REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS IN AN ORTHANT

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Abstract: We consider RBSDE in an orthant with oblique reflection and with time and space dependent coefficients, viz

$$Z(t) = \xi + \int_t^T b(s, Z(s))\ ds + \int_t^T R(s, Z(s))\ dY(s) - \int_t^T \langle U(s),\ dB(s)\rangle$$

with $Z_i(\cdot) \geq 0, Y_i(\cdot)$ nondecreasing and $Y_i(\cdot)$ increasing only when $Z_i(\cdot) = 0, 1 \leq i \leq d$. Existence of a unique solution is established under Lipschitz continuity of b, R and a uniform spectral radius condition on R. On the way we also prove a result concerning the variational distance between the "pushing parts" of solutions of auxiliary one-dimensional problem.

Running title: RBSDE's in an Orthant.

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1 Introduction

Since backward stochastic differential equations were introduced about a decade back there has been a lot of interest in them owing to wide applicability in stochastic control, differential games and economics. Recently backward stochastic differential equations with reflecting barrier have been studied by El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) and Cvitanic and Karatzas (1996) in the one dimensional case; and by Gegout-Petit and Pardoux (1996) in a convex domain in higher dimensions; these works concern the case of normal reflection at the boundary.

On the other hand, following the impetus given by queueing theory, deterministic as well as stochastic Skorokhod problem in an orthant with oblique reflection at the boundary has been studied by many authors over the last two decades; see the references in Ramasubramanian (2000).

The aim of this article is to study reflected backward stochastic differential equations (RBSDE's) in an orthant with oblique reflection at the boundary. The drift vector and the reflection matrix can be time and space dependent; existence and uniqueness are established under a uniform spectral radius condition on the reflection matrix (plus, of course, a Lipschitz continuity condition on the coefficients); such a condition has proved useful in the study of Skorokhod problem; see Harrison and Reiman (1981), Reiman (1984), Mandelbaum and Pats (1998) and Ramasubramanian (2000).

In Section 2, after describing the set up, we indicate briefly two situations from economics where RBSDE can be used as a model. The first one is a backward stochastic analogue of the subsidy-surplus model considered in Ramasubramanian (2000), and the second example is a backward stochastic (oblique) analogue of a projected dynamical system studied in Nagurney and Siokos (1997).

An auxiliary one dimensional RBSDE is discussed in Section 3. A result concerning the variational distance between the "pushing parts" of solutions of two auxiliary one dimensional equations is established, the inspiration being a deterministic analogue due to Shashiashvili (1994); see also Shashiashvili (1996). Existence of a unique solution to RBSDE is proved in Section 4 by a contraction mapping argument; the metric is given in terms of total variation and L^1- norm. As in Ramasubramanian (2000) a couple of a priori results help in confining the analysis to a smaller space. It is also shown that it is enough to have the reflection coefficients defined on the boundary.

2 RBSDE in an orthant with oblique reflection

Let $\{B(t) = (B_1(t), ..., B_d(t) : 0 \leq t \leq T\}$ be a d-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$; let $\{\mathcal{F}_t\}$ be the natural filtration generated by $\{B(t)\}$, with $\mathcal{F}_0$ containing all P - null sets.

Let $G = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ denote the d-dimensional positive orthant. We are given the following :

ξ is an $\mathcal{F}_T$ - measurable $\overline{G}$ - valued bounded random variable;

$b : \Omega \times [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $R : \Omega \times [0,T] \times R^d \longrightarrow \mathbb{M}_d(\mathbb{R})$ are both bounded measurable functions such that for each $z \in \mathbb{R}^d$,

$b(\cdot,\cdot,z) = (b_1(\cdot,\cdot,z), ..., b_d(\cdot,\cdot,z))$, $R(\cdot,\cdot,z) = ((r_{ij}\cdot(\cdot,\cdot,z)))_{1\leq i,\ j\leq d}$ are $\{\mathcal{F}_t\}$ - predictable processes; it is also assumed that $r_{ii}(\cdots) \equiv 1$ which is just a suitable normalization. (Here $\mathbb{M}_d(\mathbb{R})$ denotes the class of $d \times d$ matrices with real entries.)

A pair $\{Y(t) = (Y_1(t), ..., Y_d(t))\}, \{Z(t) = (Z_1(t), ..., Z_d(t))\}, 0 \leq t \leq T$ of $\{\mathcal{F}_t\}$ - progressively measurable continuous integrable processes is said to solve RBSDE (ξ, b, R) if there is an $\{\mathcal{F}_t\}$ - progressively measurable square integrable process $U(t) = ((U_{ij}(t)))_{1\leq i,\ j\leq d}$ such that

(i) for $i = 1, ..., d, 0 \leq t \leq T$

$$\begin{aligned} Z_i(t) = \xi_i &+ \int_t^T b_i(s, Z(s))ds + Y_i(T) - Y_i(t) \\ &+ \sum_{j\neq i} \int_t^T r_{ij}(s, Z(s))\ dY_j(s) - \int_t^T \sum_{j=1}^d U_{ij}(s)\ dB_j(s) \end{aligned} \tag{2.1}$$

(ii) $Z(t) \in \overline{G}$ for all $0 \leq t \leq T$;

(iii) $Y_i(0) = 0, Y_i(\cdot)$ continuous, nondecreasing and $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0, 1 \leq i \leq d$; that is

$$Y_i(t) = \int_0^t I_{\{0\}}(Z_i(s))dY_i(s) \tag{2.2}$$

Equation (2.1) is the analogue of Skorokhod equation. Note that the process $U(\cdot)$ need not be continuous; b is the drift and R gives the reflection directions.

We now describe briefly two situations where the above model may be applicable.

