = I - V$ with I denoting the identity matrix and V being a nonnegative matrix with zero diagonal entries; R is sometimes called a Minkowski-Leontief matrix; see Karlin (1959). If the spectral radius of V is less than unity, the system is generally known as the *open Leontief model.* In such a case $(-w)$ is interpreted as "final demand", the nonnegative solution $y = (I - V)^{-1}(-w)$ is taken as "the output needed to meet the demand without any surplus/profit", r_{ij}'s are input coefficients,

($|r_{ij}|$ being proportional to the amount of i-th good needed to produce one unit of the j-th good). For more information concerning the Leontief model see Karlin (1959), Nikaido (1967). The matrix $(I-V)^{-1}$ is called the *Leontief inverse* in economics literature.

Remark 2.2 : The case $b \equiv 0, r_{ij}$ = nonpositive constant, $i \neq j, r_{ii} = 1$ has been considered by Harrison and Reiman (1981), Reiman (1984), Chen and Mandelbaum (1991), just to mention a few. Harrison and Reiman investigate heavy traffic model for queueing networks in terms of reflected/regulated Brownian motion; R is then called the reflection matrix. Chen and Mandelbaum (1991) consider a continuous time analogue of the Leontief model. In the context of RBM and SDE's with boundary conditions, when r_{ij} are not necessarily negative, and the coefficients depend on z, there have been many studies; see Bernhard and El Kharroubi (1991), Dupuis and Ishii (1991 (a), (1993), Shashiashvili (1994)), Mandelbaum and Pats (1998).

3 Existence and uniqueness

To establish the existence of a unique solution to the deterministic Skorokhod problem formulated in the preceding section, we make the following hypotheses on b, R. The continuity assumptions below are with respect to the Euclidean norm.

(A1) : For $1 \leq i \leq d$, b_i are bounded continuous; also $(y,z) \mapsto b_i(t,y,z)$ are Lipschitz continuous, uniformly in t.

(A2) : For $1 \leq i,j \leq d, r_{ij}$ are bounded continuous; $(y,z) \mapsto r_{ij}(t,y,z)$ are Lipschitz continuous, uniformly in t. Moreover $r_{ii} \equiv 1$ for all i.

(A3) : For $i \neq j$ there exist constants v_{ij} such that $|r_{ij}(t,y,z)| \leq v_{ij}$. Set $V = ((v_{ij}))$ with $v_{ii} = 0$; we assume that $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V.

Remark 3.1 : As $\sigma(V) = \sigma(V^\dagger)$, where $V^\dagger$ is the transpose of V, by (A3) note that there exist constants $a_i > 0, 1 \leq i \leq d$, and $0 < \alpha < 1$ such that

$$\sum_{j \neq i} a_j |r_{ji}(t,y,z)| \leq \sum_{j \neq i} a_j v_{ji} \leq \alpha a_i \tag{3.1}$$

for all $1 \le i \le d, t \ge 0, y, z \in \bar{G}$; see p. 160 of Dupuis and Ishii (1991 (b)) for a proof. $\square$

For our purpose it will be convenient to have the following metric on $D_{\uparrow}([0, M] : \bar{G}) \times D([0, M] : \bar{G})$ where $M > 0$.

Let a_i's be as in Remark 3.1. Let $c_1 > 0, c_2 > 0$ be arbitrary but fixed numbers; (these will be chosen suitably later). For $(y(\cdot), z(\cdot)), (\hat{y}(\cdot), \hat{z}(\cdot)) \in D_{\uparrow} \times D$ set

$$d_M((y, z), (\hat{y}, \hat{z})) = c_1 \sum_{i=1}^{d} a_i \varphi_M(y_i - \hat{y}_i) + c_2 \sum_{i=1}^{d} a_i \psi_M(z_i - \hat{z}_i) \tag{3.2}$$

where $\varphi_M(\cdots)$ denotes the total variation over $[0, M]$ and $\psi_M(\cdots)$ denotes supremum norm over $[0, M]$. Note that $(D_{\uparrow} \times D, d_M)$ is a complete metric space.

For $w \in D([0, \infty) : \mathbb{R}^d)$ and $b(\cdot)$ satisfying (A1), set

$$h_i(t) = \sup_{0 \le s \le t} \max\{0, -w_i(s)\} = \sup_{0 \le s \le t} w_i^-(s) \tag{3.3}$$

$$\beta_i(t) = \int_0^t \sup\{b_i^-(s, y, z) : (y, z) \in \bar{G} \times \bar{G}\} ds \tag{3.4}$$

for $t \ge 0, 1 \le i \le d$. Define $(h + \beta)(\cdot) \in D_{\uparrow}$ by setting

$$(h + \beta)_i(t) = h_i(t) + \beta_i(t), t \ge 0, 1 \le i \le d \tag{3.5}$$

Also to simplify notation, given w, b, R, for $(y(\cdot), z(\cdot)) \in D_{\uparrow}([0, \infty) : \bar{G}) \times D([0, \infty) : \bar{G})$ write

$$\begin{aligned} &(X(y, z; w, b, R))_i(t) \equiv (X(y, z; w))_i(t) \qquad (3.6) \\ = \quad & w_i(t) + \int_0^t b_i(u, y(u-), z(u-)) du + \sum_{j \ne i} \int_0^t r_{ij}(u, y(u-), z(u-)) dy_j(u) \end{aligned}$$

for $i = 1, 2, \cdots, d, \quad t \ge 0$.

We now give an a priori bound for the standard subsidy.

Proposition 3.2 : Assume (A1) - (A3). Suppose (y, z) solves the Skorokhod problem corresponding to w, b, R. Then

$$y \le (I - V)^{-1}(h + \beta) \tag{3.7}$$

in the sense that

$$y_i(t) \leq ((I - V)^{-1}(h + \beta))_i(t), \quad 1 \leq i \leq d, t \geq 0 \tag{3.8}$$

Proof : By assumption, for fixed $i, (y_i, z_i)$ solves the one dimensional (normal) Skorokhod problem in $[0, \infty)$ for $(X(y, z; w))_i$. Therefore we have

$$y_i(t) = \sup_{0 \leq s \leq t} \max\{0, -(X(y, z; w))_i(s)\} \tag{3.9}$$

By (A3), (3.3), (3.4), (3.6) note that

$$-(X(y, z; w))_i(s) \leq h_i(s) + \beta_i(s) + \sum_{j \neq i} v_{ij} y_j(s)$$

Consequently as h_i, β_i, y_j are nondecreasing, by (3.9) we get

$$y_i(t) \leq h_i(t) + \beta_i(t) + \sum_{j \neq i} v_{ij} y_j(t)$$

for $1 \leq i \leq d, t \geq 0$. That is

$$((I - V)y(t))_i \leq (h + \beta)_i(t) \tag{3.10}$$

Observe that V is a nonnegative matrix; by (A3) we have that

$$(I - V)^{-1} = I + V + V^2 + \cdots \tag{3.11}$$

is a well defined nonnegative matrix. So the conclusion (3.7) is now immediate from (3.10) $\square$

Remark 3.3 : Note that h_i and β_i respectively can be taken as the maximum possible demands on Sector i due to "exogeneous consumption" and "consumption on account of innovation". So the preceding proposition states that the standard subsidy need not exceed the Leontief-output $(I - V)^{-1}(h + \beta)$ corresponding to the worst possible demand $(h + \beta)$. $\square$

From the preceding proof the following is clear.

Proposition 3.4 : Assume (A1), (A2) and that $r_{ij}(\cdots) \geq 0$ for all i, j. Let (y, z) be as in Proposition 3.2. Then $y \leq (h + \beta)$. $\square$.

Before proceeding further the following comment concerning the Lipschitz condition may be in order.

Let a_i's be as in Remark 3.1. For $x \in \mathbb{R}^d$, put $\|x\| = \sum_{i=1}^{d} a_i |x_i|$. As $a_i > 0$ for all i, this gives a norm equivalent to the Euclidean norm. So we may assume (under (A1), (A2)) without loss of generality that there is $\hat{K} > 0$ such that

$$|f(t, y, z) - f(t, \hat{y}, \hat{z})| \le \hat{K}(\|y - \hat{y}\| + \|z - \hat{z}\|) \tag{3.12}$$

with $f = b_i$ or $r_{ij}, 1 \le i, j \le d, t \ge 0, (y, z), (\hat{y}, \hat{z}) \in \bar{G} \times \bar{G}$.

A semi-constructive method of solution

The method consists of several steps. Assume (A1)-(A3).

Step. 1 : As $Yw(0-) = 0 = Zw(0-)$, at $t = 0$, solving (SP) amounts to solving the following Linear Complementarity problem (LCP) : given $w(0) \in \mathbb{R}^d$ and matrix $R(0, 0, 0)$, to find $Zw(0), Yw(0) \in \bar{G}$ such that

$$Zw(0) = w(0) + R(0, 0, 0)Yw(0) \tag{3.13}$$

$$(Zw)_i(0) \cdot (Yw)_i(0) = 0, \quad 1 \le i \le d \tag{3.14}$$

We can write $R(0, 0, 0) = I + \hat{R}(0, 0, 0)$ with spectral radius of $\hat{R}(0, 0, 0)$ less than 1; to see this use (3.1) and the remark on p.160 of Dupuis and Ishii (1991(b)). So by standard results on LCP (see Cottle, et al (1992) or Berman and Plemmons (1979)) a unique solution exists. This can be constructed by the fixed point argument given on pp. 176-177 of Shashiashvili (1994); such a method can also be gleaned from the analysis given below in a more general context. See also Step 1 of Section 4.

Step 2: We claim that it is enough to consider the case when $w(0) \in \bar{G}$. To see this consider $Zw(0) + w(\cdot) - w(0), Yw(\cdot) - Yw(0), (u, y, z) \mapsto b(u, y + Yw(0), z)$, $(u, y, z) \mapsto R(u, y+Yw(0), z)$ respectively in the place of w, Yw, b, R, and use Step 1.

Step. 3 : Let $w \in D([0, \infty) : \mathbb{R}^d)$ be fixed with $w(0) \in \bar{G}$. In such a case note that $Yw(0) = 0$. We now define a map (T, S) on $D_{\uparrow 0} \times D$, (depending on w as a parameter) as follows

$$(T, S)(y, z) \equiv (T(y, z; w, b, R), S(y, z; w, b, R)) \equiv (T(y, z; w), S(y, z; w)) \tag{3.15}$$

$$(T(y, z; w))_i(t) = \sup_{0 \le s \le t} \max\{0, -(X(y, z; w))_i(s)\} \tag{3.16}$$

$$(S(y, z; w))_i(t) = (X(y, z; w))_i(t) + (T(y, z; w))_i(t) \tag{3.17}$$

for $i = 1, 2, \cdots, d, t \geq 0, y \in D_{\uparrow 0}([0,\infty) : \bar{G}), z \in D([0,\infty) : \bar{G})$, where $X(y, z; w)$ is given by (3.6). Observe that any solution to the (SP) will be a fixed point for the map (T, S); (this has been partly indicated in the proof of Proposition 3.2 as well). It is also clear from (3.16), (3.17) that $(T(y, z; w))_i(\cdot)$ can increase only when $(S(y, z; w))_i(\cdot)$ is zero. The map T as a function of y is similar to the one considered by Harrison and Reiman (1981) and by Mandelbaum, Massey and Pats (1995). As $(S(y, z; w))_i(t) \geq 0$ is clear from (3.16), (3.17) the following proposition is immediate.

Proposition 3.5 : (T, S) is a map from $D_{\uparrow 0} \times D$ into itself with $T(y, z; w) \in D_{\uparrow 0}([0,\infty) : \bar{G})$ and $S(y, z; w) \in D([0,\infty) : \bar{G})$. If w is continuous then $C_{\uparrow 0}([0,\infty) : \bar{G}) \times C([0,\infty) : \bar{G})$ is invariant under (T, S). □

Another invariant set is provided by the following result, which enables the fixed point approach to work.

Proposition 3.6 : Assume (A1)-(A3); let $w(0) \in \bar{G}$ and $(h + \beta)$ be as in (3.5). Let $y \in D_{\uparrow 0}([0,\infty) : \bar{G})$ be such that $y \leq (I - V)^{-1}(h + \beta)$ in the sense of (3.8). Then

$$T(y, z; w) \leq (I - V)^{-1}(h + \beta) \tag{3.18}$$

for any $z \in D([0,\infty) : \bar{G})$; that is $(T(y, z; w))_i(t) \leq ((I - V)^{-1}(h + \beta))_i(t)$ for $1 \leq i \leq d, t \geq 0, z \in D([0,\infty) : \bar{G})$.

