Non-rigidity of Peterson Graph

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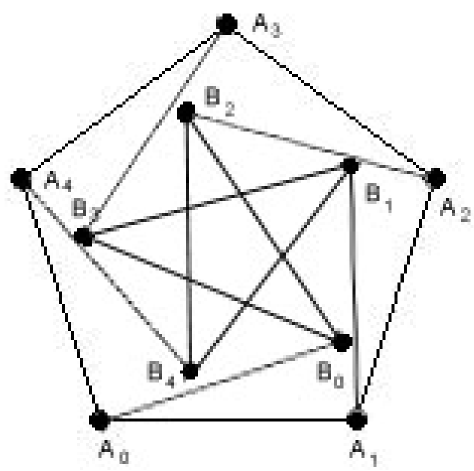
1 Introduction

Theorem 1 *Peterson Graph can be immersed in* $\mathbb{R}^2$ *with egdes as line segments of equal length.*

Proof We define an immersion for $\theta \in (0, \frac{\pi}{5}]$. Draw a regular pentagon $A_0A_1A_2A_3A_4$ with each edge of unit length. Then draw edges A_jB_j of unit length such that $\angle B_jA_jA_{j+1} = \theta$ (the indexing being done modulo 5). Join B_j to B_{j+2}, and this is an immersion of Peterson's Graph.



By symmetry, each B_jB_{j+2} has equal length, say $a(\theta)$. Clearly, a is a continuous function of θ and $\lim_{\theta\to 0} a(\theta) > 1$. At $\theta = \frac{\pi}{5}$, $A_{j+2}B_{j+2}A_jA_{j+1}$ form a rhombus giving $\angle B_jA_jB_{j+2} = \frac{\pi}{5} < \angle A_jB_jB_{j+2}$ and hence $a(\theta) < 1$. By continuity $\exists\theta_0$ with $a(\theta_0) = 1$, and so we are done.

2 Main Problem

Rigidity of the Peterson Graph actually does not make any sense (since Peterson Graph is just a topological space). But we can ask the question of whether an immersion of the Graph, with edges as line segments, is rigid or not. This definitely depends on the immersion.

We do not expect a general immersion to be rigid. To fix an immersion, we have 20 variables for the co-ordiantes of 10 vertices. But we have 15 quadratic constraints for the 15 edges, 2 linear constraints for translation (by which we put both the co-ordinates of some arbitrary vertex as 0), and 1 linear constraint for rotataion (by which we put the y co-ordinate of some other vertex as 0). We want to have the final solution space to have dimension (Zariski) ≥ 1. Since our base field is $\mathbb{R}$, which is not closed (algebraically), we cannot assume that each equation will decrease solution space by atmost 1, but however it gives us a good reason to believe that a general immersion, will not be rigid. For this, let us prove the following lemmas.

Lemma 1 *Let $v, w : U \to \mathbb{R}^2$ be analytic in a neighbourhood U of (ϕ_0, θ_0) with $||v(\phi_0, \theta_0) - w(\phi_0, \theta_0)||_2 \in (0, 2)$, and let $(x_0, y_0) \in \mathbb{R}^2$ be such that $||(x_0, y_0) - v(\phi_0, \theta_0)||_2 = ||(x_0, y_0) - w(\phi_0, \theta_0)||_2 = 1$.*

Then $\exists u : W \to \mathbb{R}^2$ analytic in some neighbourhood W of (ϕ_0, θ_0), with $u(\phi_0, \theta_0) = (x_0, y_0)$ and $||u(\phi, \theta) - v(\phi, \theta)||_2 = ||u(\phi, \theta) - w(\phi, \theta)||_2 = 1$.

Proof Trivial, by Implicit function theorem, or by direct computation.

Lemma 2 *With the situation as in previous lemma, let $\frac{\partial w}{\partial \theta}(\phi_0, \theta_0) = 0$ and $\frac{\partial v}{\partial \theta}(\phi_0, \theta_0).(v(\phi_0, \theta_0) - u(\phi_0, \theta_0)) \neq 0$.*

Then $\frac{\partial u}{\partial \theta}(\phi_0, \theta_0) \neq 0$ and $\frac{\partial u}{\partial \theta}(\phi_0, \theta_0).(u(\phi_0, \theta_0) - w(\phi_0, \theta_0)) = 0$

Proof Trivial, by taking partial derivative with respect to θ of $(||u - v||_2)^2$ and $(||u - w||_2)^2$ at (ϕ_0, θ_0).

Lemma 3 *In the equilateral immersion of **Section 1**, A_j, B_j and B_{j+2} are not in a straight line.*

Proof Assuming so, we get a quadrilateral $A_j A_{j+1} A_{j+2} B_{j+2}$ with

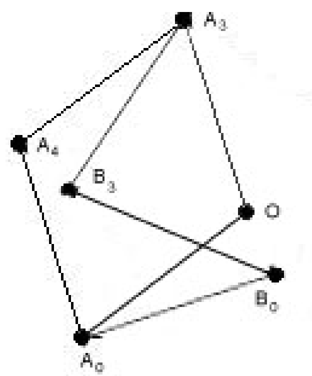
$$|A_j B_{j+2}| = 2, |A_j A_{j+1}| = |A_{j+1} A_{j+2}| = |A_{j+2} B_{j+2}| = 1$$

$$\angle A_j A_{j+1} A_{j+2} = \frac{3\pi}{5}, \angle A_{j+2} B_{j+2} A_j = \frac{4\pi}{5}$$

Then $|A_j A_{j+2}| \leq |A_j A_{j+1}| + |A_{j+1} A_{j+2}| = 2$, and as $\angle A_{j+2} B_{j+2} A_j > \frac{\pi}{2} > \angle A_j A_{j+2} B_{j+2}$, so $|A_j A_{j+2}| > |A_j B_{j+2}| = 2$, so we get a contradiction.