Remark 2.1: Following Ramasubramanian (2000), RBSDE (ξ, b, R) can be viewed upon as a subsidy-surplus model. We consider an economy with d interdependent sectors, with the following interpretations :

$Z_i(t)$ = current surplus in Sector i at time t;
$Y_i(t)$ = cumulative subsidy given to Sector i over $[0, t]$;
ξ_i = desired surplus in Sector i at time T;
of course, $Z_i(t), Y_i(t)$ depend on"information" only upto time t;
$\int_s^t b_i(u, Z(u))du$ = net production of Sector i over $[s, t]$ due to evolution of the system; this being negative indicates there is net consumption;
$\int_s^t r_{ij}^-(u, Z(u))dY_j(u)$ = amount of subsidy for Sector j mobilized from Sector i over $[s, t]$;
$\int_s^t r_{ij}^+(u, Z(u))dY_j(u)$ = amount of subsidy mobilized for Sector j which is actually used in Sector i (but not as subsidy in Sector i) over $[s, t]$.

The condition (iii) in RBSDE (ξ, b, R) means that subsidy for Sector i can be mobilized only when Sector i has no surplus; this is a *natural minimality condition.* The uniform spectral radius condition $(A3)$ which is imposed in Section 4 would mean that the subsidy mobilized from external sources is nonzero; so this would be an "open" system in the jargon of economics; see also Section 2 of Ramasubramanian (2000). This suggests that the above situation may be called a *stochastic differential subsidy-surplus model* a la Duffie and Epstein (1992).

Remark 2.2: We give another interpretation. Suppose the system represents d traders each specializing in a different commodity. For this model we assume $r_{ij}(\cdots) \le 0$, $i \ne j$. Here

$Z_i(t)$ = current price of Commodity i at time t; there is a price floor viz. prices cannot be negative;
$Y_i(t)$ = cumulative "tatonnement" (adjustment) involved in the price of Commodity i over $[0, t]$;
$b_i(t, Z(t))dt$ = infinitesimal change in price of Commodity i due to evolution of the system;
ξ_i = desired price level of Commodity i at time T

Condition (iii) (that is, (2.2)) of RBSDE (ξ, b, R) then means that tatonnement / adjustment $dY_i(\cdot)$ can take place only if the price of Commodity i is zero. In such a case

$\int_s^t r_{ij}^-(u, Z(u))dY_j(u)$ = tatonnement from Trader i when price of Commodity j is zero.

Note that $dY_j(\cdot)$ can be viewed upon as a sort of artificial / forced infinitesimal consumption when the price of Commodity j is zero to boost up the price; hence $r_{ij}^-(t, Z(t))\ dY_j(t)$ is the contribution of Trader i towards this forced consumption. As before, (A3) implies that there is nonzero "external tatonnement", like perhaps governmental intervention / consumption to boost prices when prices crash.

In the context of the Skorokhod problem with normal reflection, a similar interpretation is given in Nagurney and Siokos (1997, pp.76-80) in connection with financial networks; these authors call the model as a "projected dynamical system"; see also Dupuis and Nagurney (1993). One dimensional RBSDE, (of course, with normal reflection), has been proposed as a model for pricing of American contingent claims in El Karoui and Quenez (1997, pp.229-231).

Since "tatonnement" can be viewed upon as a "subsidy", the above may also be taken as a special case of Remark 2.1.

3 Auxiliary one dimensional problem

In this section we look at an auxiliary one dimensional problem needed for studying the d - dimensional problem.

Let $(\Omega, \mathcal{F}, P), \{B(t)\}, \{\mathcal{F}_t\}, 0 \leq t \leq T$ be as in Section 2. We are given the following:

ς is an $\mathcal{F}_T$ - measurable bounded nonnegative random variable;

$f : \Omega \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}, \quad g_j : \Omega \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}, \quad 1 \leq j \leq k$,are bounded measurable functions such that for each $z \in \mathbb{R}$, $f(\cdot,\cdot,z)$, $g_j(\cdot,\cdot,z)$ are $\{\mathcal{F}_t\}$ - predictable;

$A_j, 1 \leq j \leq k$ are $\{\mathcal{F}_t\}$ - progressively measurable integrable continuous nondecreasing processes.

A pair $\{L(t)\}, \{M(t)\}, 0 \leq t \leq T$ of real valued $\{\mathcal{F}_t\}$ - progressively measurable continuous integrable processes is said to solve the auxiliary one-dimensional problem corresponding to (ς, f, g_j, A_j) if there exists an $\{\mathcal{F}_t\}$ - progressively measurable square integrable process $\{V(t) = (V_1(t), ..., V_d(t))\}, 0 \leq t \leq T$ such that

(i) the Skorokhod equation holds, viz.

$$M(t) = \varsigma + \int_t^T f(s, M(s))ds + \sum_{j=1}^k \int_t^T g_j(s, M(s))dA_j(s)$$
$$+ L(T) - L(t) - \int_t^T \sum_{\ell=1}^k V_\ell(s) \, dB_\ell(s) \tag{3.1}$$

(ii) $M(t) \geq 0$ for all $0 \leq t \leq T$;
(iii) $L(0) = 0$, $L(\cdot)$ nondecreasing, $L(\cdot)$ can increase only when $M(\cdot) = 0$.

Proceeding as in the proof of Proposition 4.2 and Remark 4.3 of El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) the following result can be proved.