Proof : Observe that $(h + \beta)$ and $(I - V)^{-1}(h + \beta)$ are in $D_{\uparrow 0}$. Next note that

$$\begin{aligned}
&-\sum_{j\neq i} \int_0^s r_{ij}(u, y(u-), z(u-)) dy_j(u) \leq \sum_{j\neq i} v_{ij} y_j(s) \\
&\leq \sum_{j\neq i} v_{ij}((I - V)^{-1}(h + \beta))_j(s) = (V(I - V)^{-1}(h + \beta))_i(s) \\
&= ((I - V)^{-1}(h + \beta))_i(s) - (h_i(s) + \beta_i(s)) \qquad (3.19)
\end{aligned}$$

whence it follows that

$$-(X(y, z; w))_i(s) \leq ((I - V)^{-1}(h + \beta))_i(s) \tag{3.20}$$

This completes the proof. □

Step . 4 : Set

$$\zeta(t) = t + \sum_{j=1}^{d} ((I-V)^{-1}(h+\beta))_j(t) \tag{3.21}$$

Let $(y,z),(\hat{y},\hat{z}) \in D_{\uparrow 0} \times D$ with $y \leq (I-V)^{-1}(h+\beta), \hat{y} \leq (I-V)^{-1}(h+\beta)$. For convenience denote $T(y,z;w), T(\hat{y},\hat{z};w), S(y,z;w), S(\hat{y},\hat{z};w)$ respectively by $Ty, T\hat{y}, Sz, S\hat{z}$; also write $K = \hat{K}\left(\sum\limits_{i=1}^{d} a_i\right)$ where $\hat{K}$ is the Lipschitz constant given by (3.12), a_i's are as in Remark 3.1. By the lemma of Shashiashvili ((1994) pp. 170-175) concerning variational distance between maximal functions, and using (3.12), (3.21), (A3) we get

$$\begin{aligned}
&\varphi_t((Ty)_i - (T\hat{y})_i) \\
&\leq \varphi_t((X(y,z;w))_i(\cdot) - (X(\hat{y},\hat{z};w))_i(\cdot)) \\
&\leq \int_0^t |b_i(u,y(u),z(u)) - b_i(u,\hat{y}(u),\hat{z}(u))| du \\
&\quad + \sum_{j\neq i} \int_0^t |r_{ij}(u,y(u-),z(u-)) - r_{ij}(u,\hat{y}(u-),\hat{z}(u-))| dy_j(u) \\
&\quad + \varphi_t\left(\sum_{j\neq i} \int_0^{\cdot} r_{ij}(u,\hat{y}(u-),\hat{z}(u-)) d(y_j - \hat{y}_j)(u)\right) \\
&\leq \hat{K} \int_0^t [\|y(u) - \hat{y}(u)\| + \|z(u) - \hat{z}(u)\|] du \\
&\quad + \hat{K} \int_0^t [\|y(u-) - \hat{y}(u-)\| + \|z(u-) - \hat{z}(u-)\|] \left(\sum_{j\neq i} dy_j(u)\right) \\
&\quad + \sum_{j\neq i} v_{ij} \varphi_t(y_j - \hat{y}_j) \\
&\leq \hat{K} \int_0^t \left[\sup_{0\leq u\leq s} \left(\sum_{\ell=1}^{d} a_\ell |z_\ell(u) - \hat{z}_\ell(u)|\right)\right] \left(ds + \sum_{j\neq i} dy_j(s)\right)
\end{aligned}$$

$$+\hat{K}\int_0^t \left[\sup_{0\le u\le s}\left(\sum_{\ell=1}^d a_\ell |y_\ell(u)-\hat{y}_\ell(u)|\right)\right]\left(ds+\sum_{j\neq i} dy_j(s)\right)$$

$$+\sum_{j\neq i} v_{ij}\varphi_t(y_j-\hat{y}_j)$$

$$\le \hat{K}\zeta(t)\left(\sum_{j=1}^d a_j\psi_t(z_j-\hat{z}_j)\right)$$

$$+\hat{K}\zeta(t)\left(\sum_{j=1}^d a_j\varphi_t(y_j-\hat{y}_j)\right)+\sum_{j\neq i} v_{ij}\varphi_t(y_j-\hat{y}_j) \tag{3.22}$$

Multiplying (3.22) by a_i, adding and using (3.1) we obtain for any $t \ge 0$,

$$\sum_{i=1}^d a_i\varphi_t((Ty)_i-(T\hat{y})_i) \tag{3.23}$$

$$\le [\alpha+K\zeta(t)]\left(\sum_{j=1}^d a_j\varphi_t(y_j-\hat{y}_j)\right)+K\zeta(t)\left(\sum_{j=1}^d a_j\psi_t(z_j-\hat{z}_j)\right)$$

Next using (3.17), (3.1), (3.12), (3.23) we can similarly get

$$\sum_{i=1}^d a_i\psi_t((Sz)_i-(S\hat{z})_i) \le 2\{\text{ r.h.s. of(3.23) }\} \tag{3.24}$$

Consequently for any $t>0$, with d_t defined by (3.2) we have

$$d_t((T,S)(y,z),(T,S)(\hat{y},\hat{z}))$$

$$\le \left[\frac{c_1+2c_2}{c_2}K\zeta(t)\right]\left[c_2\sum_{j=1}^d a_j\psi_t(z_j-\hat{z}_j)\right]$$

$$+\left[\frac{c_1+2c_2}{c_1}(\alpha+K\zeta(t))\right]\left[c_1\sum_{j=1}^d a_j\varphi_t(y_j-\hat{y}_j)\right] \tag{3.25}$$

Therefore if we choose c_1, c_2 such that

$$\frac{c_1}{c_2} > \frac{(1+2\alpha)}{(1-\alpha)} \tag{3.26}$$

and $t>0$ such that

$$\zeta(t) \le \frac{\alpha c_2}{K(c_1-c_2)}, \tag{3.27}$$

(such a t exists as $\zeta(t) \downarrow 0$ as $t \downarrow 0$ since $w(0) \in \bar{G}$), then (T, S) is a contraction map on the complete metric space

$$\hat{D}_t := \{(y, z) \in (D_{\uparrow 0} \times D, d_t) : y \leq (I - V)^{-1}(h + \beta)\}. \tag{3.28}$$

So there exists a unique fixed point in $(\hat{D}_t, d_t)$. However, any fixed point (y, z) in $D_{\uparrow 0} \times D$ is a solution for (SP), and hence $y \leq (I - V)^{-1}(h + \beta)$ by Proposition 3.2. Therefore (T, S) has a unique fixed point in $(D_{\uparrow 0}([0, t] : \bar{G}) \times D([0, t] : \bar{G}), d_t)$ and the fixed point lies in $\hat{D}_t$. Thus (SP) is uniquely solved on $[0, t]$.

Step. 5 : Observe that c_1, c_2 are chosen so that (3.26) holds and fixed. Let

$$t_1 = \sup\{t > 0 : \zeta(t) \leq \frac{\alpha c_2}{K(c_1 - c_2)}\} \tag{3.29}$$

Note that $t_1 > 0$ as $w(0) \in \bar{G}$. If $\zeta(t_1) \leq \alpha c_2 / K(c_1 - c_2)$, then by the preceding step the problem is solved uniquely on $[0, t_1]$. If $\zeta(t_1) > \alpha c_2 / K(c_1 - c_2)$ then ζ has a jump at t_1 (which can happen only if some of the w_i's have negative jumps of sufficient magnitude at t_1). As $\zeta(t) < \alpha c_2 / K(c_1 - c_2)$ for all $t < t_1$, the problem has unique solution over $[0, t]$ for every $t < t_1$. Consequently $Yw(t_1-), Zw(t_1-)$ are well defined and are at our disposal. Now to extend the solution to $[0, t_1]$ we need to find $Yw(t_1), Zw(t_1) \in \bar{G}$ such that the following hold:

$$\begin{aligned} (Zw)(t_1) &= (Zw)(t_1-) + w(t_1) - w(t_1-) \\ &\quad + (R(t_1, Yw(t_1-), Zw(t_1-))\Delta Yw(t_1) \end{aligned} \tag{3.30}$$

$$(Zw)_i(t_1) \cdot \Delta(Yw)_i(t_1) = 0, \quad 1 \leq i \leq d \tag{3.31}$$

where $\Delta Yw(t_1) = Yw(t_1) - Yw(t_1-)$. This is again a linear complementarity problem with $Zw(t_1), \Delta Yw(t_1)$ as the unknowns. Since

$$R(t_1, Yw(t_1-), Zw(t_1-)) = I + \hat{R}(t_1, Yw(t_1-), Zw(t_1-))$$

with the spectral radius of $\hat{R}(t_1, Yw(t_1-), Zw(t_1-))$ less than 1, as in Step 1 we have a unique solution (which again can be constructed in principle). Thus (SP) has been uniquely solved on the interval $[0, t_1]$.

Step 6 : Set

$$\bar{w}(s) = (Zw)(t_1) + w(s + t_1) - w(t_1), (\bar{Y}w)(s) = Yw(s + t_1) - Yw(t_1),$$

$$\bar{Z}w(s) = Zw(s+t_1), \bar{b}(s,y,z) = b(s+t_1, y+Yw(t_1), z),$$

$$\bar{R}(s,y,z) = R(s+t_1, y+Yw(t_1), z), s \geq 0, y, z \in \bar{G}.$$

To solve SP(w,b,R) on $[t_1,\infty)$ it is enough to solve SP$(\bar{w},\bar{b},\bar{R})$ on $[0,\infty)$ where $\bar{w}(0) \in \bar{G}$ and $\bar{b}, \bar{R}$ satisfy (A1) - (A3). Using the analysis as in Steps 3-5, this problem can be uniquely solved, say, on $[0, t_2 - t_1]$. This means that the original (SP) is uniquely solved on $[0, t_2]$. Repeating this procedure one gets $0 < t_1 < t_2 < \cdots < t_n < \cdots$ such that the Skorokhod problem is uniquely solved on $[0, t_n]$ for each n.

Step 7 : Let $t_\infty = \sup\{t$: (SP) corresponding to (w,b,R) has a unique solution on $[0,s] \quad \forall 0 \leq s \leq t\}$. If $t_\infty < \infty$, then there exists $t_n \uparrow t_\infty$ such that the problem is well posed on $[0,t_n]$ for each n. So by Proposition 3.2 we have for $1 \leq i \leq d$,

$$(Yw)_i(t_\infty-) = \lim_{n\to\infty} (Yw)_i(t_n) \leq ((I-V)^{-1}(h+\beta))_i(t_\infty-) < \infty \quad (3.32)$$

Consequently by (2.2) and (A3) we get $(Zw)_i(t_\infty-) < \infty$ for all i. Thus $Zw(t_\infty-)$ and $Yw(t_\infty-)$ are well defined and the problem is uniquely solvable on $[0,t]$ for any $t < t_\infty$. Now, as in Step 5 the solution can be uniquely extended on $[0,t_\infty]$; and then, as in Step 6 the solution can be uniquely extended to $[0,\hat{t}]$ for some $\hat{t} > t_\infty$. This contradicts maximality of t_∞.

Thus we have proved:

Theorem 3.7 : Assume (A1) - (A3). Then there is a unique solution (Yw, Zw) for the Skorokhod problem corresponding to (w,b,R). Moreover

$$(Yw)_i(t) \leq ((I-V)^{-1}(h+\beta))_i(t) \qquad (3.33)$$

$$(Zw)_i(t) \leq w_i^+(t) + \gamma_i(t) + ((I+V)(I-V)^{-1}(h+\beta))_i(t) \qquad (3.34)$$

where V is as in (A3), h, β are given by (3.3), (3.4) and

$$\gamma_i(t) = \int_0^t \sup\{b_i^+(s,y,z) : (y,z) \in \bar{G} \times \bar{G}\} ds.$$

Also, if w is continuous then so are Yw, Zw. $\square$

Note that (3.34) follows from (3.33) and (2.2).

