Solutions to September-October problems

Solution to 1 :
(i) Let us first show that 0 is a limit point of the set $\{f(a) : f \in S\}$. Fix $\epsilon > 0$. Let S_n, S_n^+ denote, respectively, the sets of polynomials in S of degrees $\leq n$ and, those in S_n with non-negative coefficients. Clearly, there are 2^{n+1} elements in S_n^+. If $f \in P_n^+$, then

$$0 \leq f(a) \leq 1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}.$$

If $f \neq g$ are in S_n^+, then $f - g \in S_n$ and, therefore, by the pigeon hole principle, there exist $f_n \neq g_n$ in S_n^+ with

$$0 < |f_n(a) - g_n(a)| \leq \frac{1}{2^{n+1}} \frac{a^{n+1} - 1}{a - 1}.$$

As $1 < a < 2$, the right hand side can be made smaller than ϵ for large n and thus 0 is indeed a limit point.
Now, let us show that this already implies that the values are dense on the whole line. Let n, k be arbitrary positive integers. Choose $f \in S$ with $0 < f(a) < a^{-n-k}$. Choose $m \geq n$ such that $a^{-m-1-k} \leq f(a) < a^{-m-k}$. Thus, we have

$$a^{-k-1} \leq a^m f(a) < a^{-k} \cdots\cdots (\heartsuit)$$

Now, $\pm t^m f(t)$ is also in S. Using this simple observation, we shall construct a sequence of polynomials $\{f_N\}$ in S each of which satisfies $a^{-k-1} \leq f_N(a) < a^{-k}$ and for which no pair has common terms. To do this, start with $f_1, f_2, \cdots, f_l \in S$ satisfying the above properties that $a^{-k-1} \leq f_r(a) < a^{-k}$ for $r = 1, \cdots l$ and no two have common terms. To choose f_{l+1} in S, get m such that $(\heartsuit)$ is satisfied for some f in S and put $f_{l+1}(t) = t^m f(t)$. Thus the sequence has been defined inductively and satisfies

$$Na^{-k-1} \leq f_1(a) + f_2(a) + \cdots + f_N(a) \leq Na^{-k}.$$

Note that the polynomial $f_1(t) + \cdots + f_N(t) \in S$ as well. As N, k are arbitrary and $a > 1$, it is clear that elements of $\{f(a) : f \in S\}$ approximate arbitrarily any non-negative real number and, hence, any real number.
(ii) Order the polynomials in S lexicographically according to decreasing exponents of the terms. More precisely, say that $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ is of smaller rank than $g(t) = b_0 + b_1 t + \cdots + b_m t^m$ if the first $k \geq 0$ for which $|a_k| \neq |b_k|$ satisfies $a_k = 0 \neq b_k$. In case $|a_l| = |b_l|$ for all l, it is convenient to define f to be of smaller rank than g if $f(e) < g(e)$. Of course, e is just an essentially arbitrary choice.
Now, let us look at the values $\{f(b) : f \in S\}$. The special property to keep in mind about b is that $b^2 = b+1$. In particular, we will keep using $b^{d+2} = b^{d+1}+b^d$.

Now, if s is a value at b, let f_s denote the polynomial of smallest rank in S with $f_s(b) = s$. Write

$$f_s(t) = \pm(t^{h(s)} + \cdots + t^{\alpha(s)} - t^{\beta} \pm \cdots)$$

In the above expression, $h(s)$ is the highest term, and $\alpha(s)$ is the first place where it changes sign. We claim that all the terms with exponents less than $\alpha(s)$ occur and they occur with alternating signs. Now,

$$f_s(b) = \pm(b^{h(s)} + \cdots + b^{\alpha(s)} - b^{\beta} \pm \cdots)$$

First, we observe that if $\beta = \alpha(s) - 2$, then the terms $t^{\alpha(s)} - t^{\alpha(s)-2}$ can be replaced by $t^{\alpha(s)-1}$ of smaller rank, a contradiction. If $\beta < \alpha(s) - 2$, then the term $t^{\alpha(s)} - t^{\beta}$ can be replaced by $-t^{\alpha(s)-1} - t^{\alpha(s)-2} - t^{\beta}$ whose rank is smaller. Therefore, $\beta = \alpha(s) - 1$. Further, if $\gamma > 1$ and if $\pm(t^{\gamma} - t^{\gamma-1})$ occurs, then $\pm t^{\gamma-2}$ also occurs where the sign is that of t^{γ}. If not, one can replace the terms $t^{\gamma} - t^{\gamma-1} + 0.t^{\gamma-2}$ by $\pm t^{\gamma-2}$ or the terms $\pm(t^{\gamma} - t^{\gamma-1} - t^{\gamma-2})$ by 0 which have smaller rank, a contradiction. Hence, finally we have

$$s = f_s(b) = \pm(b^h + b^{h-1} + \cdots b^{\alpha} + (-1)^{\alpha} \sum_{r=0}^{\alpha-1} (-1)^r b^r)$$

$$= \pm(b^h + b^{h-1} + \cdots b^{\alpha} - \frac{(b^{\alpha} \pm 1)}{b+1}).$$

Evidently, $|s| \to \infty$ as $h \to \infty$, which implies that the values $\{f(b) : f \in S\}$ is discrete.