Lemma 3.1 *Let $\{L(t)\}, \{M(t)\}, 0 \leq t \leq T$ be a solution to the auxiliary one dimensional problem. Let $\{\ell(t)\}, 0 \leq t \leq T$ denote the local time at 0 of the continuous semimartingale $\{M(t)\}$. Then*

$$0 \leq dL(t) \leq I_{\{0\}}(M(t)) \left\{ |f(t,0)|dt + \sum_{j=1}^{k} |g_j(t,0)| \; dA_j(t) \right\} \tag{3.2}$$

$$0 \leq d\ell(t) \leq I_{\{0\}}(M(t)) \left\{ |f(t,0)|dt + \sum_{j=1}^{k} |g_j(t,0)| \; dA_j(t) \right\} \tag{3.3}$$

If in addition $A_j, 1 \leq j \leq k$ are absolutely continuous then

$$dL(t) + \frac{1}{2} d\ell(t) = I_{\{0\}}(M(t)) \left[f(t,0) + \sum_{j=1}^{k} g_j(t,0) \frac{dA_j(t)}{dt} \right]^{-} dt \tag{3.4}$$

The next result concerns the variational distance between the L - parts of the solutions of two auxiliary one-dimensional equations; it has been motivated by a deterministic analogue due to Shashiashvili (1994) in the context of Skorokhod problem. For our purposes it suffices to consider the case when $f : \Omega \times [0,T] \longrightarrow \mathbb{R}$ is $\{\mathcal{F}_t\}$ - predictable and $g_j \equiv 0$ for all j.

To be more precise, for $k = 1, 2$ let $f^{(k)} : \Omega \times [0,T] \longrightarrow \mathbb{R}$ be bounded $\{\mathcal{F}_t\}$ - predictable process, $\varsigma^{(k)}$ bounded nonnegative $\{\mathcal{F}_T\}$ - measurable random variable; let $\{L^{(k)}(t)\}, \{M^{(k)}(t)\}, 0 \leq t \leq T$ solve the auxiliary one-dimensional problem corresponding to $(\varsigma^{(k)}, f^{(k)}, 0, 0)$ so that

$$M^{(k)}(t) = \varsigma^{(k)} + \int_t^T f^{(k)}(s)ds + L^{(k)}(T) - L^{(k)}(t) - \int_t^T \left\langle V^{(k)}(s), dB(s) \right\rangle, \tag{3.5}$$

$$\int M^{(k)}(s) dL^{(k)}(s) = 0, \tag{3.6}$$

$M^{(k)}(t) \geq 0, L^{(k)}(0) = 0, L^{(k)}(\cdot)$ continuous nondecreasing, $k = 1, 2$, $0 \leq t \leq T$; (all these hold a.s.). By Proposition 5.1 of El Karoui et al. (1997) unique square integrable $M^{(k)}(\cdot), L^{(k)}(\cdot), V^{(k)}(\cdot)$ exist solving the above. Clearly $L^{(1)}(\cdot) - L^{(2)}(\cdot)$ is of bounded variation a.s.; in fact, by the preceding lemma $L^{(1)}, L^{(2)}$ are absolutely continuous; let $\lambda^1(\cdot), \lambda^{(2)}(\cdot)$ denote their respective derivatives. Let $|d(L^{(1)} - L^{(2)})|(\cdot)$ denote the measure given by the total variation of $(L^{(1)} - L^{(2)})(\cdot)$.

Theorem 3.2 *For any $\theta \geq 0$, $0 \leq s \leq t \leq T$*

$$
\begin{aligned}
&E\int_s^t (e^{\theta r} - 1)\ |\lambda^{(1)}(r) - \lambda^{(2)}(r)|dr \\
&= E\int_s^t (e^{\theta r} - 1)\ |d(L^{(1)} - L^{(2)})|(r) \\
&\leq E[(e^{\theta t} - 1)\ |M^{(1)}(t) - M^{(2)}(t)| - (e^{\theta s} - 1)\ |M^{(1)}(s) - M^{(2)}(s)|] \\
&- E\int_s^t e^{\theta r}\theta\ \ |M^{(1)}(r) - M^{(2)}(r)|dr \\
&+ E\int_s^t (e^{\theta r} - 1)\ \ |f^{(1)}(r) - f^{(2)}(r)|dr
\end{aligned}
\tag{3.7}
$$

Proof. : All equalities / inequalities below are satisfied a.s. We denote

$\widehat{\xi}(\cdot) = \xi^{(1)}(\cdot) - \xi^{(2)}(\cdot)$ for $\xi = \lambda, L, f, V$. Proceeding as in the proof of equation (13) in Shashiashvili (1994, pp.171-173) using

$$
I_{(0,\infty)}(\widehat{M}(r))dL^{(1)}(r) = 0, \quad I_{(-\infty,0)}(\widehat{M}(r))dL^{(2)}(r) = 0
$$

we get for $0 \leq s \leq t \leq T$

$$
\int_s^t |d(\widehat{L})|(r) = \int_s^t [-\mathrm{sgn}(\widehat{M}(r)) + I_{\{0\}}(\widehat{M}(r))\chi(r)]d(\widehat{L})(r) \tag{3.8}
$$

where $\chi(\cdot)$ is $\{\mathcal{F}_t\}$ - progressively measurable function taking only the values $+1, -1$ and the function sgn is defined by

$$
\mathrm{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}
$$

Progressive measurability of $\chi(\cdot)$ follows by the proof of Radon-Nikodym theorem and p.171 of Shashiashvili (1994). Therefore denoting the integrand on the r.h.s. of (3.8) by $J(r)$ and using (3.5) for $k = 1, 2$, we get

$$\begin{aligned}
\int_s^t (e^{\theta r} - 1)|\widehat{\lambda}(r)|dr &= \int_s^t (e^{\theta r} - 1)|d(\widehat{L})|(r) \\
&= \int_s^t (e^{\theta r} - 1)J(r)d(\widehat{L})(r) \\
&= -\int_s^t (e^{\theta r} - 1)J(r)(\widehat{f})(r)dr + \int_s^t (e^{\theta r} - 1)J(r)\left\langle \widehat{V}(r), dB(r)\right\rangle \\
&- \int_s^t (e^{\theta r} - 1)J(r)d\widehat{M}(r) \\
&= I_1 + I_2 + I_3
\end{aligned} \tag{3.9}$$