One can explicitly solve the Skorokhod problem in the following situation

Example 3.8 : Suppose $b_i(\cdot,\cdot,\cdot) \leq 0$ for all $i, r_{ij}(\cdot,\cdot,\cdot) \leq 0$ for $i \neq j, w_i(\cdot) = -h_i(\cdot), 1 \leq i,j \leq d$ where h_i are nonnegative nondecreasing functions. Since $w_i(t) + \int_0^t b_i(s,\cdot,\cdot)ds$ in a nonincreasing negative function for each i, the nature of the Skorokhod problem suggests that one may try out $Zw \equiv 0$. In such a case the Skorokhod equation (2.2) gives for $1 \leq i \leq d, 0 \leq s \leq t$

$$\begin{aligned} (Yw)_i(t) - (Yw)_i(s) + \sum_{j\neq i} \int_{(s,t]} r_{ij}(u, Yw(u-), 0) d(Yw)_j(u) \\ = h_i(t) - h_i(s) + \int_{(s,t]} b_i^-(u, Yw(u-), 0)du \qquad (3.35) \end{aligned}$$

Or equivalently

$$R(s, Yw(s-), 0)d(Yw)(s) = dh(s) + b^-(s, Yw(s-), 0)ds \qquad (3.36)$$

where $h(\cdot) = (h_1(\cdot), \cdots, h_d(\cdot)), b^-(\cdot,\cdot,\cdot) = (b_1^-(\cdot,\cdot,\cdot), \cdots, b_d^-(\cdot,\cdot,\cdot))$. Thus Yw can be taken as the solution of the vector integral equation

$$\begin{aligned} y(t) = h(0) \quad &+ \int_0^t R^{-1}(s, y(s-), 0)dh(s) \qquad (3.37) \\ &+ \int_0^t R^{-1}(s, y(s-), 0)b^-(s, y(s-), 0)ds \end{aligned}$$

In addition to (A2) , (A3) assume that $R^{-1}(s, y, 0)$ is Lipschitz continuous in y uniformly in s. So by a Picard iteration scheme, the integral equation (3.37) has a unique solution. As $r_{ij}(\cdot,\cdot,\cdot) \leq 0$ for $i \neq j$ by (A3) and an analogue of (3.11) note that $R^{-1}(\cdot,\cdot,\cdot)$ is a nonegative matrix valued function. Thus $y_i(\cdot)$ given by (3.37) is nonnegative and nondecreasing. (It is easily verified that if we take Yw as the solution of (3.37) and $Zw \equiv 0$, then they solve SP(w, b, R)). This class of examples admits the following interpretation. The equation (3.37) gives the standard subsidy needed just to keep a system afloat when there is no production (due to exogeneous evolution or innovations) but there is an ever increasing demand due to consumption (due to exogeneous evolution, innovation and allocation of subsidies). Such a situation need not be as bad as it may sound! If one considers, for example, only the collection of all welfare/no surplus sectors then (3.37) gives the standard subsidy needed to be mobilised from other sectors/external sources. □

In view of the lemma of Shashiashvili ((1994), pp. 170-175) concerning the variational distance between maximal functions, and Propositon 3.2, the following result is easily proved proceeding as in Step 4 above.

Proposition 3.9 : Let $w, \hat{w} \in D([0,\infty) : \mathbb{R}^d)$ be such that $w - \hat{w}$ is of bounded variation over every finite interval. Let (Yw, Zw) (resp. $(Y\hat{w}, Z\hat{w})$) be the solution of (SP) corresponding to (w, b, R) (resp. $(\hat{w}, b, R)$). If $\hat{w}(\cdot) \to w(\cdot)$ in the variation norm over $[0,t]$, then $Y\hat{w}(\cdot) \to Yw(\cdot)$ in the variation norm and $Z\hat{w}(\cdot) \to Zw(\cdot)$ in the sup norm over $[0,t]$. In particular for $x \in \mathbb{R}^d, w \in D([0,\infty) : \mathbb{R}^d)$ denote $w_x(\cdot) = w(\cdot) + x$; then for any $t > 0, Yw_x(\cdot) \to Yw(\cdot)$ in the variation norm and $Zw_x(\cdot) \to Zw(\cdot)$ in the sup norm over $[0,t]$, as $x \to 0$. □

Remark 3.10 : Assumptions (A1), (A2) are standard; (3.1) has been assumed by Shashiashvili (1994) when R depends on z alone. Note that (A3) is a sort of uniform Hawkins-Simon condition known in input-output analysis. If we assume that $R(s,y,z) = I + \hat{R}(s,y,z)$ with spectral radius of $\hat{R}(s,y,z)$ less than 1 for each s, y, z, then (3.1) holds with a_i, α depending on s, y, z; (this is precisely condition (3.8) on p. 557 of Dupuis and Ishii (1993) when R depends on z alone). Using continuity of R it is not difficult to show that for any s, y, z one can find a neighbourhood around s, y, z such that (3.1) holds with the same a_i, α in that neighbourhood. So by a localisation argument one can establish the existence of a unique solution to (SP) in such a case; (but analogue of (3.33) can be asserted only locally)

If r_{ij}, b_i are functions only of time t, one can obtain an estimate on $\sum\limits_i \varphi_t((Ty)_i - (T\hat{y})_i)$ in terms of $\sum\limits_{i=1}^d \varphi_t(y_i - \hat{y}_i)$ in (3.22); it is enough to consider only T rather than (T,S), and the contraction map argument goes through without any need for restriction to $\hat{D}_t$; see Mandelbaum, Massey and Pats (1995). □

Remark 3.11 : In our approach, only Step 7 is not constructive; (this explains the adjective in the sub-sectional title). To make it fully constructive one has to show that given $t > 0$ there exists n such that $t < t_n$ where t_j's are as in Step 6. It may be noted that both Dupuis and Ishii (1993) and Shashiashvili (1994) use a compactness argument (invoking Ascoli theorem) to prove existence of a solution. □

4 Comparison result

The objective in this section is to prove a comparison result for the Skorokhod problem. Comparison results for ordinary differential equations and stochastic differential equations are well known; see Birkhoff and Rota (1978), Ikeda and Watanabe (1981); such results for one dimensional s.d.e.'s with (normal) reflection at 0 have been obtained recently by Zhang (1994).

In our context we will be dealing with a partial order, viz $x \leq \tilde{x} \Leftrightarrow x_i \leq \tilde{x}_i$ for all i. This partial order has been found very useful in queueing networks, LCP, (SP) and concerning RBM in the orthant; (for example, the notion of completely-$\mathcal{S}$ property plays a crucial role in the semimartingale formulation of RBM; see Bernard and EL Kharroubi (1991), Williams (1995) and the references therein).

The main result of this section is

Theorem 4.1 : Let b, R (resp. $\tilde{b}, \tilde{R}$) satisfy (A1) - (A3). Assume that for $t \geq 0, 1 \leq i, j \leq d$

$$\tilde{b}_i(t, \tilde{y}, \tilde{z}) \leq b_i(t, y, z) \tag{4.1}$$

$$\tilde{r}_{ij}(t, \tilde{y}, \tilde{z}) \leq r_{ij}(t, y, z) \leq 0, \quad i \neq j \tag{4.2}$$

whenever $\tilde{y} \geq y, \tilde{z} \leq z$. Let $w, \tilde{w}$ be such that

$$\tilde{w}_i(t_2) - \tilde{w}_i(t_1) \leq w_i(t_2) - w_i(t_1) \tag{4.3}$$

for any $0 \leq t_1 \leq t_2$, and $\tilde{w}_i(0) \leq w_i(0)$ for $i = 1, 2, \cdots, d$. Let (Yw, Zw) (resp. $(\tilde{Y}\tilde{w}, \tilde{Z}\tilde{w})$) solve the Skorokhod problem corresponding to (w, b, R) (resp. $(\tilde{w}, \tilde{b}, \tilde{R})$). Then

$$(Yw)_i(t) \leq (\tilde{Y}\tilde{w})_i(t) \tag{4.4}$$

$$(Yw)_i(t_2) - (Yw)_i(t_1) \leq (\tilde{Y}\tilde{w})_i(t_2) - (\tilde{Y}\tilde{w})_i(t_1) \tag{4.5}$$

$$(Zw)_i(t) \geq (\tilde{Z}\tilde{w})_i(t) \tag{4.6}$$

for any $t \geq 0, 0 \leq t_1 \leq t_2, 1 \leq i \leq d$. $\square$

Proof of the theorem basically involves a comparison of the map (T, S) defined by (3.15) - (3.17) and the corresponding map $(\tilde{T}, \tilde{S})$ for $(\tilde{w}, \tilde{b}, \tilde{R})$. For this we initially use the following simplified set-up.

Set-up I : 1. $w, \tilde{w}$ are as in Theorem 4.1; also $b(\cdot) \equiv \tilde{b}(\cdot) \equiv 0$.

2. $R(t) = ((r_{ij}(t))), \tilde{R}(t) = ((\tilde{r}_{ij}(t)))$ are r.c.l.l. functions of t alone, satisfy (A3), $r_{ii}(\cdot) \equiv 1 \equiv \tilde{r}_{ii}(\cdot)$, and

$$\tilde{r}_{ij}(t) \leq r_{ij}(t) \leq 0, \qquad i \neq j, t \geq 0 \tag{4.7}$$

3. $y, \tilde{y} \in D_{\uparrow 0}([0, \infty) : \bar{G})$ and

$$y_i(t_2) - y_i(t_1) \leq \tilde{y}_i(t_2) - \tilde{y}_i(t_1) \tag{4.8}$$

for all $0 \leq t_1 \leq t_2, 1 \leq i \leq d$.

Now define for $t \geq 0, 1 \leq i \leq d$

$$(Xy)_i(t) = w_i(t) + \sum_{j \neq i} \int_0^t r_{ij}(u-) dy_j(u) \tag{4.9}$$

$$(\tilde{X}\tilde{y})_i(t) = \tilde{w}_i(t) + \sum_{j \neq i} \int_0^t \tilde{r}_{ij}(u-) d\tilde{y}_j(u) \tag{4.10}$$

$$(Ty)_i(t) = \sup_{0 \leq s \leq t} \max\{0, -(Xy)_i(s)\} \tag{4.11}$$

$$(\tilde{T}\tilde{y})_i(t) = \sup_{0 \leq s \leq t} \max\{0, -(\tilde{X}\tilde{y})_i(s)\} \tag{4.12}$$

$$(Sy)_i(t) = (Xy)_i(t) + (Ty)_i(t) \tag{4.13}$$

$$(\tilde{S}\tilde{y})_i(t) = (\tilde{X}\tilde{y})_i(t) + (\tilde{T}\tilde{y})_i(t) \tag{4.14}$$

Note that $Ty, \tilde{T}\tilde{y} \in D_{\uparrow 0}$ and $Sy, \tilde{S}\tilde{y} \in D([0, \infty) : \bar{G})$.

Lemma 4.2 : Under set-up I, for $1 \leq i \leq d, t \geq 0$

$$(Ty)_i(t) \leq (\tilde{T}\tilde{y})_i(t) \tag{4.15}$$

Proof : Easily verified $\square$

Lemma 4.3 : Assume set-up I. If $(\tilde{T}\tilde{y})_i(t_1) = (\tilde{T}\tilde{y})_i(t_2)$ for some $1 \leq i \leq d, 0 \leq t_1 \leq t_2$ then $(Ty)_i(t_1) = (Ty)_i(t_2)$.

Proof : If $(\tilde{T}\tilde{y})_i(t_1) = (\tilde{T}\tilde{y})_i(t_2) = 0$, then the result follows by the preceding lemma. So we may assume that $(\tilde{T}\tilde{y})_i(t_1) > 0$. Observe that

$$(\tilde{T}\tilde{y})_i(t_2) = \max\{(\tilde{T}\tilde{y})_i(t_1), \sup_{t_1 \leq s \leq t_2} \{-(Xy)_i(s)\}\} \tag{4.16}$$

So there exists $\tilde{s}_1 \leq t_1$ such that one of the following two cases obtains, viz. for all $s \in [t_1, t_2]$

$$(\tilde{T}\tilde{y})_i(t_1) = -(\tilde{X}\tilde{y})_i(\tilde{s}_1) \geq -(\tilde{X}\tilde{y})_i(s) \tag{4.17}$$

or

$$(\tilde{T}\tilde{y})_i(t_1) = -(\tilde{X}\tilde{y})_i(\tilde{s}_1-) \geq -(\tilde{X}\tilde{y})_i(s) \tag{4.18}$$

We consider the case when (4.18) holds; the case when (4.17) holds can be similarly dealt with . By (4.3), the inequality in (4.18), (4.7), (4.8) we get

$$\begin{aligned} w_i(s) - w_i(\tilde{s}_1-) &\geq \tilde{w}_i(s) - \tilde{w}_i(\tilde{s}_1-) \\ &\geq \sum_{j \neq i} \int_{[\tilde{s}_1, s)} (-\tilde{r}_{ij}(u-)) d\tilde{y}_j(u) \\ &\geq \sum_{j \neq i} \int_{[\tilde{s}_1, s)} (-r_{ij}(u-)) dy_j(u) \end{aligned} \tag{4.19}$$

for all $t_1 \leq s \leq t_2$. From (4.19) it follows that $-(Xy)_i(\tilde{s}_1-) \geq -(Xy)_i(s)$ for all $t_1 \leq s \leq t_2$. The required conclusion is now clear. □

Lemma 4.4 : Under set-up I, for $1 \leq i \leq d, 0 \leq t_1 \leq t_2$,

$$(Ty)_i(t_2) - (Ty)_i(t_1) \leq (\tilde{T}\tilde{y})_i(t_2) - (\tilde{T}\tilde{y})_i(t_1) \tag{4.20}$$

Proof : We may assume that the left side of (4.20) is > 0, for otherwise there is nothing to prove. This forces the right side of (4.20) also to be > 0 by the preceding lemma. So by (4.16) there exists the *least* number $\tilde{s}_2$ in $[t_1, t_2]$ such that one of the following holds, viz.