Lemma 4 *With the same notation, complete the rhombus $OA_3A_4A_0$. Let u_j denote the vector A_jB_j, w denote the vector B_3B_0 and let v_j be a vector orthogonal to u_j, such that the orientation of the basis $\{u_0, v_0\}$ is same as that of $\{u_3, v_3\}$.*

Then $(v_0 - v_3)$ is not orthogonal to w.



Proof Assuming the contrary, $(v_0 - v_3).w = 0$.

Also w lies in the angle formed by u_0 and u_3. Thus $\exists r_0, r_3 \geq 0$ (with both not 0 as w has unit norm) with $w = r_0 u_0 + r_3 u_3$.

Let $c_j > 0$ be such that $c_j v_j$ has unit norm. There exists linear transformation T (which is rotation by $\frac{\pi}{2}$ or $-\frac{\pi}{2}$) with $c_j v_j = T u_j$. So $w.Tw = 0$ gives $w.(c_0 r_0 v_0 + c_3 r_3 v_3) = 0$.

But u_0 and u_3 are linearly independent, and hence are v_0 and v_3. $c_j r_j \geq 0$ (with both not 0) gives $c_0 r_0 v_0 + c_3 r_3 v_3$ and $v_0 - v_3$ linearly independent. $w \neq 0$ being orthogonal to both of them gives a contradiction.

Now, we try to prove that the equilateral immersion, as described in **Section 1** is not rigid. For this, let us define θ_0 as in **Section 1** and $\phi_0 = \frac{3\pi}{5}$. We give an immersion for each (ϕ, θ) in some neighbourhood of (ϕ_0, θ_0), as follows.

Put A_0 at $(0,0)$ and A_1 at $(1,0)$. Let A_4, A_2 and B_0 be such that $\angle A_4 A_0 A_1 = \angle A_0 A_1 A_2 = \phi$ and $\angle B_0 A_0 A_1 = \theta$. Clearly the co-ordinates of all these 5 points are analytic functions of ϕ and θ. For $(\phi, \theta) = (\phi_0, \theta_0)$, length of $A_4 A_2 \in (0, 2)$. Hence by **Lemma 1**, let A_3 be the (unique) point which is at a distance of 1 from A_2 and A_4, with co-ordinates as analytic functions of ϕ and θ, and agreeing with A_3 of **Section 1** for $(\phi, \theta) = (\phi_0, \theta_0)$.

Similarly construct B_2 to be of unit distance from each of B_0 and A_2, and inductively after constructing B_j, construct B_{j+2} at unit distance from B_j and A_{j+2} (indices obviously being read modulo 5). Join A_j to A_{j+1}, A_j to B_j and B_j to B_{j+2}. All the edges (except possibly $B_0 B_3$) are of length 1, and at $(\phi, \theta) = (\phi_0, \theta_0)$ the immersion coincides with the one described in **Section 1**.

Thus, now we are in a setting to prove the following theorem, which will settle the problem.

Theorem 2 *If $f(\phi, \theta) = |B_0 B_3|^2$, then $\exists g : U \to \mathbb{R}$ with U a neighbourhood of ϕ_0 such that $g(\phi_0) = \theta_0$ and $f(\phi, g(\phi)) = 1$.*

Proof Let a_j and b_j be the co-ordinate functions (from a neighbourhood of (ϕ_0, θ_0) to $\mathbb{R}^2$) of A_j and B_j respectively.

By a repeated application of the **Lemma 1**, we know that b_0 and b_3 are analytic and hence f is analytic function of (ϕ, θ).

For $j \in \{0, 2, 4, 1\}$ and with $(\phi, \theta) = (\phi_0, \theta_0)$, assume $\frac{\partial b_j}{\partial \theta}(\phi_0, \theta_0) \neq 0$ and it is orthogonal to $A_j B_j$, then by **Lemma 3** $\angle A_j B_j B_{j+2} \neq \pi$, so $\frac{\partial b_j}{\partial \theta}(\phi_0, \theta_0)$ is not orthogonal to $B_j B_{j+2}$. Since B_{j+2} was defined as the point at unit distance from B_j and A_{j+2}, so by **Lemma 2**, $\frac{\partial b_{j+2}}{\partial \theta}(\phi_0, \theta_0) \neq 0$ and is orthogonal to $A_{j+2} B_{j+2}$.

Since the above assumption is clearly true for $j = 0$, so by induction it is true for $j = 3$. Now at (ϕ_0, θ_0), there could be two cases. As θ increases, the direction of B_j's rotation around A_j, could be the same or reverse of the direction of B_{j+2}'s rotation around A_{j+2}. In either case however, the directions of rotation of B_0 and B_3 (around A_0 and A_3 respectively) are the same. As θ increases, B_0 rotates anti-clockwise, and hence so does B_3.

Thus by **Lemma 4**, $(\frac{\partial b_0}{\partial \theta} - \frac{\partial b_3}{\partial \theta})(\phi_0, \theta_0)$ is not orthogonal to $B_0 B_3$. This gives $\frac{\partial f}{\partial \theta}(\phi_0, \theta_0) \neq 0$. As $f(\phi_0, \theta_0) = 1$, hence by Implicit function theorem, $\exists g : U \to \mathbb{R}$ with the required properties.