As $|J(r)| \leq 1$ it is clear that

$$E(I_1) \leq E\int_s^t (e^{\theta r} - 1)\ |\widehat{f}(r)|\ dr, \tag{3.10}$$

and I_2 being an Ito integral

$$E(I_2) = 0 \tag{3.11}$$

Let $t \mapsto \ell(t, a)$ denote the local time of the continuous semimartingale $\widehat{M}$ at $a \in \mathbb{R}$. By the version of Ito-Tanaka-Meyer formula given in Exercise 1.25, Chapter VI (p.219) of Revuz and Yor (1991) we get

$$\begin{aligned}
d((e^{\theta r} - 1)|\widehat{M}|(r)) &= |\widehat{M}|\ (r)\ \theta\ e^{\theta r}dr + (e^{\theta r} - 1)\ d\ |\widehat{M}|\ (r) \\
&= |\widehat{M}|\ (r)\ \theta\ e^{\theta r}dr + \frac{(e^{\theta r} - 1)}{2}[d\ell(r, 0) + d\ell(r, 0-)] \\
&+ (e^{\theta r} - 1)[\mathrm{sgn}(\widehat{M})(r)d(\widehat{M})(r)]
\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_s^t (e^{\theta r} - 1)\ \mathrm{sgn}(\widehat{M}(r))d(\widehat{M})(r) \\
&= (e^{\theta t} - 1)\ |\widehat{M}|(t) - (e^{\theta s} - 1)|\widehat{M}|\ (s) - \int_s^t e^{\theta r}\ \theta\ |\widehat{M}|\ (r)\ dr \\
&- \frac{1}{2}\left[\int_s^t (e^{\theta r} - 1)\ d\ell\ (r, 0) + \int_s^t (e^{\theta r} - 1)\ d\ell\ (r, 0-)\right]
\end{aligned} \tag{3.12}$$

By Theorem 1.7, Chapter VI of Revuz and Yor (1991)

$$\begin{aligned}
&\left| \int_s^t -(e^{\theta r} - 1)\chi(r) I_{\{0\}}(\widehat{M})(r)) d(\widehat{M})(r) \right| \\
&\leq \frac{1}{2} \int_s^t (e^{\theta r} - 1) |d(\ell(\cdot, 0) - \ell(\cdot, 0-))|(r) \\
&\leq \frac{1}{2} \left[\int_s^t (e^{\theta r} - 1) d\ell(r, 0) + \int_s^t (e^{\theta r} - 1) d\ell(r, 0-) \right] \qquad (3.13)
\end{aligned}$$

By (3.12), (3.13)

$$I_3 \leq (e^{\theta t} - 1)\ |\widehat{M}|(t) - (e^{\theta s} - 1)\ |\widehat{M}|\ (s) - \int_s^t e^{\theta r} \theta\ |\widehat{M}|\ (r)\ dr \qquad (3.14)$$

Taking expectation in (3.9) and (3.14), and using (3.10), (3.11) the required estimate (3.7) is now immediate. ∎

4 Existence and Uniqueness

We make the following assumptions on the coefficients b, R.

(**A1**) : For $1 \leq i \leq d$, $z \mapsto b_i(\omega, t, z)$ is Lipschitz continuous, uniformly over (ω, t); there is a constant β_i such that $|b_i(\omega, t, z)| \leq \beta_i$ for all ω, t, z. Denote $\beta = (\beta_1, ..., \beta_d)$.

(**A2**) : For $1 \leq i, j \leq d$, $z \mapsto r_{ij}(\omega, t, z)$ is Lipschitz continuous, uniformly over (ω, t). Also $r_{ii} \equiv 1$ for all i.

(**A3**) : For $i \neq j$ there exists constant v_{ij} such that $|r_{ij}(\omega, t, z)| \leq v_{ij}$. Set $V = ((v_{ij}))$ with $v_{ii} = 0$; we assume that $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V.

If $\sigma(V) < 1$ observe that

$$(I - V)^{-1} = I + V + V^2 + V^3 + ...$$

is a matrix with nonnegative entries; here I is the $(d \times d)$ identity matrix.

We first establish an a priori estimate.

Proposition 4.1 *Assume (A1) - (A3) and let* ξ *be a bounded* $\mathcal{F}_T$ *- measurable* $\overline{G}$ *- valued random variable. Suppose* $\{Y(t)\}, \{Z(t)\}, 0 \leq t \leq T$ *solve RBSDE* (ξ, b, R)*. Then*

$$0 \leq dY(t) \leq (I - V)^{-1}\beta \; dt \tag{4.1}$$

in the sense that

$$0 \leq dY_i(t) \leq ((I - V)^{-1}\beta)_i \; dt, \; 1 \leq i \leq d \tag{4.2}$$

In particular $dY_i(\cdot)$ *is absolutely continuous, and hence the local time at* 0 *of* $Z_i(\cdot)$ *is also absolutely continuous for each* $i = 1, ..., d$.

Proof. : For each fixed $i = 1, ..., d$ note that $\{L(t) = Y_i(t)\}, \{M(t) = Z_i(t)\}$, $0 \leq t \leq T$ is a solution to the auxiliary one-dimensional problem corresponding to $\varsigma = \xi_i$, $f(\omega, s, z) = b_i(\omega, s, \overline{Z}_{i,z}(s, \omega))$, $g_j(\omega, s, z) = r_{ij}(\omega, s, \overline{Z}_{i,z}(s, \omega))$, $dA_j(s) = dY_j(s), j \neq i, 1 \leq i, j \leq d$ where $\overline{Z}_{i,z} = (Z_1, ..., Z_{i-1}, z, Z_{i+1}, ..., Z_d)$.