$$(\tilde{T}\tilde{y})_i(t_1) < -(\tilde{X}\tilde{y})_i(\tilde{s}_2) = (\tilde{T}\tilde{y})_i(\tilde{s}_2) = (\tilde{T}\tilde{y})_i(t_2) \tag{4.21}$$

or

$$(\tilde{T}\tilde{y})_i(t_1) < -(\tilde{X}\tilde{y})_i(\tilde{s}_2-) = (\tilde{T}\tilde{y})_i(\tilde{s}_2) = (\tilde{T}\tilde{y})_i(t_2) \tag{4.22}$$

Note that $\tilde{s}_2$ could depend on i, but i is fixed for this discussion. Because of the strict inequality in the above and right continuity of the functions concerned we have $\tilde{s}_2 > t_1$. Similarly by the analogue of (4.16) for Ty, there exists the *least* number s_2 in $[t_1, t_2]$ such that one of the following holds, viz.

$$(Ty)_i(t_1) < -(Xy)_i(s_2) = (Ty)_i(s_2) = (Ty)_i(t_2) \tag{4.23}$$

or

$$(Ty)_i(t_1) < -(Xy)_i(s_2-) = (Ty)_i(s_2) = (Ty)_i(t_2) \tag{4.24}$$

Again, as before $s_2 > t_1$. By the last equality in (4.21) or (4.22) and the preceding lemma we get $(Ty)_i(t_2) = (Ty)_i(\tilde{s}_2)$. Therefore by minimality of s_2 it follows that

$$t_1 < s_2 \le \tilde{s}_2 \le t_2 \tag{4.25}$$

Now if $(\tilde{T}\tilde{y})_i(t_1) = 0$ the result follows by Lemma 4.2. So assume $(\tilde{T}\tilde{y})_i(t_1) > 0$. Let $\tilde{s}_1$ denote the least element in $[0, t_1]$ such that

$$0 < -(\tilde{X}\tilde{y})_i(\tilde{s}_1) = (\tilde{T}\tilde{y})_i(\tilde{s}_1) = (\tilde{T}\tilde{y})_i(t_1) \tag{4.26}$$

or

$$0 < -(\tilde{X}\tilde{y})_i(\tilde{s}_1-) = (\tilde{T}\tilde{y})_i(\tilde{s}_1) = (\tilde{T}\tilde{y})_i(t_1) \tag{4.27}$$

Observe that

$$0 \le \tilde{s}_1 \le t_1 < s_2 \le \tilde{s}_2 \le t_2 \tag{4.28}$$

We consider only the case when (4.22), (4.24), (4.27) hold; (the other seven cases can be treated analogously). We have

$$\begin{aligned}
(\tilde{T}\tilde{y})_i(t_2) - (\tilde{T}\tilde{y})_i(t_1) &= (\tilde{T}\tilde{y})_i(\tilde{s}_2) - (\tilde{T}\tilde{y})_i(\tilde{s}_1) \\
&\ge (\tilde{T}\tilde{y})_i(s_2) - (\tilde{T}\tilde{y})_i(\tilde{s}_1) \\
&\ge [-(\tilde{X}\tilde{y})_i(s_2-)] - [-(\tilde{X}\tilde{y})_i(\tilde{s}_1-)] \\
&= -[\tilde{w}_i(s_2-) - \tilde{w}_i(\tilde{s}_1-)] + \sum_{j\neq i} \int_{[\tilde{s}_1, s_2)} (-\tilde{r}_{ij}(u-))d\tilde{y}_j(u) \\
&\ge -[w_i(s_2-) - w_i(\tilde{s}_1-)] + \sum_{j\neq i} \int_{[\tilde{s}_1, s_2)} (-r_{ij}(u-))dy_j(u) \\
&= (Ty)_i(t_2) - [-(Xy)_i(\tilde{s}_1-)] \\
&\ge (Ty)_i(t_2) - (Ty)_i(t_1)
\end{aligned} \tag{4.29}$$

where we have used (4.28), (4.3), (4.7), (4.8), definitions of $Ty, \tilde{T}\tilde{y}$, besides (4.22), (4.24), (4.27). This completes the proof. □

Lemma 4.5 : Under set-up I, for $1 \leq i \leq d, t \geq 0$

$$(Sy)_i(t) \geq (\tilde{S}\tilde{y})_i(t) \tag{4.30}$$

Proof : Fix i, t. Consider first the case when $(\tilde{T}\tilde{y})_i(t) = 0$. Then $(Ty)_i(t) = 0$ by Lemma 4.2. So

$$\begin{aligned}(Sy)_i(t) - (\tilde{S}\tilde{y})_i(t) &= w_i(t) - \tilde{w}_i(t) + \sum_{j\neq i}\int_0^t (r_{ij}(u-) - \tilde{r}_{ij}(u-))dy_j(u) \\ &\quad + \sum_{j\neq i}\int_0^t (-\tilde{r}_{ij}(u-))d(\tilde{y}_j - y_j)(u) \\ &\geq 0 \end{aligned} \tag{4.31}$$

by (4.7), (4.8) and as $\tilde{w}_i(\cdot) \leq w_i(\cdot)$.

Now suppose $(\tilde{T}\tilde{y})_i(t) > 0$. Then there exists $t_0 \leq t$ such that

$$(\tilde{T}\tilde{y})_i(t) = -(\tilde{X}\tilde{y})_i(t_0) > 0 \tag{4.32}$$

or

$$(\tilde{T}\tilde{y})_i(t) = -(\tilde{X}\tilde{y})_i(t_0-) > 0 \tag{4.33}$$

We consider only the case when (4.33) holds. Since $[-(Xy)_i(t_0-)] \leq (Ty)_i(t)$, by (4.3), (4.7), (4.8), (4.13), (4.14), (4.33) we get

$$\begin{aligned} & (Sy)_i(t) - (\tilde{S}\tilde{y})_i(t) \\ \geq\ & [w_i(t) - w_i(t_0-)] - [\tilde{w}_i(t) - \tilde{w}_i(t_0-)] \\ & + \sum_{j\neq i}\int_{t_0}^t (r_{ij}(u-) - \tilde{r}_{ij}(u-))dy_j(u) + \sum_{j\neq i}\int_{t_0}^t (-\tilde{r}_{ij}(u-))d(\tilde{y}_j - y_j)(u) \\ \geq\ & 0, \end{aligned} \tag{4.34}$$

completing the proof. $\square$

Remark 4.6 : Let $w, \tilde{w}, b, \tilde{b}, R, \tilde{R}$ be as in Theorem 4.1. For $y \in D_{\uparrow 0}([0,\infty) : \bar{G}), z \in D([0,\infty) : \bar{G})$ let $T(y,z;w,b,R), S(y,z;w,b,R), T(y,z;\tilde{w},\tilde{b},\tilde{R})$, $S(y,z;\tilde{w},\tilde{b},\tilde{R})$ be defined using (3.6), (3.15)-(3.17). For notational convenience put

$T^{(1)}(y,z) = T(y,z;w,b,R), \qquad \tilde{T}^{(1)}(y,z) = T(y,z;\tilde{w},\tilde{b},\tilde{R}),$

$$S^{(1)}(y,z) = S(y,z;w,b,R), \qquad \tilde{S}^{(1)}(y,z) = S(y,z;\tilde{w},\tilde{b},\tilde{R}),$$

and for $n = 2, 3, \cdots$ set

$$T^{(n)}(y,z) = T(T^{(n-1)}y, S^{(n-1)}z; w, b, R),$$

$$S^{(n)}(y,z) = S(T^{(n-1)}y, S^{(n-1)}z; w, b, R),$$

$$\tilde{T}^{(n)}(y,z) = T(\tilde{T}^{(n-1)}y, \tilde{S}^{(n-1)}z; \tilde{w}, \tilde{b}, \tilde{R}),$$

$$\tilde{S}^{(n)}(y,z) = S(\tilde{T}^{(n-1)}y, \tilde{S}^{(n-1)}z; \tilde{w}, \tilde{b}, \tilde{R}).$$

Under the hypotheses of Theorem 4.1, with Lemmas 4.4 and 4.5 applied repeatedly we obtain for $1 \leq i \leq d, t \geq 0, 0 \leq t_1 \leq t_2$

$$(T^{(n)}(y,z))_i(t) \leq (\tilde{T}^{(n)}(y,z))_i(t) \tag{4.35}$$

$$(T^{(n)}(y,z))_i(t_2) - (T^{(n)}(y,z))_i(t_1) \leq (\tilde{T}^{(n)}(y,z))_i(t_2) - (\tilde{T}^{(n)}(y,z))_i(t_1) \tag{4.36}$$

$$(S^{(n)}(y,z))_i(t) \geq (\tilde{S}^{(n)}(y,z))_i(t) \tag{4.37}$$

for $n = 1, 2, 3, \cdots$ □

Proof of Theorem 4.1 : The proof is in a few steps roughly paralleling the steps in the proof of Theorem 3.7.

Step 1: We consider (SP) at $t = 0$; so the set up is as in Step 1 of Section 3. Define $\tau : \bar{G} \to \bar{G}, \quad \sigma : \bar{G} \to \bar{G}$ by

$$\begin{aligned}
(\tau y)_i &= \max\{0, -[w_i(0) + \sum_{j \neq i} r_{ij}(0,0,0) y_j]\} \\
(\sigma y)_i &= w_i(0) + \sum_{j \neq i} r_{ij}(0,0,0) y_j + (\tau y)_i
\end{aligned}$$

Similarly $\tilde{\tau}, \tilde{\sigma}$ are defined by

$$\begin{aligned}
(\tilde{\tau} y)_i &= \max\{0, -[\tilde{w}_i(0) + \sum_{j \neq i} \tilde{r}_{ij}(0,0,0) y_j]\} \\
(\tilde{\sigma} y)_i &= \tilde{w}_i(0) + \sum_{j \neq i} \tilde{r}_{ij}(0,0,0) y_j + (\tilde{\tau} y)_i
\end{aligned}$$

Set $\tau^{(n)}y = \tau(\tau^{(n-1)}y), \sigma^{(n)}y = \sigma(\sigma^{(n-1)}y)$, $\tilde{\tau}^{(n)}y = \tilde{\tau}(\tilde{\tau}^{(n-1)}y), \tilde{\sigma}^{(n)}y = \tilde{\sigma}(\tilde{\sigma}^{(n-1)}y)$, for $n = 2, 3, \cdots$ with of course $\tau^{(1)}y = \tau y, \sigma^{(1)}y = \sigma y$, $\tilde{\tau}^{(1)}y = \tilde{\tau}y, \tilde{\sigma}^{(1)}y = \tilde{\sigma}y$.

Observe that $\tau, \tilde{\tau}$ are the analogues of $T, \tilde{T}$ of (4.11), (4.12) when $w, \tilde{w}, R, \tilde{R}$ are constants and y is degenerate. As $R, \tilde{R}$ satisfy (A3), by a contraction mapping argument it is easily seen that $\tau^{(n)}y \to Yw(0), \tilde{\tau}^{(n)}y \to \tilde{Y}\tilde{w}(0)$, $\sigma^{(n)}y \to Zw(0), \tilde{\sigma}^{(n)}y \to \tilde{Z}\tilde{w}(0)$, as $n \to \infty$. By the proofs of Lemmas 4.2 - 4.5 and Remark 4.6 it follows that $\tau^{(n)}y \leq \tilde{\tau}^{(n)}y, \quad \sigma^{(n)}y \geq \tilde{\sigma}^{(n)}y$ componentwise for each n. Hence $Yw(0) \leq \tilde{Y}\tilde{w}(0)$ and $Zw(0) \geq \tilde{Z}\tilde{w}(0)$.

Step 2 : Now the set up is as in Steps 2-4 of Section 3. Let $T^{(n)}, S^{(n)}, \tilde{T}^{(n)}, \tilde{S}^{(n)}$ be as in Remark 4.6. By Step 4 of Section 3 there is $t > 0$ such that $T^{(n)}(y,z)(\cdot) + Yw(0) \to Yw(\cdot)$ in variation norm over $[0, s], S^{(n)}(y,z)(\cdot) \to Zw(\cdot)$ uniformly over $[0, s]$ for all $s < t$ for sufficiently large class of (y, z); (for example one can take $y(\cdot) \equiv 0, z(\cdot) \equiv 0$; this is because (T, S) is a contraction on $\hat{D}_s$). Similarly there is $\tilde{t} > 0$ such that $\tilde{T}^{(n)}(y,z)(\cdot) + \tilde{Y}\tilde{w}(0) \to \tilde{Y}\tilde{w}(\cdot)$ in variation norm and $\tilde{S}^{(n)}(y,z)(\cdot) \to \tilde{Z}\tilde{w}(\cdot)$ uniformly over $[0, s]$ for all $s < \tilde{t}$ for large class of (y, z). Take $t_0 = t \wedge \tilde{t}$. By Step 1 and Remark 4.6 it is now seen that the theorem holds on $[0, t_0)$.