Solution to 2 :
This can be easily proved by partial fractions. But, in fact, it is a special case of a class of identities involving arctan function which can be easily proved by the method of telescoping. Let us see how. Indeed, consider any real function $f(x)$ with fixed sign and look at the function $g(x) = \frac{f(x+1)-f(x)}{1+f(x+1)f(x)}$. Then, by telescoping property of the sum, one has

$$\sum_{r=1}^{n} tan^{-1} g(r) = tan^{-1} f(n+1) - tan^{-1} f(1)$$

for each n. Taking limits as $n \to \infty$ and assuming f has a limit $f(\infty)$ (which includes the possibility of limit being ∞), we have

$$\sum_{r=1}^{\infty} tan^{-1} g(r) = tan^{-1} f(\infty) - tan^{-1} f(1).$$

The assumption about f having same sign is necessary; otherwise, one will have to add multiples of π etc. to get proper identities. Looking at the particular case $f(x) = ax$ for fixed a and in $x > 0$, gives us $g(x) = \frac{a}{1+a^2x+a^2x^2}$ and, so we get

$$\sum_{r=1}^{\infty} tan^{-1}\frac{a}{1+a^2r+a^2r^2} = \frac{\pi}{2} - tan^{-1}a.$$

Differentiating with respect to a yields,

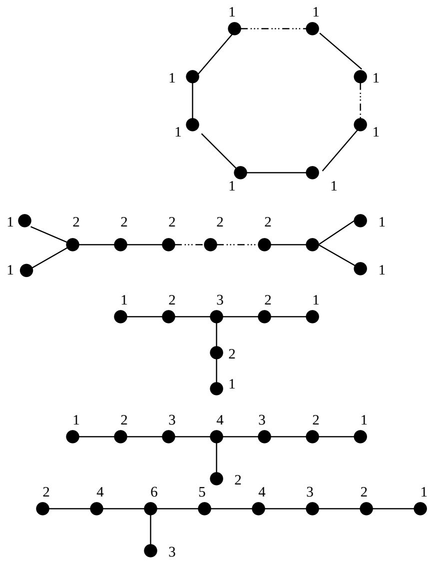
$$\sum_{r=1}^{\infty} \frac{a^2r^2+a^2r-1}{a^4r^4+2a^4r^3+(2a^2+a^4)r^2+2a^2r+1+a^2} = \frac{1}{1+a^2}.$$

The special case $a = 1$ gives us the identity posed. One can get several such identities by taking different such functions $f(x)$.

Solution to 3 :
This is yet another appearance of the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (!)
Actually, this combinatorial description coincides with the following group-theoretic description. In the group $SL(n, \mathbf{C})$, any matrix A which satisfies $A^{n+1} = I$ is diagonalizable and its eigenvalues are $(n+1)$-th roots of unity. Note that the product of the eigenvalues of A is also required to be 1. Therefore, the number of conjugacy classes $[A]$ of $SL(n, \mathbf{C})$ satisfying $A^{n+1} = I$ equals the number of multi-sets in $\mathbf{Z}/(n+1)\mathbf{Z}$ whose sum is zero.

Solution to 4 :
Both questions can be rephrased in terms of matrices. Associate to a graph (to be found!) a symmetric matrix A of the following type (and size to be the number of vertices !). The diagonal entries are all equal to 2 and, for $i \neq j$, the entry $a_{ij} = -1$ if and only if the vertices corresponding to i, j are connected. The first problem is to find all such 'connected' matrices A and all positive solutions x (column vector with all entries positive) such that $Ax = 0$. Clearly, a solution vector gives a labelling. Similarly, for the second problem, one has to solve $Ax = \mathbf{2}$ for positive solutions x, where the $\mathbf{2}$ on the right hand side above is the column vector with all entries equal to 2. The solutions to these linear algebra problems are not all that obvious, but they turn out to give the following graphs answering the two questions. Any one interested in seeing a working of the linear algebra assertion may either consult Victor Kac's 'Infinite-dimensional Lie algebras', chapter 4 or write to this email.

Graphs for (i)



Graphs for (ii)