By Lemma 3.1 and our hypotheses

$$\begin{aligned} 0 \leq dY_i(t) &\leq I_{\{0\}}(Z_i(t)) \left\{ |b_i(t, Z(t))| \; dt + \sum_{j \neq i} |r_{ij}(t, Z(t))| \; dY_j(t) \right\} \\ &\leq \beta_i \; dt + \sum_{j \neq i} v_{ij} \; dY_j(t). \end{aligned}$$

Consequently

$$dY_i(t) - \sum_{j \neq i} v_{ij} \; dY_j(t) \;\; \leq \beta_i \; dt, \quad 1 \leq i \leq d$$

The above can be expressed as

$$((I - V)dY)_i(t) \;\; \leq (\beta)_i \;\; dt, \quad 1 \leq i \leq d \tag{4.3}$$

As $\sigma(V) < 1$ we can get (4.1), (4.2) from (4.3). The last assertion is now a consequence of Lemma 3.1. ■

Remark 4.2 *As* $\sigma(V^\dagger) = \sigma(V)$*, where* $V^\dagger$ *denotes transpose of* V*, by (A3) it follows that there are constants* $a_j > 0, 1 \leq j \leq d$ *and* $0 < \alpha < 1$ *such that*

$$\sum_{i \neq j} a_i |r_{ij}(\omega, t, z)| \leq \sum_{i \neq j} a_i \; v_{ij} \leq \alpha a_j \tag{4.4}$$

for all $j = 1, ..., d, \omega \in \Omega$, $0 \leq t \leq T$, $z \in \mathbb{R}^d$*; see, for example, Dupuis and Ishii (1991) for a proof.* □

Let $\theta > 0$ be a constant. Let $\mathcal{H}$ denote the space of all (equivalence classes of) $\{\mathcal{F}_t\}$ - progressively measurable processes $\{Y(t) := (Y_1(t), ..., Y_d(t))\}$, $\{Z(t) := (Z_1(t), ..., Z_d(t))\}, 0 \leq t \leq T$ such that

(i) $Z_i(t) \geq 0$, $0 \leq t \leq T$, $1 \leq i \leq d$,

(ii) $Y_i(0) = 0$, $Y_i(\cdot)$ is nondecreasing, $1 \leq i \leq d$,

(iii) $E\sum_{i=1}^{d}\int_0^T e^{\theta t} a_i |Z_i(t)|\ dt \quad < \infty$

(iv) $E\sum_{i=1}^{d}\int_0^T e^{\theta t} a_i \varphi_t(Y_i) dt \quad < \quad \infty$

where $\varphi_t(g)$ denotes the total variation of g over $[t, T]$. The constant $\theta > 0$ will be chosen suitably later; the constants a_i are as in Remark 4.2.

For $((Y, Z), (\widehat{Y}, \widehat{Z})) \in \mathcal{H}$ define the metric

$$\begin{aligned} d((Y,Z),(\widehat{Y},\widehat{Z})) &= E\sum_{i=1}^{d}\int_0^T e^{\theta t} a_i\ |Z_i(t) - \widehat{Z}_i(t)|\ dt \\ &+ E\sum_{i=1}^{d}\int_0^T e^{\theta t} a_i\ \varphi_t(Y_i - \widehat{Y}_i)\ dt \end{aligned} \tag{4.5}$$

Note that $(\mathcal{H}, d)$ is a complete metric space.

Let $\widetilde{\mathcal{H}}$ denote the collection of all $(Y, Z) \in \mathcal{H}$ such that there is an $\{\mathcal{F}_t\}$ - progressively measurable process $D(t) = (D_1(t), ..., D_d(t))$, $0 \leq t \leq T$ with $0 \leq D_i(t) \leq ((I - V)^{-1}\beta)_i$ a.s. and $Y_i(t) = \int_0^t D_i(s)\ ds$, $0 \leq t \leq T$.

Observe that $\widetilde{\mathcal{H}}$ is a closed subspace of $\mathcal{H}$ and hence $(\widetilde{\mathcal{H}}, d)$ is a complete metric space.

Let $(Y, Z), (\widehat{Y}, \widehat{Z}) \in \widetilde{\mathcal{H}}$ with $D_i, \widehat{D}_i$ being respective derivatives of $Y_i, \widehat{Y}_i$.

Since

$$\varphi_t(Y_i - \widehat{Y}_i) = \int_t^T |D_i(s) - \widehat{D}_i(s)| ds \tag{4.6}$$

using integration by parts and (4.5) we have

$$d((Y,Z),(\widehat{Y},\widehat{Z})) = E\sum_{i=1}^{d}\int_0^T e^{\theta t} a_i \; |Z_i(t) - \widehat{Z}_i(t)| \; dt$$

$$+ E\sum_{i=1}^{d}\int_0^T \frac{(e^{\theta t}-1)}{\theta} a_i \; |D_i(t) - \widehat{D}_i(t)| \; dt \tag{4.7}$$

As $a_i > 0$ for all i, note that $(\widetilde{\mathcal{H}}, d)$ can be identified with a closed subspace of $L^1_+((\Omega \times [0,T],\, dP \times e^{\theta t} dt) \to \mathbb{R}^d) \times L^1_+((\Omega \times [0,T],\, dP \times \frac{(e^{\theta t}-1)}{\theta} dt) \to \mathbb{R}^d)$.

In view of Proposition 4.1 we need to seek a solution only in $\widetilde{\mathcal{H}}$.