Step 3 : With the theorem holding on $[0, t_0)$ we want to show that it holds on $[0, t_0]$; see Step 5 of Section 3. Analogous to Step 1, define $\tau, \tilde{\tau}$ on $\bar{G}$ by

$$\begin{aligned}
(\tau y)_i &= \max\{0, -[(Zw)_i(t_0-) + w_i(t_0) - w_i(t_0-) \\
&\quad + \sum_{j \neq i} r_{ij}(t_0, Yw(t_0-), Zw(t_0-))y_j]\} \\
(\tilde{\tau} y)_i &= \max\{0, -[(\tilde{Z}\tilde{w})_i(t_0-) + \tilde{w}_i(t_0) - \tilde{w}_i(t_0-) \\
&\quad + \sum_{j \neq i} \tilde{r}_{ij}(t_0, \tilde{Y}\tilde{w}(t_0-), \tilde{Z}\tilde{w}(t_0-))y_j]\}
\end{aligned}$$

Similarly $\sigma, \tilde{\sigma}$ can be defined with obvious modifications. As in Step 1, $\tau^{(n)}y := \tau(\tau^{(n-1)}y) \to Yw(t_0) - Yw(t_0-), \quad \sigma^{(n)}y := \sigma(\sigma^{(n-1)}y) \to Zw(t_0)$, $\tilde{\tau}^{(n)}y := \tilde{\tau}(\tilde{\tau}^{(n-1)}y) \to \tilde{Y}\tilde{w}(t_0) - \tilde{Y}\tilde{w}(t_0-), \tilde{\sigma}^{(n)}y := \tilde{\sigma}(\tilde{\sigma}^{(n-1)}y) \to \tilde{Z}\tilde{w}(t_0)$ as $n \to \infty$. By an argument as in Step 1 it is seen that the theorem holds on $[0, t_0]$.

Step 4 : To complete the proof, in view of Step 7 of Section 3, it just remains to show that the theorem holds on $[0, t_0 + \hat{t})$ for some $\hat{t} > 0$ if it holds on

$[0, t_0]$. For this as in Step 6 of Section 3 we consider the functions

$$b(u + t_0, y + Yw(t_0), z), \quad \tilde{b}(u + t_0, y + \tilde{Y}\tilde{w}(t_0), z)$$

$$R(u + t_0, y + Yw(t_0), z), \quad \tilde{R}(u + t_0, y + \tilde{Y}\tilde{w}(t_0), z)$$

as functions of (u, y, z). By the preceding step and our hypotheses, note that the situation is very similar to the one in Step 2; hence the proof can be completed proceeding as in Step 2. □

Remark 4.7 ; In addition to the hypotheses of Theorem 4.1, assume that $w_i(\cdot) = -h_i(\cdot), \tilde{w}_i(\cdot) = -\tilde{h}_i(\cdot)$, where $h_i, \tilde{h}_i$ are nonnegative nondecreasing functions, and that strict inequality holds in the first inequality of (4.2) or in (4.3). Then strict inequality holds in (4.4) for all $t > 0$ and in (4.5) for all $0 \leq t_1 < t_2$; this is easily seen from the proof. In such a case note that $Zw \equiv 0 \equiv \tilde{Z}\tilde{w}$. □

Following the terminology in the theory of ordinary differential equations, one can say that $\mathrm{SP}(w, b, R)$ is *stable* if Zw is attracted to the origin, that is, for every $\in > 0$ there is $M < \infty$ such that $|Zw(t)| < \in$ for all $t \geq M$. The next corollary is immediate from Theorem 4.1.

Corollary 4.8 : Assume the notation and hypotheses of Theorem 4.1. If $\mathrm{SP}(w, b, R)$ is stable then so is $\mathrm{SP}(\tilde{w}, \tilde{b}, \tilde{R})$. □

Another immediate corollary from (4.4), (4.5) is

Corollary 4.9: Under the hypotheses of Theorem 4.1, for $i = 1, 2, \cdots, d$ the measure $d(Yw)_i$ is dominated by $d(\tilde{Y}\tilde{w})_i$; that is, for any Borel set $A \subseteq [0, \infty)$,

$$\int_A d(Yw)_i(t) \leq \int_A d(\tilde{Y}\tilde{w})_i(t),$$

and the Radon-Nikodym derivative $\frac{d(Yw)_i}{d(\tilde{Y}\tilde{w})_i} \leq 1$. □

Remark 4.10 : In addition to the hypotheses of Theorem 4.1, suppose $b_i(\cdot, \cdot, \cdot) \leq 0, \quad w_i(\cdot) = -h_i(\cdot), \quad \tilde{w}_i(\cdot) = -\tilde{h}_i(\cdot)$ where $h_i(\cdot), \tilde{h}_i(\cdot)$ are nonnegative nondecreasing functions for $1 \leq i \leq d$. Note that $dh_i(\cdot) \leq d\tilde{h}_i(\cdot), 1 \leq i \leq d$. Let $\frac{dYw}{d\tilde{Y}\tilde{w}}(t)$ (resp. $\frac{dh}{d\tilde{h}}(t)$) denote the $(d \times d)$ diagonal matrix whose i-th diagonal enry is $\frac{d(Yw)_i}{d(\tilde{Y}\tilde{w})_i}(t)$ (resp. $\frac{dh_i}{d\tilde{h}_i}(t)$). Then by (3.36), (3.37) in Example

3.8 one can see that

$$
\begin{aligned}
& \int_s^t \frac{dYw}{d\tilde{Y}\tilde{w}}(u) d\tilde{Y}\tilde{w}(u) = Yw(t) - Yw(s) \\
= & \int_s^t R^{-1}(u, Yw(u-), 0) \frac{dh}{d\tilde{h}}(u) \tilde{R}(u, \tilde{Y}\tilde{w}(u-), 0) d\tilde{Y}\tilde{w}(u) \\
& + \int_s^t R^{-1}(u, Yw(u-), 0) \left[b^-(u, Yw(u-), 0) - \frac{dh}{d\tilde{h}}(u) \tilde{b}^-(u, \tilde{Y}\tilde{w}(u-), 0) \right] du
\end{aligned} \tag{4.38}
$$

In particular if $b(\cdots) \equiv 0 \equiv \tilde{b}(\cdots)$, and $R(u, y, 0)$ is independent of y, then

$$
\frac{dYw}{d\tilde{Y}\tilde{w}}(s) = R^{-1}(s, 0, 0) \frac{dh}{d\tilde{h}}(s) \tilde{R}(s, \tilde{Y}\tilde{w}(s-), 0) \tag{4.39}
$$

for almost all s w.r.t. $d\tilde{Y}\tilde{w}(\cdot)$. □

What happens if the condition of nonpositivity of $r_{ij}, \tilde{r}_{ij}, i \neq j$ in (4.2) is dropped? The following examples indicate that one cannot expect very general results.

Example 4.11 : Let $d = 2$ and $0 < q < \hat{q} < \alpha < 1 < \alpha^{-1}$; set $r_{11} = 1 = r_{22}$, $r_{12} = q = r_{21}$, $\hat{r}_{11} = 1 = \hat{r}_{22}$, $\hat{r}_{12} = \hat{q} = \hat{r}_{21}$, $w_1(t) = -f(t)$, $w_2(t) = -\alpha f(t)$, $t \geq 0$, where $f(\cdot)$ is a nonnegative nondecreasing function, $b(\cdots) \equiv 0 \equiv \hat{b}(\cdots)$, $\hat{w}(\cdot) = w(\cdot)$. Then $\mathrm{SP}(w, 0, R)$ is solved by $Yw(\cdot) = y(\cdot), Zw \equiv 0$ where

$$
y_1(t) = \frac{(1 - q\alpha)}{(1 - q^2)} f(t), \quad y_2(t) = \frac{(\alpha - q)}{(1 - q^2)} f(t).
$$

Similarly $\mathrm{SP}(w, 0, \hat{R})$ is solved by $\hat{Y}w(\cdot) = \hat{y}(\cdot), \hat{Z}w \equiv 0$ where

$$
\hat{y}_1(t) = \frac{(1 - \hat{q}\alpha)}{(1 - \hat{q}^2)} f(t), \quad \hat{y}_2(t) = \frac{(\alpha - \hat{q})}{(1 - \hat{q}^2)} f(t).
$$

It is easily seen that $y_1(t) \geq \hat{y}_1(t) \Leftrightarrow \alpha \geq (q + \hat{q})/(1 + q\hat{q})$ and $y_2(t) \geq \hat{y}_2(t) \Leftrightarrow \alpha^{-1} \geq (q + \hat{q})/(1 + q\hat{q})$. Observe that $(q + \hat{q})/(1 + q\hat{q}) < 1$. Consequently, if $\hat{q} < \alpha < (q + \hat{q})/(1 + q\hat{q})$ then $y_2(\cdot) > \hat{y}_2(\cdot)$ but $y_1(\cdot) < \hat{y}_1(\cdot)$. If $(q + \hat{q})/(1 + q\hat{q}) \leq \alpha \leq 1$ then $y_i(\cdot) \geq \hat{y}_i(\cdot), i = 1, 2$. □

Example 4.12 : Let $d = 3, q, \hat{q}, \alpha$ be as in Example 4.11, $0 < q_0 < 1$ and $f(\cdot), g(\cdot)$ be nonnegative nondecreasing functions. Set

$$R = \begin{pmatrix} 1 & q & 0 \\ q & 1 & 0 \\ q_0 & 0 & 1 \end{pmatrix}, \hat{R} = \begin{pmatrix} 1 & \hat{q} & 0 \\ \hat{q} & 1 & 0 \\ q_0 & 0 & 1 \end{pmatrix}, w(t) = \begin{pmatrix} -f(t) \\ -\alpha f(t) \\ g(t) \end{pmatrix}$$

Then $\mathrm{SP}(w, 0, R)$ is solved by $Yw(\cdot) = (y_1(\cdot), y_2(\cdot), 0), Zw(\cdot) = (0, 0, z_3(\cdot))$, and $\mathrm{SP}(w, 0, \hat{R})$ is solved by $\hat{Y}w(\cdot) = (\hat{y}_1(\cdot), \hat{y}_2(\cdot), 0), \hat{Z}w(\cdot) = (0, 0, \hat{z}_3(\cdot))$ where $y_1, y_2, \hat{y}_1, \hat{y}_2$ are as in Example 4.11 and $z_3(\cdot) = g(\cdot) + q_0 y_1(\cdot), \hat{z}_3(\cdot) = g(\cdot) + q_0 \hat{y}_1(\cdot)$. If $\hat{q} < \alpha < (q + \hat{q})/(1 + q\hat{q})$ then $\hat{z}_3(t) > z_3(t)$ for all t; but if $(q + \hat{q})/(1 + q\hat{q}) < \alpha \leq 1$, then $\hat{z}_3(t) < z_3(t)$ for all t. $\square$

5 Feasibility and minimality

Recall that the standard subsidy Yw can be mobilised only when the corresponding sector is empty. There can of course be situations when it may be advisable not to have such a restriction. This leads to the following natural definition.

Let (w, b, R) be as in Section 2. Let $(y, z) \in D_{\uparrow}([0, \infty) : \bar{G}) \times D([0, \infty) : \bar{G})$ be such that

$$\begin{aligned} z_i(t) = w_i(t) \quad &+ \int_0^t b_i(u, y(u-), z(u-))du + y_i(t) \\ &+ \sum_{j \neq i} \int_0^t r_{ij}(u, y(u-), z(u-))dy_j(u) \end{aligned} \tag{5.1}$$

is nonnegative for $1 \leq i \leq d, t \geq 0$; then y is called a *feasible subsidy*, z a *feasible surplus*, (y, z) a *feasible solution* corresponding to (w, b, R).

If b, R satisfy (A1), (A2), for given $w \in D([0, \infty) : \mathbb{R}^d), y \in D_{\uparrow}([0, \infty) : \bar{G})$ the system of integral equations (5.1) has a unique solution in $D[0, \infty)$; this can be seen by a Picard iteration; (moreover if w, y are continuous then so is the solution of (5.1)). For feasibility we demand that the solution be nonnegative in each coordinate. Clearly the solution to the Skorokhod problem is a feasible solution.

The next result shows that the Leontief output (corresponding to the worst possible demand) forms a feasible subsidy.

Theorem 5.1 : Let $w \in D([0,\infty) : I\!R^d)$, b, R satisfy (A1) - (A3); let V be as in (A3), h, β be given respectively by (3.3), (3.4). Then $(I-V)^{-1}(h+\beta)$ is a feasible subsidy for (w, b, R).

Proof : Define $\hat{R}(u,y,z) = ((\hat{r}_{ij}(u,y,z)))$ by $\hat{r}_{ij}(\cdot,\cdot,\cdot) = r_{ij}(\cdot,\cdot,\cdot), i \neq j, \hat{r}_{ii}(\cdots) \equiv 0$. Then we can write for any u, y, z

$$R(u,y,z) = (I-V) + (V + \hat{R}(u,y,z)) \tag{5.2}$$

Note that by (A3), $V + \hat{R}(u,y,z)$ is a nonnegative matrix.