Let b, R satisfy (A1) - (A3) and ξ be a bounded $\mathcal{F}_T$ - measurable $\overline{G}$ - valued random variable. Let $(Y,Z) \in \widetilde{\mathcal{H}}$. For fixed $1 \le i \le d$ set

$$f_i(\omega,t) = b_i(\omega,t,Z(t,\omega)) + \sum_{j \ne i} r_{ij}(\omega,t,Z(t,\omega)) D_j(t,\omega) \tag{4.8}$$

By our assumption note that f_i is bounded. So by Proposition 5.1 of El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) there exists a unique pair $\widehat{Y}_i(t)$, $\widehat{Z}_i(t)$, $0 \le t \le T$ of nonnegative $\{\mathcal{F}_t\}$ - progressively measurable square integrable processes solving the auxiliary one dimensional problem such that

$$\widehat{Z}_i(t) = \xi_i + \int_t^T f_i(s)\; ds + \widehat{Y}_i(T) - \widehat{Y}_i(t) - \int_t^T \left\langle \widehat{U}_{i\cdot}(s),\; dB(s) \right\rangle \tag{4.9}$$

for some $\{\mathcal{F}_t\}$ - progressively measurable square integrable process $\{\widehat{U}_{i\cdot}(t)\}$; (of course, $\widehat{Y}_i(0) = 0$, $\widehat{Y}_i(\cdot)$ is nondecreasing and can increase only when $\widehat{Z}_i(\cdot) = 0$).

Set $\widehat{Y}(t) = (\widehat{Y}_1(t), ..., \widehat{Y}_d(t))$, $\widehat{Z}(t) = (\widehat{Z}_1(t), ..., \widehat{Z}_d(t))$, $0 \le t \le T$.

Lemma 4.3 *Assume (A1) - (A3); let ξ be bounded. If $(Y,Z) \in \widetilde{\mathcal{H}}$ then $(\widehat{Y},\widehat{Z}) \in \widetilde{\mathcal{H}}$.*

Proof. : As $\widehat{Z}_i(\cdot)$ is square integrable it is clear that

$$E\sum_{i=1}^{d}\int_0^T e^{\theta t} a_i \; |\widehat{Z}_i(t)| \; dt \qquad < \quad \infty.$$

By Lemma 3.1 $\widehat{Y}_i(\cdot)$ is absolutely continuous. To complete the proof it is enough to prove that $0 \leq d\widehat{Y}(t) \leq (I-V)^{-1}\beta\ dt$ is in the sense of (4.2). Again by Lemma 3.1, (4.8), (A1) and (A3)

$$
\begin{aligned}
0 \leq d\widehat{Y}_i(t) &\leq |b_i(t,0)|\ dt + \sum_{j\neq i} |r_{ij}(t,0)|\ D_j(t)\ dt \\
&\leq \beta_i\ dt + \sum_{j\neq i} v_{ij}\ D_j(t)\ dt
\end{aligned}
$$

As $v_{ii} \equiv 0$ and $dY(t) \leq (I-V)^{-1}\beta\ dt$ the above can be written as

$$
\begin{aligned}
0 \leq d\widehat{Y}(t) &\leq \beta\ dt + V(I-V)^{-1}\beta\ dt \\
&= [I + V(I-V)^{-1}]\beta\ dt = (I-V)^{-1}\beta\ dt
\end{aligned}
$$

completing the proof. ■

Note: Analogues of Proposition 4.1 and Lemma 4.3 for (deterministic) Skorokhod problem have been proved in Ramasubramanian (2000). □

Before we state our main result a comment concerning Lipschitz continuity may be in order. On $\mathbb{R}^d$ define the norm $||\ z\ || = \sum_{i=1}^{d} a_i\ |z_i|$; since $a_i > 0$ for all i, this norm is equivalent to the Euclidean norm. So we may as well assume that Lipschitz continuity in (A1), (A2) is with respect to this norm; that is, there is a constant $K > 0$ such that

$$
|f(\omega,t,z) - f(\omega,t,\widehat{z})| \quad \leq \quad K\ ||z - \widehat{z}|| \tag{4.10}
$$

for all $z, \widehat{z} \in \mathbb{R}^d$, $\omega \in \Omega, 0 \leq t \leq T$, $f = b_i,\ r_{ij}, 1 \leq i,\ j \leq d$ where $||z - \widehat{z}|| = \sum a_i |z_i - \widehat{z}_i|$.

Consequently by (4.7), for $(Y,Z), (\widehat{Y},\widehat{Z}) \in \widetilde{\mathcal{H}}$

$$
\begin{aligned}
d((Y,Z),(\widehat{Y},\widehat{Z})) = E\int_0^T e^{\theta t} ||Z(t) - \widehat{Z}(t)||\ dt \\
+ E\int_0^T \frac{(e^{\theta t}-1)}{\theta}\ ||D(t) - \widehat{D}(t)||\ dt
\end{aligned} \tag{4.11}
$$

Theorem 4.4 *Assume (A1) - (A3) and let ξ be a bounded $\mathcal{F}_T$ - measurable $\overline{G}$ - valued random variable. Then there is a unique $(Y,Z) \in \mathcal{H}$ solving RBSDE (ξ, b, R). Moreover Y, Z are continuous processes and*
$0 \leq dY_i(t) \leq ((I-V)^{-1}\beta)_i\ dt$, $0 \leq t \leq T$, $1 \leq i \leq d$.

Proof. : In view of Proposition 4.1 it is enough to prove that the map $(Y,Z) \mapsto (\widehat{Y},\widehat{Z})$ is a strict contraction on $\widetilde{\mathcal{H}}$ where $(\widehat{Y},\widehat{Z})$ is as in the discussion prior to Lemma 4.3; by Lemma 4.3 $(\widehat{Y},\widehat{Z}) \in \widetilde{\mathcal{H}}$ whenever (Y,Z) does.