Let $z(\cdot)$ be the unique solution for the vector integral equation

$$
\begin{aligned}
z(t) = w(t) \quad &+ \quad \int_0^t b(u, y(u-), z(u-))du \\
&+ \quad \int_0^t R(u, y(u-), z(u-))dy(u)
\end{aligned}
\tag{5.3}
$$

where $y(\cdot) = (I - V)^{-1}(h + \beta)$. By (5.2) we have

$$
\begin{aligned}
&\int_0^t R(u, y(u-), z(u-))dy(u) \\
&= \int_0^t [(I - V) + (V + \hat{R}(u, y(u-), z(u-)))]d(I - V)^{-1}(h + \beta)(u) \\
&\geq (h + \beta)(t)
\end{aligned}
\tag{5.4}
$$

From the definition of $(h + \beta)$ and (5.3), (5.4) it now follows that $z(\cdot)$ is nonnegative. □

Remark : In the case when $b \equiv 0, r_{ij}$ are constants with $r_{ij} \leq 0, i \neq j$ the preceding result is contained in the proof of the proposition in the appendix of Reiman (1984). □

Example 5.2 : Let r_{ij}, b_i be functions only of t, y (and independent of z), $r_{ij}(\cdot, \cdot) \leq 0, i \neq j$. For $w \in D([0, \infty) : I\!R^d)$ let h be given by (3.3). Let $b_i(\cdot, \cdot) = b_i^+(\cdot, \cdot) - b_i^-(\cdot, \cdot)$ and $b^-(\cdot, \cdot) = (b_1^-(\cdot, \cdot), \cdots, b_d^-(\cdot, \cdot))$. Assume (A1)-(A3). Moreover let $R^{-1}(s, y)$ be Lipschitz continuous in y, uniformly in s. Let $y(\cdot)$ be the solution of the vector integral equation

$$
\begin{aligned}
y(t) = h(0) \quad &+ \quad \int_0^t R^{-1}(s, y(s-))dh(s) \\
&+ \quad \int_0^t R^{-1}(s, y(s-))b^-(s, y(s-))ds
\end{aligned}
\tag{5.5}
$$

As in Example 3.8 we have $y \in D_{\uparrow}([0, \infty) : \bar{G})$. Now note that

$$
w_i(t) + \int_0^t b_i(s, y(s-))ds + y_i(t) + \sum_{j \neq i} \int_0^t r_{ij}(s, y(s-))dy_j(s)
$$

$$
= w_i(t) + \int_0^t b_i(s, y(s-))ds + h_i(t) + \int_0^t b_i^-(s, y(s-))ds \geq 0.
$$

Thus $y(\cdot)$ given by (5.5) is a feasible subsidy. □

A natural off-shoot of the notion of feasible subsidy is that of minimality. However in view of the comparison results of the preceding section (especially (4.4), (4.5)) four different notions of minimality seem possible; we list them all though only two seem to be amenable to analysis.

Let $\tilde{D}$ be a (closed) subset of $D_{\uparrow}([0,\infty) : \bar{G})$. Let $y^{(0)}(\cdot) = (y_1^{(0)}(\cdot), \cdots, y_d^{(0)}(\cdot)) \in \tilde{D}$ be a feasible subsidy for (w, b, R). We say

(i) $y_i^{(0)}(\cdot)$ is *I - minimal* in $\tilde{D} \Leftrightarrow y_i^{(0)}(0) \leq y_i(0), y_i^{(0)}(t_2) - y_i^{(0)}(t_1) \leq y_i(t_2) - y_i(t_1)$ whenever $0 \leq t_1 \leq t_2, 1 \leq i \leq d$ for any feasible subsidy $y \in \tilde{D}$;

(ii) $y^{(0)}(\cdot)$ is *II - minimal* in $\tilde{D} \Leftrightarrow y_i^{(0)}(t) \leq y_i(t), t \geq 0, 1 \leq i \leq d$ for any feasible subsidy $y \in \tilde{D}$;

(iii) $y^{(0)}(\cdot)$ is *III - minimal* in $\tilde{D} \Leftrightarrow y \in \tilde{D}$ is a feasible subsidy such that $y_i(t) \leq y_i^{(0)}(t)$ for all $t \geq 0, 1 \leq i \leq d$ implies $y \equiv y^{(0)}$.

(iv) $y^{(0)}(\cdot)$ is *IV - minimal* in $\tilde{D} \Leftrightarrow y \in \tilde{D}$ is a feasible subsidy such that $y_i(0) \leq y_i^{(0)}(0), y_i(t_2) - y_i(t_1) \leq y_i^{(0)}(t_2) - y_i^{(0)}(t_1)$ for all $0 \leq t_1 \leq t_2, 1 \leq i \leq d$ implies $y \equiv y^{(0)}$. $\square$

From the definitions the following implications among the minimality notions are clear : $I \Rightarrow II \Rightarrow III \Rightarrow IV$. (For example suppose $y^{(0)}$ is III-minimal. Let y be a feasible subsidy such that $y_i(t_2) - y_i(t_1) \leq y_i^{(0)}(t_2) - y_i^{(0)}(t_1)$ for all $0 \leq t_1 \leq t_2$ and $y_i(0) \leq y_i^{(0)}(0)$. Then $y_i(t) \leq y_i^{(0)}(t) \;\; \forall t$, and by III - minimality $y \equiv y^{(0)}$. Thus $y^{(0)}$ is IV - minimal.) Also uniqueness is obvious in case of I or II minimality.

An immediate question is the connection between standard subsidy and minimal feasible subsidy. We begin with a result connecting II - minimality and (SP) which generalises a result of Reiman (1984); see also Chen and Mandelbaum (1991).

Theorem 5.3 : In addition to (A1) - (A3) assume the following :

(i) $b_i(\cdot, \cdot)$ are functions of (t, y) (independent of z) such that

$$b_i(t, \hat{y}) \geq b_i(t, y) \tag{5.6}$$

whenever $\hat{y} \leq y$ for $t \geq 0, 1 \leq i \leq d$;

(ii) $r_{ij}(\cdot)$ are functions of t (independent of y, z) such that $t \mapsto r_{ij}(t)$ is nondecreasing, and $r_{ij}(t) \leq 0, i \neq j$.

Let $w \in D([0,\infty) : I\!R^d)$ be such that $w(0) \in \bar{G}$. Then Yw is the unique II-minimal element in $D_{\uparrow 0}([0,\infty) : \bar{G})$. □

For proving we need the following lemmas

Lemma 5.4 : Let $r_{ij}(\cdot)$ be as in the theorem. Let $y, \hat{y} \in D_{\uparrow 0}$ be such that $\hat{y}_i(t) \leq y_i(t)$ for all $t \geq 0, 1 \leq i \leq d$. Then

$$-\int_0^t r_{ij}(u) d\hat{y}_j(u) \leq -\int_0^t r_{ij}(u) dy_j(u) \tag{5.7}$$

Proof : As $r_{ij}(\cdot)$ is monotone, Stieltjes integral w.r.t $dr_{ij}(\cdot)$ is meaningful; so the lemma can be proved by integration by parts. □

Lemma 5.5 : Let b, R be as in Theorem 5.3. If $y', y" \in D_{\uparrow 0}$ are feasible then so is $\hat{y}(\cdot)$ given by $\hat{y}_i(t) = \min\{y'_i(t), y"_i(t)\}$.

Proof : For $t \geq 0$, let $\mathcal{I}(t) = \{1 \leq i \leq d : y'_i(t) \leq y"_i(t)\}$ and $\mathcal{J}(t) = \{1 \leq i \leq d : y"_i(t) < y'_i(t)\}$. For $i \in \mathcal{I}(t)$, by integration by parts and proceeding as in the proof of Proposition 1 of the Appendix of Reiman ((1984), p. 457), we get

$$\begin{aligned}
\hat{y}_i(t) \quad &+ \quad \sum_{j \neq i} \int_0^t r_{ij}(u) d\hat{y}_j(u) \\
= y'_i(t) \quad &+ \quad \sum_{\substack{j \in \mathcal{I}(t) \\ j \neq i}} r_{ij}(t) y'_j(t) + \sum_{j \in \mathcal{J}(t)} r_{ij}(t) y"_j(t) - \sum_{j \neq i} \int_0^t \hat{y}_j(u) dr_{ij}(u) \\
\geq y'_i(t) \quad &+ \quad \sum_{\substack{j \in \mathcal{I}(t) \\ j \neq i}} r_{ij}(t) y'_j(t) + \sum_{j \in \mathcal{J}(t)} r_{ij}(t) y'_j(t) - \sum_{j \neq i} \int_0^t y'_j(u) dr_{ij}(u) \\
= y'_i(t) \quad &+ \quad \sum_{j \neq i} \int_0^t r_{ij}(u) dy'_j(u)
\end{aligned} \tag{5.8}$$

Consequently by feasibility of $y'(\cdot)$ and (5.6) we have

$$w_i(t) + \int_0^t b_i(u, \hat{y}(u-)) du + \hat{y}_i(t) + \sum_{j \neq i} \int_0^t r_{ij}(u) d\hat{y}_j(u)$$

$$
\begin{aligned}
&\geq \quad w_i(t) + \int_0^t b_i(u, y'(u-))du + y_i'(t) + \sum_{j \neq i} \int_0^t r_{ij}(u) dy_j'(u) \\
&\geq \quad 0 \qquad (5.9)
\end{aligned}
$$

for $i \in \mathcal{I}(t)$. By symmetry, analogue of (5.9) holds also for $i \in \mathcal{J}(t)$. □

Proof of theorem : Let $y \in D_{\uparrow 0}$ be a feasible subsidy. In view of Theorem 5.1 and Lemma 5.5 we may assume $y \leq (I - V)^{-1}(h + \beta)$. By (5.1) and definition of feasibility we get

$$
\begin{aligned}
&y_i(t) \geq (Ty)_i(t) \qquad (5.10) \\
&:= \sup_{0 \leq s \leq t} \max\{0, -[w_i(s) + \int_0^s b_i(u, y(u-))du + \sum_{j \neq i} \int_0^s r_{ij}(u) dy_j(u)]\}
\end{aligned}
$$

Therefore by Lemma 5.4 and (5.6) we get $T^2 y := T(Ty) \leq Ty \leq y$. Applying T repeatedly we get that $Yw(t) \leq y(t)$ for $0 \leq t < t_1$ where t_1 is as in Step 5 of Section 3. A similar iterative procedure for the linear complementarity problem (3.30)-(3.31) gives the result for all $t \in [0, t_1]$. Repeating this procedure and using the set up as in Steps 6 and 7 of Section 3 one can complete the proof. □

Remark 5.6 : It is not clear if the preceding result can be significantly improved even when $r_{ij}(\cdot, \cdot)$ is a nonpositive function of t, y (and independent of z) for $i \neq j$. To get an analogue of (5.7) for $y, \tilde{y} \in D_{\uparrow 0}$ with $y(\cdot) \leq \tilde{y}(\cdot)$, proceeding formally one wants the following chain :

$$
\begin{aligned}
-\int_0^t & r_{ij}(u, y(u)) dy_j(u) \leq -\int_0^t r_{ij}(u, \tilde{y}(u)) dy_j(u) \\
&= \quad -r_{ij}(t, \tilde{y}(t)) y_j(t) + \int_0^t y_j(u) d(r_{ij}(u, \tilde{y}(u))) \\
&\leq \quad -r_{ij}(t, \tilde{y}(t)) \tilde{y}_j(t) + \int_0^t \tilde{y}_j(u) d(r_{ij}(u, \tilde{y}(u))) \\
&= \quad -\int_0^t r_{ij}(u, \tilde{y}(u)) d\tilde{y}_j(u) \qquad (5.11)
\end{aligned}
$$

The first inequality follows if $r_{ij}(u, \tilde{\xi}) \leq r_{ij}(u, \xi)$ for $\xi \leq \tilde{\xi}$, whereas the second inequality follows if $r_{ij}(\cdot, \cdot)$ is nondecreasing in u and $r_{ij}(u, \tilde{\xi}) \geq r_{ij}(u, \xi)$ for $\xi \leq \tilde{\xi}$.

Similar problems seem to arise if r_{ij}, b_i depend on z as well. □

Remark 5.7 : Easy examples can be given even in one dimension to show that II-minimality need not imply I - minimality; the latter may be too strong a requirement (when $\tilde{D}$ is sufficiently large) which might hold only when $y \equiv 0$ is a feasible subsidy. $\square$

We need the following result on feasible solutions to establish a connection between (SP) and IV - minimality.