Let $(Y^{(1)},Z^{(1)}),(Y^{(2)},Z^{(2)}) \in \widetilde{\mathcal{H}}$. Let $(\widehat{Y}^{(1)},\widehat{Z}^{(1)}),(\widehat{Y}^{(2)},\widehat{Z}^{(2)}) \in \widetilde{\mathcal{H}}$ be obtained by solving the associated auxiliary one dimensional problems as in the discussion prior to Lemma 4.3; see (4.8), (4.9). So there exist matrix valued $\{\mathcal{F}_t\}$ - progressively measurable square integrable processes $\widehat{U}^{(1)},\widehat{U}^{(2)}$ such that

$$\widehat{Z}_i^{(k)}(t) = \xi_i + \int_t^T b_i(s,Z^{(k)}(s))\ ds + \sum_{j\neq i}\int_t^T r_{ij}(s,Z^{(k)}(s))\ D_j^{(k)}(s)\ ds \\ + \int_t^T \widehat{D}_i^{(k)}(s)\ ds - \int_t^T \left\langle \widehat{U}_{i\cdot}^{(k)}(s), dB(s)\right\rangle \tag{4.12}$$

where $d\widehat{Y}_i^{(k)}(t) = \widehat{D}_i^{(k)}(t)\ dt$, $dY_i^{(k)}(t) = D_i^{(k)}(t)\ dt$, for $i=1,...,d$, $k=1,2$.

Applying Theorem 3.2 to $(\widehat{Y}_i^{(k)},\widehat{Z}_i^{(k)})$, $k=1,2$ for a fixed i, using an analogue of (4.6), integration by parts and (4.12) we get

$$\begin{aligned}
&E\int_0^T e^{\theta t}\theta\ |\widehat{Z}_i^{(1)}(t)-\widehat{Z}_i^{(2)}(t)|\ dt + E\int_0^T e^{\theta t}\theta\ \varphi_t(\widehat{Y}_i^{(1)}-\widehat{Y}_i^{(2)})dt \\
&= E\int_0^T e^{\theta t}\theta\ |\widehat{Z}_i^{(1)}(t)-\widehat{Z}_i^{(2)}(t)|\ dt + E\int_0^T (e^{\theta t}-1)\ |\widehat{D}_i^{(1)}(t)-\widehat{D}_i^{(2)}(t)|\ dt \\
&\le E\int_0^T (e^{\theta t}-1)\ |b_i(t,Z^{(1)}(t))-b_i(t,Z^{(2)}(t))|\ dt \\
&+ E\int_0^T (e^{\theta t}-1)\ \left|\sum_{j\neq i}(r_{ij}(t,Z^{(1)}(t))\ D_j^{(1)}(t) - r_{ij}(t,Z^{(2)}(t))\ D_j^{(2)}(t))\right| dt \\
&\le E\int_0^T (e^{\theta t}-1)\ |b_i(t,Z^{(1)}(t))-b_i(t,Z^{(2)}(t))|\ dt \\
&+ E\int_0^T (e^{\theta t}-1)\sum_{j\neq i}|r_{ij}(t,Z^{(1)}(t))-r_{ij}(t,Z^{(2)}(t))|D_j^{(1)}(t)dt \\
&+ E\int_0^T (e^{\theta t}-1)\sum_{j\neq i}|r_{ij}(t,Z^{(2)}(t))|\ |D_j^{(1)}(t)-D_j^{(2)}(t)|\ dt
\end{aligned} \tag{4.13}$$

As $(Y^{(1)}, Z^{(1)}) \in \widetilde{\mathcal{H}}$ note that $\sum\limits_j D_j^{(1)}(t) \leq \sum\limits_j ((I-V)^{-1}\beta)_j \leq K_1$ for some constant K_1. So by the Lipschitz condition (4.10) and (A3) we now get

$$E\left[\int_0^T e^{\theta t}\ \theta\ |\widehat{Z}_i^{(1)}(t) - \widehat{Z}_i^{(2)}(t)|\ dt + \int_0^T e^{\theta t}\ \theta\ \varphi_t\ (\widehat{Y}_i^{(1)} - \widehat{Y}_i^{(2)})\ dt\right]$$

$$\leq K(K_1+1)\ E\int_0^T (e^{\theta t}-1)\ ||Z^{(1)}(t) - Z^{(2)}(t)||\ dt$$

$$+ E\int_0^T (e^{\theta t}-1)\sum_{j\neq i} v_{ij}\ |D_j^{(1)}(t) - D_j^{(2)}(t)|\ dt \tag{4.14}$$

Multiplying (4.14) by a_i, adding and using (4.4)

$$\theta d((\widehat{Y}^{(1)}, \widehat{Z}^{(1)}),\ (\widehat{Y}^{(2)}, \widehat{Z}^{(2)}))$$

$$\leq (\textstyle\sum a_i)\ \ K(K_1+1)\ E\displaystyle\int_0^T (e^{\theta t}-1)\ ||Z^{(1)}(t) - Z^{(2)}(t)||\ dt$$

$$+ E\int_0^T (e^{\theta t}-1)\sum_i \sum_{j\neq i} a_i\ v_{ij}\ |D_j^{(1)}(t) - D_j^{(2)}(t)|\ dt$$

$$\leq (\textstyle\sum a_i)\ \ K(K_1+1)\ E\displaystyle\int_0^T e^{\theta t}\ ||Z^{(1)}(t) - Z^{(2)}(t)||\ dt$$

$$+ \alpha\ E\int_0^T (e^{\theta t}-1)\ ||D^{(1)}(t) - D^{(2)}(t)||\ dt \tag{4.15}$$

Choose θ large enough that $\frac{1}{\theta}\,(\sum a_i)\ K(K_1+1) \leq \alpha$. Then we get using (4.11), (4.15)

$$d((\widehat{Y}^{(1)}, \widehat{Z}^{(1)}), (\widehat{Y}^{(2)}, \widehat{Z}^{(2)})) \leq \alpha\ d((Y^{(1)}, Z^{(1)}), (Y^{(2)}, Z^{(2)})). \tag{4.16}$$

As $\alpha < 1$ this shows that $(Y, Z) \mapsto (\widehat{Y}, \widehat{Z})$ is a strict contraction on $\widetilde{\mathcal{H}}$, completing the proof. ■

While considering diffusions with boundary conditions, usually the reflection terms are specified only for z on the boundary. More precisely, for $1 \leq j \leq d$ denoting the j-th face of the orthant by $\partial_j G = \{x \in \overline{G} : x_j = 0\}$, note that the column vector $r_{\cdot j}(\cdots)$ denotes the direction of reflection on $\partial_j G$; so $r_{ij}(t, z)$ is generally defined only for $z \in \partial_j G$. Of course bounded Lipschitz continuous function on $\partial_j G$ can be extended to $\overline{G}$ or $\mathbb{R}^d$ with the same Lipschitz constant and the same bounds needed in (A3); see Shashiashvili (1994) for example. But there is no unique way of extension to $\overline{G}$ or $\mathbb{R}^d$. However our next result indicates that it does not matter which extension we take, only the values on the boundary determine the process.

Theorem 4.5 *Let b satisfy (A1). Let $R^{(1)}(\cdots) = ((r_{ij}^{(1)}(\cdots)))$, $R^{(2)}(\cdots) = ((r_{ij}^{(2)}(\cdots)))$ satisfy (A2), (A3) with the same Lipschitz constant and the same $((v_{ij}))$. Let $(Y^{(k)}, Z^{(k)}) \in \mathcal{H}$ solve RBSDE $(\xi, b, R^{(k)})$, $k = 1, 2$. Suppose $r_{ij}^{(1)}(t, z) = r_{ij}^{(2)}(t, z)$ for $z \in \partial_j G$, $1 \leq i \leq d$, $0 \leq t \leq T$ for $j = 1, ..., d$. Then $(Y^{(1)}, Z^{(1)}) = (Y^{(2)}, Z^{(2)})$.*

Proof. : Proceeding as in the proof of Theorem 4.4 with obvious modifications we get

$$
\begin{aligned}
&E\left[\int_0^T e^{\theta t}\ \theta\ |Z_i^{(1)}(t) - Z_i^{(2)}(t)|\ dt + \int_0^T e^{\theta t}\ \theta\ \varphi_t\ (Y_i^{(1)} - Y_i^{(2)})dt\right] \\
&\leq E\int_0^T (e^{\theta t} - 1)\ |b_i(t, Z^{(1)}(t)) - b_i(t, Z^{(2)}(t))|\ dt \\
&+ E\int_0^T (e^{\theta t} - 1)\sum_{j\neq i} |r_{ij}^{(1)}(t, Z^{(1)}(t)) - r_{ij}^{(2)}(t, Z^{(1)}(t))|\ D_j^{(1)}(t)\ dt \\
&+ E\int_0^T (e^{\theta t} - 1)\sum_{j\neq i} |r_{ij}^{(2)}(t, Z^{(1)}(t)) - r_{ij}^{(2)}(t, Z^{(2)}(t))|\ D_j^{(1)}(t) \\
&+ E\int_0^T (e^{\theta t} - 1)\sum_{j\neq i} |r_{ij}^{(2)}(t, Z^{(2)}(t))|\ |D_j^{(1)}(t) - D_j^{(2)}(t)|\ dt
\end{aligned} \tag{4.17}
$$

For any j note that $D_j^{(1)}(\cdot) > 0$ only if $Z_j^{(1)}(\cdot) = 0$, that is only if $Z^{(1)}(\cdot) \in \partial_j G$. So by our hypothesis the second term on the r.h.s. of (4.17) is zero. Therefore imitating the proof of (4.14) - (4.16) with the same choice of θ we get

$$d((Y^{(1)}, Z^{(1)}), (Y^{(2)}, Z^{(2)})) \leq \alpha d((Y^{(1)}, Z^{(1)}), (Y^{(2)}, Z^{(2)}))$$

As $0 < \alpha < 1$ the result now follows. ■

Entirely analogous arguments yield the following continuity result.

Proposition 4.6 *Let $\xi^{(n)}, b^{(n)}, R^{(n)},\ n = 0,1,2,....$ satisfy the hypotheses of Theorem 4.4 with the same bound, Lipschitz constant, $((v_{ij}))$. Let $(Y^{(n)}, Z^{(n)}) \in \mathcal{H}$ solve RBSDE $(\xi^{(n)}, b^{(n)}, R^{(n)})$ for $n = 0,1,2,...$ Assume $E|\xi^{(n)} - \xi^{(0)}| \to 0$, $\sup_{t,z}|b_i^{(n)}(t,z) - b_i^{(0)}(t,z)| \to 0, \sup_{t,z}|r_{ij}^{(n)}(t,z) - r_{ij}^{(0)}(t,z)| \to 0$ as $n \to \infty$ for all i,j. Then $(Y^{(n)}, Z^{(n)}) \to (Y^{(0)}, Z^{(0)})$ in $\mathcal{H}$.*

We conclude with a few comments.

Remark 4.7: From the uniqueness of (Y, Z) it is clear that $\{U(t)\}$ is also unique. It is also clear that Y and Z are square integrable.

Remark 4.8: An important feature of BSDE as well as RBSDE with normal reflection is the dependence of the drift b on the "control" variable $U(\cdot)$ as well; in these cases the appropriate metric is given by an L^2 - norm; see El Karoui et al (1997) and the references therein. However when one considers the case of oblique reflection (with $r_{ij} \neq 0$) the suitable metric seems to be in terms of the L^1 - norm given by (4.5). It is not quite clear to the author how dependence of b_i, r_{ij} on $U(\cdot)$ and $Y(\cdot)$ can be handled.

Remark 4.9: In view of Theorem 4.1 of El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) and Theorem 4.1 of Ramasubramanian (2000) a natural question is : Is there a comparison result for RBSDE in an orthant vis-a-vis the usual partial order ? Note that Theorem 4.1 of El Karoui et al (1997) gives monotonicity property of only the M - part of the solution of the auxiliary one dimensional problem. If in addition one can have monotonicity property of the L - part of the solution (perhaps in the opposite direction !) then the analysis in Section 4 of Ramasubramanian (2000) can possibly be modified to give a comparison result.

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