Proposition 5.8 : In addition to (A1) - (A3) assume that

$$b_i(t, \bar{y}, \bar{z}) \geq b_i(t, y, z) \tag{5.12}$$

$$r_{ij}(t, \bar{y}, \bar{z}) \geq r_{ij}(t, y, z) \geq 0 \tag{5.13}$$

whenever $\bar{y} \geq y, \bar{z} \geq z$ for all $t \geq 0$. Let $(y(\cdot), z(\cdot))$ be a feasible solution corresponding to (w, b, R). Let $\eta(\cdot) \in D_{\uparrow}([0, \infty) : \bar{G})$ and set $\bar{y}(\cdot) = y(\cdot) + \eta(\cdot)$. Let $\bar{z}(\cdot)$ denote the solution of the integral equation (5.1) with $y(\cdot)$ replaced by $\bar{y}(\cdot)$. Then $(\bar{y}(\cdot), \bar{z}(\cdot))$ is also a feasible solution for (w, b, R). Moreover $\bar{z}(\cdot) \geq z(\cdot)$, and in fact $\bar{z}_i(t) > z_i(t)$ if $\eta_i(t) > 0$.

Proof : For $\xi \in D([0, \infty) : I\!R^d)$ define

$$\begin{aligned} \bar{S}\xi(t) &= w(t) + \int_0^t b(u, \bar{y}(u-), \xi(u-))du \\ &\quad + \int_0^t R(u, \bar{y}(u-), \xi(u-))d\bar{y}(u) \end{aligned} \tag{5.14}$$

Note that $(\bar{S})^n \xi \to \bar{z}$ in sup norm in $D([0, t] : I\!R^d)$ for any ξ, and t. In particular $(\bar{S})^n z \to \bar{z}$.

For $1 \leq i \leq d, t \geq 0$ by (5.12), (5.13) observe that

$$\begin{aligned} (\bar{S}z)_i(t) - z_i(t) &= \int_0^t (b_i(u, \bar{y}(u-), z(u-)) - b_i(u, y(u-), z(u-)))du \\ &\quad + \bar{y}_i(t) - y_i(t) + \sum_{j \neq i} \int_0^t r_{ij}(u, \bar{y}(u-), z(u-))d\bar{y}_j(u) \\ &\quad - \sum_{j \neq i} \int_0^t r_{ij}(u, y(u-), z(u-))dy_j(u) \\ &\geq \eta_i(t) + \sum_{j \neq i} \int_0^t r_{ij}(u, \bar{y}(u-), z(u-))d\eta_j(u) \end{aligned}$$

$$
\begin{aligned}
& +\sum_{j\neq i}\int_0^t (r_{ij}(u,\bar{y}(u-),z(u-)) - r_{ij}(u,y(u-),z(u-)))dy_j(u) \\
\geq \quad & 0 \qquad (5.15)
\end{aligned}
$$

Now by induction on n, (5.12), (5.13) we get

$$
\begin{aligned}
&((\bar{S})^{n+1}z)_i(t) - ((\bar{S})^n z)_i(t) \\
&= \int_0^t (b_i(u, \bar{y}(u-), (\bar{S})^n z(u-)) - b_i(u, \bar{y}(u-), (\bar{S})^{n-1} z(u-)))du \\
&\quad + \sum_{j \neq i} \int_0^t (r_{ij}(u, \bar{y}(u-), (\bar{S})^n z(u-)) - r_{ij}(u, \bar{y}(u-), (\bar{S})^{n-1} z(u-)))d\bar{y}_j(u) \\
&\geq 0 \qquad (5.16)
\end{aligned}
$$

It is now clear that $(\bar{y}(\cdot), \bar{z}(\cdot))$ is feasible. Finally, if $\eta_i(t) > 0$ for some i, t then strict positivity holds in (5.15), and as $(\bar{S})^n z \uparrow \bar{z}$ it follows that $\bar{z}_i(t) > z_i(t)$. □

Corollary 5.9 : Under the hypotheses of the preceding theorem, any feasible solution can be approximated strictly from above by feasible solutions. □

Theorem 5.10 : Assume (A1) - (A3), (5.12), (5.13). Let (Yw, Zw) solve SP(w, b, R). Then Yw is IV-minimal in $D_\uparrow([0,\infty) : \bar{G})$ for (w, b, R).

Proof : Let $(y(\cdot), z(\cdot))$ be a feasible solution such that $y_i(0) \leq (Yw)_i(0)$, $y_i(t_2) - y_i(t_1) \leq (Yw)_i(t_2) - (Yw)_i(t_1)$ for any $0 \leq t_1 \leq t_2, 1 \leq i \leq d$. Set $\eta_i(\cdot) = (Yw)_i(\cdot) - y_i(\cdot)$. Note that $\eta(\cdot) \in D_\uparrow([0,\infty) : \bar{G})$. We need to show that $\eta(\cdot) \equiv 0$.

Fix $1 \leq i \leq d$. Let $t_0 = \inf\{t : \eta_i(t) > 0\}$. If $t_0 = \infty$ there is nothing to prove; so $t_0 < \infty$. Note that $\eta_i(s) > 0$ for all $s > t_0$. Now by the preceding proposition $(Zw)_i(s) > z_i(s) \geq 0$ for all $s > t_0$. Consequently $(Yw)_i$ can not increase on (t_0, ∞), and hence by right continuity $(Yw)_i(s) = (Yw)_i(t_0)$ for all $s \geq t_0$. Therefore $y_i(s) = y_i(t_0)$ for all $s \geq t_0$.

If $\eta_i(t_0) = 0$, it now follows that $\eta_i(s) = 0$ for $s \geq t_0$ contradicting $t_0 < \infty$.

So suppose $\eta_i(t_0) > 0$. By definition of t_0 note that $\eta_i(t_0-) = 0$. Also by the preceding proposition $(Zw)_i(t_0) > z_i(t_0) \geq 0$; so $(Yw)_i$ can not increase at t_0 and hence $(Yw)_i(t_0) = (Yw)_i(t_0-)$. So we now get

$$
\begin{aligned}
0 < \eta_i(t_0) &= (Yw)_i(t_0) - y_i(t_0) = (Yw)_i(t_0-) - y_i(t_0) \\
&= y_i(t_0-) - y_i(t_0) \leq 0
\end{aligned}
$$

which is a contradiction.

Thus $t_0 = \infty$ completing the proof. $\square$

In the converse direction we have the following.

Theorem 5.11 : In addition to (A1) - (A3), assume that b_i, r_{ij} are independent of z and that

$$b_i(t,y) \le b_i(t,\tilde{y}) \tag{5.17}$$

$$r_{ij}(t,y) \le r_{ij}(t,\tilde{y}) \le 0, \quad i \ne j \tag{5.18}$$

whenever $0 \le \tilde{y} \le y$. If $y(\cdot)$ is the II-minimal subsidy for (w,b,R) in $D_\uparrow([0,t]:\bar{G})$ for all $t \ge 0$, then $y(\cdot) = Yw(\cdot)$.

Note: Existence of II-minimal subsidy is not asserted in the preceding theorem.

Proof : Let $z(\cdot)$ denote the solution of the integral equation (5.1) corresponding to $y(\cdot)$. As $(y(\cdot), z(\cdot))$ is feasible it just remains to show that $y_i(\cdot)$ can increase only when $z_i(\cdot)$ is zero.

If possible let there be $1 \le i \le d, t_0 \ge 0, \delta > 0$ such that $\inf\{z_i(s) : t_0 \le s \le t_o + \delta\} > 0, \quad y_i(t_0+\delta) > y_i(t_0)$. Choose $\eta(\cdot) \in D_\uparrow([0,\infty) : \bar{G})$ such that $\eta_j(\cdot) \equiv 0, j \ne i, \quad \eta_i(t) = 0, t < t_0, \quad 0 < \eta_i(t_0+\delta) < \inf\{z_i(s) : t_0 \le s \le t_0+\delta\}$, and such that $\tilde{y}(\cdot) := y(\cdot) - \eta(\cdot)$ is in $D_\uparrow([0,\infty) : \bar{G})$. Let $\tilde{z}(\cdot)$ be the solution of (5.1) with $y(\cdot)$ replaced by $\tilde{y}(\cdot)$. By (5.17), (5.18) we have

$$\begin{aligned} z_i(t) - \tilde{z}_i(t) &= \int_{t_0}^{t} (b_i(u, y(u-)) - b_i(u, \tilde{y}(u-)))du \\ &\quad + \eta_i(t) + \sum_{j \ne i} \int_{t_0}^{t} (r_{ij}(u, y(u-)) - r_{ij}(u, \tilde{y}(u-)))dy_j(u) \\ &\le \eta_i(t) \end{aligned} \tag{5.19}$$

for $t_0 \le t \le t_0 + \delta$; and for $k \ne i$ similarly

$$\begin{aligned} z_k(t) - \tilde{z}_k(t) &= \int_{t_0}^{t} (b_k(u, y(u-)) - b_k(u, \tilde{y}(u-)))du \\ &\quad + \sum_{j \ne k} \int_{t_0}^{t} (r_{kj}(u, y(u-)) - r_{kj}(u, \tilde{y}(u-)))dy_j(u) \\ &\quad + \int_{t_0}^{t} r_{ki}(u, \tilde{y}(u-))d\eta_i(u) \\ &\le 0 \end{aligned} \tag{5.20}$$

From (5.19), (5.20) it follows that $\tilde{z}(\cdot) \geq 0$ on $[0, t_0 + \delta]$ and hence $(\tilde{y}(\cdot), \tilde{z}(\cdot))$ is a feasible solution on $[0, t_0 + \delta]$.

As $\tilde{y}(\cdot) \not\equiv y(\cdot)$ and $\tilde{y}(\cdot) \leq y(\cdot)$ this contradicts minimality of $y(\cdot)$ on $[0, t_0 + \delta]$. This completes the proof. $\square$

Remark 5.12 : By (5.10) we see that $y \geq Ty$ whenever y is feasible (even if b, R do not satisfy the additional assumptions of Theorem 5.3). Now let $t_1 > 0$ be such that (T, S) is a contraction on $\hat{D}_{t_1}$ (as in Step 4 of Section 3). Let $\tilde{D}_{t_1} = \{y \in D_{\uparrow 0}([0, t_1] : \bar{G}) : y$ is a feasible subsidy for (w, b, R) on $[0, t_1]$ and $(T^2 y)_i(t_2) - (T^2 y)_i(t_1) \leq (Ty)_i(t_2) - (Ty)_i(t_1)$ for all $0 \leq t_1 \leq t_2\}$

Then under the hypotheses of Theorem 4.1 it can be seen that Yw is II-minimal for (w, b, R) in $\tilde{D}_{t_1}$ (on $[0, t_1]$); this follows by Remark 4.6. Though $\tilde{D}_{t_1}$ is nonempty (as $Yw \in \tilde{D}_{t_1}$) it is not clear how large $\tilde{D}_{t_1}$ is. In this case observe that $T^k y$ is feasible on $[0, t_1]$ for each k and $T^k y \downarrow Yw$ for any $y \in \tilde{D}_{t_1}$. $\square$.

From the definitions it is clear that the II-minimal subsidy can not be improved in *any* sector; on the other hand a IV-minimal subsidy can not be beaten in *all* sectors simultaneously (in the sense of domination as in Corollary 4.9). We present below an example which shows that the standard subsidy can be bettered in *some* sectors. This example (in view of Theorem 5.10), shows also that IV-minimality does not imply II-minimality, and that the condition $r_{ij}(\cdot) \leq 0, i \neq j$ in Theorem 5.3 can not be dropped.

Example 5.13 : Take $d = 2, b \equiv 0, r_{12} > 0, r_{21} > 0$ with $r_{12} r_{21} < 1, w_1(t) = t, w_2(t) = -t$. Then (5.1) becomes

$$\begin{aligned} z_1(t) &= t + y_1(t) + r_{12} y_2(t) \\ z_2(t) &= -t + y_2(t) + r_{21} y_1(t) \end{aligned} \tag{5.21}$$

In this case $(Yw)_1(\cdot) \equiv 0, (Yw)_2(t) = t, (Zw)_1(t) = (1 + r_{12})t, (Zw)_2(\cdot) \equiv 0$.

Let $\beta_1 \geq 0, \beta_2 \geq 0$ be such that $\beta_2 + r_{21}\beta_1 \geq 1$. Put $y_1(t) = \beta_1 t, y_2(t) = \beta_2 t$; by (5.21) it is clear that $y(\cdot) = (y_1(\cdot), y_2(\cdot))$ is a feasible subsidy. Choosing $0 \leq \beta_2 < 1$ and β_1 appropriately large it is clear that the standard subsidy can be improved in Sector 2, but at the cost of increasing the subsidy in the other sector. $\square$

6 Stochastic case

In this section we consider briefly the situation when the exogeneous net evolution is not deterministic but is given by a Levy process; existence of a unique solution for the corresponding stochastic Skorokhod problem can be read off from Theorem 3.7.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, P)$ be a filtered probability space; let $\{B(t) : t \geq 0\}$ be a d-dimensional $\mathcal{F}_t$-adapted Levy process. (We assume that $\mathcal{F}_0$ contains all P-null sets). Let the drift function $b(\cdot,\cdot,\cdot)$ and reflection function $R(\cdot,\cdot,\cdot)$ be as in Section 2. We seek processes $Y(\cdot,\cdot)$ and $Z(\cdot,\cdot)$ such that the following hold:

(i) $\{Z(t) : t \geq 0\}$ is an $\mathcal{F}_t$-adapted r.c.l.l. process taking values in $\bar{G}$;

(ii) $Y(\cdot,\omega) \in D_{\uparrow}([0,\infty) : \bar{G})$ for a.a. $\omega \in \Omega$; $Y(t)$ is $\mathcal{F}_t$-adapted;

(iii) for $1 \leq i \leq d, t \geq 0$, for a.a.$\omega$,

$$Y_i(t,\omega) = \int_0^t I_{\{0\}}(Z_i(s,\omega))dY_i(s,\omega); \tag{6.1}$$

(iv) Skorokhod equation is satisfied, viz. for $P-$a.a.ω, $1 \leq i \leq d, t \geq 0$,

$$\begin{aligned} Z_i(t,\omega) &= B_i(t,\omega) + \int_0^t b_i(u, Y(u-,\omega), Z(u-,\omega))du \qquad (6.2)\\ &+ Y_i(t,\omega) + \sum_{j \neq i} \int_0^t r_{ij}(u, Y(u-,\omega), Z(u-,\omega))dY_j(u,\omega) \end{aligned}$$

As sample paths of $B(\cdot,\cdot)$ are r.c.l.l., from the methods of Section 3 it is clear that, under (A1)- (A3), there exists a unique pair Y, Z of $\mathcal{F}_t$-adapted r.c.l.l. processes satisfying (i) - (iv) above. It is also clear that analogues of (3.33), (3.34) hold.

Theorem 6.1 : Assume (A1)-(A3). Let $\{B(t) : t \geq 0\}$ be a d-dimensional $\mathcal{F}_t$-adapted Levy process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Then there exists a unique pair Y, Z of $\mathcal{F}_t$-adapted r.c.l.l. processes solving the stochastic Skorokhod problem corresponding to $(B(\cdot), b, R)$; also

$$Y_i(t,\omega) \leq ((I - V)^{-1}(L + \beta))_i(t,\omega) \tag{6.3}$$

$$Z_i(t,\omega) \leq B_i^+(t,\omega) + \gamma_i(t) + ((I+V)(I-V)^{-1}(L+\beta))_i(t,\omega) \qquad (6.4)$$

where V, β, γ_i are as in Theorem 3.7, $(L+\beta)$ is defined componentwise by

$$(L+\beta)_i(t,\omega) = L_i(t,\omega) + \beta_i(t) \qquad (6.5)$$

$$L_i(t,\omega) = \sup_{0\leq s\leq t} \max\{0, -B_i(s,\omega)\} \qquad (6.6)$$

Moreover $\{(Y(t), Z(t)) : t \geq 0\}$ is an $\mathcal{F}_t$-adapted $\bar{G} \times \bar{G}$-valued Feller continuous strong Markov process. Further Y, Z are continuous if B is.

Proof : Only the penultimate assertion needs to be proved. Let $t_1 \geq 0$ be fixed. For $t > t_1$ note that for a.a.ω

$$\begin{aligned} Z_i(t,\omega) \quad &= \quad Z_i(t_1,\omega) + B_i(t,\omega) - B_i(t_1,\omega) \\ &\quad + \int_{t_1}^{t} \hat{b}_i(u, \hat{Y}(u-,\omega), Z(u-,\omega))du + \hat{Y}_i(t,\omega) \\ &\quad + \sum_{j\neq i} \int_{t_1}^{t} \hat{r}_{ij}(u, \hat{Y}(u-,\omega), Z(u-,\omega))d\hat{Y}_j(u,\omega) \end{aligned}$$

where $\hat{Y}(u) = Y(u) - Y(t_1), \hat{b}_i(u,\hat{y},z) = b_i(u,\hat{y}+Y(t_1,\omega),z), \hat{r}_{ij}(u,\hat{y},z) = r_{ij}(u,\hat{y}+Y(t_1,\omega),z)$. By our proof of Theorem 3.7 it is clear that $(\hat{Y}(t), Z(t))$ is measurable w.r.t. $\sigma\{Y(t_1), Z(t_1), B(s) - B(t_1) : s \geq t_1\}$, and hence $(Y(t), Z(t))$ is also $\sigma\{Y(t_1), Z(t_1), B(s) - B(t_1) : s \geq t_1)\}$- measurable for any $t \geq t_1$. The Markov property of $(Y(t), Z(t))$ now follows. Feller continuity and hence strong Markov property are consequences of Proposition 3.9. $\square$.

Note : If b and R are independent of y, then it is clear that $\{Z(t) : t \geq 0\}$ is also a strong Markov process. Markov property of $(Y(t), Z(t))$ is a generalisation of Exercise (2.18) of Chapter VI of Revuz and Yor (1991), p. 227; (see also Exercise (1.13) of Chapter X, p. 377 of the same reference).

Corollary 6.2 : Let $b_i(\cdots) \leq 0$ for all $i, r_{ij}(\cdots) \leq 0$ for all $i \neq j$, and $\{B(t) : t \geq 0\}$ be the d-dimensional Brownian motion process. Then $Y_i(t,\omega) \to \infty$ as $t \to \infty$ for a.a.ω for all i.

Proof : If B is the Brownian motion note that L_i is the Brownian local time at 0 for each i and $L_i \uparrow \infty$ almost surely. Observe that $(L, B+L)$ solves the stochastic Skorokhod problem for (B, O, I) (zero drift and normal reflection). The required conclusion follows now by (4.4) of Theorem 4.1. $\square$

For the remainder of the section we shall assume that $\{B(t) : t \geq 0\}$ is a d-dimensional Brownian motion; in such a case Y and Z are continuous processes.

Remark 6.3 : Let $\{B(t) : t \geq 0\}$ be an $\mathcal{F}_t$-adapted d-dimensional Brownian motion; let $(Y(\cdot), Z(\cdot))$ be the solution of the stochastic (SP) corresponding to $(B(\cdot), b, R)$. For any sufficiently smooth function f on $[0, \infty) \times \bar{G} \times \bar{G}$ it follows by Ito's formula that

$$\begin{aligned} f(t, Y(t), Z(t)) &- \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_z + b(s, Y(s), Z(s)) \cdot \nabla_z \right) f(s, Y(s), Z(s)) ds \\ - \int_0^t &((\nabla_y f)(s, Y(s), Z(s)) + R(s, Y(s), Z(s))(\nabla_z f)(s, Y(s), Z(s))) \cdot dY(s) \\ &= \text{an } \ \mathcal{F}_t - \ \text{ adapted martingale} \end{aligned} \tag{6.7}$$

where the subscript in the differential operators indicate the variables w.r.t. which differentiation is applied. Consequently, if b and R do not depend on t and A denotes the infinitesimal generator of the continuous Markov process $\{(Y(t), Z(t)) : t \geq 0\}$ then

$$Af(y, z) = \frac{1}{2}\Delta_z f(y, z) + b(y, z) \cdot \nabla_z f(y, z), \tag{6.8}$$

$y, z \in \bar{G}$ for any twice continuously differentiable function f such that

$$\nabla_y f(y, z) + R(y, z)\nabla_z f(y, z) = 0, z \in \partial G, y \in \bar{G} \tag{6.9}$$

Note that we have used the fact that $dY(\cdot)$ can charge only when $Z(\cdot) \in \partial G$. $\square$

Let b, R and $\tilde{b}, \tilde{R}$ be as in Theorem 4.1. Let $(Y(\cdot), Z(\cdot))$ (resp. $(\tilde{Y}(\cdot), \tilde{Z}(\cdot))$) solve the stochastic (SP) corresponding to $(B(\cdot), b, R)$ (resp. $(B(\cdot), \tilde{b}, \tilde{R})$). Denote

$$\tau = \inf\{t \geq 0 : |Z(t)| \leq 1\} \tag{6.10}$$
$$\tilde{\tau} = \inf\{t \geq 0 : |\tilde{Z}(t)| \leq 1\} \tag{6.11}$$

In view of Theorem 4.1 the following is immediate.

Proposition 6.4 : Assume the hypotheses of Theorem 4.1 concerning b, R and $\tilde{b}, \tilde{R}$. Let $B(\cdot)$ denote a d-dimensional Brownian motion. Then

$$P(\tau < \infty | Z(0) = x) \leq P(\tilde{\tau} < \infty | \tilde{Z}(0) = x) \tag{6.12}$$
$$E(\tau | Z(0) = x) \geq E(\tilde{\tau} | \tilde{Z}(0) = x) \tag{6.13}$$

for any $x \in \bar{G}, |x| > 1$. In fact if the processes $Z(\cdot)$ and $\tilde{Z}(\cdot)$ start from the same point then $\tau \geq \tilde{\tau}$ a.s. $\square$

Under (A1) - (A3), using (6.7), it is not difficult to prove that the process $Z(\cdot)$ exits from a bounded set in finite time with probability one; (cf. Balaji and Ramasubramanian (1998), Ramasubramanian (1986)). Therefore, even though $Z(\cdot)$ may not be Markov, the condition $P(\tau < \infty) = 1$ can be taken to be equivalent to recurrence of $Z(\cdot)$; (cf. Ramasubramanian (1983)).

Now (6.12) implies that (under the assumptions of Proposition 6.4) if $Z(\cdot)$ is recurrent then $\tilde{Z}(\cdot)$ is also recurrent. This is quite intuitive; if a process in an orthant with reflection at the boundary is recurrent then so is any other process with more "inward looking" reflection and drift.

Similar comments concerning (6.13) vis-a-vis positive recurrence also apply.

Sufficient conditions for recurrence, positive recurrence, etc. when b, R are constants are given in Dupuis and Williams (1994) in terms of stability of deterministic (SP), in Chen (1996) in terms of certain algebraic conditions; when b, R depend only on z sufficient conditions in terms of Lyapunov functions are given in Balaji and Ramasubramanian (1997, 1998).

7 Assorted comments

We conclude by mentioning a few interesting questions, along with some comments.

1. Let $w, \tilde{w} \in D(0, \infty) : I\!R^d)$; assume (A1) - (A3). Clearly $Yw - Y\tilde{w}$ is of bounded variation on any finite interval. It would be interesting to have a useful estimate for the variation of $Yw - Y\tilde{w}$ that could give (Lipschitz) continuity of Y and Z; (see Dupuis and Ishii (1991(a)), Dupuis and Ramanan (1997) for the importance of Lipschitz continuity of the solution map). Can such an estimate be given, for example, in terms of (variation of) $h - \tilde{h}$ (or something similar) and Leontief inverse $(I - V)^{-1}$? (h is defined by (3.3) and $\tilde{h}$ can be defined analogously for $\tilde{w}$). Observe that the estimate of Shashiashvili (1994) needs the assumption that $w - \tilde{w}$ is of bounded variation on a finite interval. Such an estimate might also make the study of SDE's in $\bar{G}$ with reflection (and nonconstant dispersion) simpler. (Of course, fairly

general results for SDE's with reflection when coefficients depend only on t, z, using nonlinear PDE methods, are available in Dupuis and Ishii (1993)). $\square$.

2. Let $\tilde{D} \subseteq D_{\uparrow 0}$ be a closed subset; let

$$y_i^{(0)}(t) = \inf\{y_i(t) : y \in \tilde{D}, y \text{ feasible subsidy for } (w, b, R)\}$$

Is $y^{(0)}$ feasible? If yes, then it is II-minimal in $\tilde{D}$. Or what conditions on b, R would make $y^{(0)}$ feasible for a reasonable $\tilde{D}$? Under the hypotheses of Theorem 5.3 such a result follows with $\tilde{D} = D_{\uparrow 0}, y^{(0)} = Yw$; see also Chen and Mandelbaum (1991), Reiman (1984). Answer to this question might depend on getting conditions under which an analogue of Lemma 5.5 holds. $\square$

3. It may be noted that the condition (2.1), viz. that the standard subsidy for Sector i can not be mobilised when Sector i has a strictly positive surplus, is basically a minimality criterion. In Section 5 we have investigated the connection between this notion of minimality and some other natural notions of minimality under certain monotonicity/regularity assumptions. However, for the notions of minimality defined in Section 5, it is not clear if minimal feasible subsidy exists (and is unique) for nonconstant, nonmonotonic b_i, r_{ij} in general. On this score the model in terms of the Skorokhod problem has a tremendous advantage, as existence of a unique solution is guaranteed under fairly general conditions. Thus the subsidy-surplus model outlined in Section 2 can be considered as the continuous-time, feedback form analogue of the classical (open) Leontief type input-output model, and the solution (Yw, Zw) of the (SP) is the analogue of the efficient point of the Leontief model.

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