Solutions to January - March 2007 problems

Solution 1 (Ashay Burungale).
Writing $a_{n+1} = \frac{[(1+\sqrt{2})^{2n+1}]+2}{4}$, note that $a_{n+1} = \frac{(1+\sqrt{2})^{2n+1}-(\sqrt{2}-1)^{2n+1}+2}{4}$. This follows since the last expression is seen to be an integer by the binomial expansion and since this expression is obtained from the real number $\frac{(1+\sqrt{2})^{2n+1}+2}{4}$ by subtracting the real number $\frac{(\sqrt{2}-1)^{2n+1}}{4}$ which is between 0 and 1. Now, clearly by induction on n, we have

$$a_{n+1} = 6a_n - a_{n-1} - 2.$$

As $a_2 = 4$ and a_n is non-decreasing, we have that a_{even} is even ≥ 4. To see that a_{odd} is composite, rewrite

$$a_{n+1} = \frac{(1+\sqrt{2})^{2n+1} - (\sqrt{2}-1)^{2n+1} + 2}{4}$$

$$= \frac{(1+\sqrt{2})^{n+1} - (\sqrt{2}-1)^{n+1}}{2\sqrt{2}} \frac{(1+\sqrt{2})^{n} + (\sqrt{2}-1)^{n}}{\sqrt{2}}.$$

If $n+1$ is odd, then both the fractions above are integers as seen by the binomial expansion.

Solution 2 (Ashay Burungale).
Now $xx'(xx')xx' = xx'(xx'x)x' = xx'xx' = xx'$ which shows (by uniqueness)
$xx' = (xx')'$ (1)
From this, it is easy to see that $(xx')^d = xx'$ for all d. Also $x'xx' = x'$ since $x(x'xx')x = xx'x = x$. Therefore, by the characterizing property of $(x')'$,
we have $(x')' = x$ (2)
Notice that (2) implies the version $x'x = (x'x)'$ of (1). Further, $(xy)'xx'$ has the characterizing property of $(xy)'$ since

$$(xy)((xy)'xx')(xy) = xy(xy)'xy = xy.$$

Therefore, $(xy)'xx' = (xy)'$ (3)
Now, xx' also has the property characterizing $((xy)(xy)')'$ since

$$xy(xy)'(xx')xy(xy)' = (xy(xy)'xy)(xy)' = xy(xy)'.$$

Therefore, $xx' = (xy(xy)')' = xy(xy)'$ *by* (1) (4)

We also get from $(x(yx)')y(x(yx)') = x(yx)'(yx)(yx)' = x(yx)'$ that
$y = (x(yx)')'$ (5)
We notice that yxx' also satisfies the characterizing property of $(x(yx)')'$ because

$$x(yx)'yxx'x(yx)' = x(yx)'yx(yx)' = x(yx)'.$$

Thus, we have $yxx' = (x(yx)')'$ (6)
From (5),(6) we get $y = yxx'$ for all x, y which implies that

$$y'y(xx')y'y = y'(yxx')y'y = y'yy'y = y'y;$$

hence $xx' = (y'y)' = y'y \; \forall \;\; x, y$ (7)
In other words, $e := xx'$ is a uniquely defined element not depending on the choice of x. It is a left identity by definition. Indeed, we have also for any x that $xe = x(x'(x')') = x(x'x) = x$ and e is a right identity as well. For any x, we have $xx' = e$ and so x' is a right inverse. Finally, $x'x = e$ means x' is also the right inverse. This proves that G is a group.

Solution 3
The eigenvalues x, y, z of such a matrix would be roots of

$$T^3 - (a^2 + b^2 + c^2)T - 2abc = 0.$$

Thus, we are looking for a, b, c non-zero integers with distinct a^2, b^2, c^2 for which the above polynomial has all zeroes to be integers. Thus, one really needs to study the system of equations

$$x + y + z = 0$$

$$xy + yz + zx = -(a^2 + b^2 + c^2)$$

$$xyz = 2abc$$

This defines actually a 2-dimensional surface S in the projective 5-space $\mathbf{P}^5$. In fact, the equation $x + y + z = 0$ shows that it is contained even in $\mathbf{P}^4$. In order to find integral (or rational) points on S, it is useful to look for rational curves on it; that is, find polynomials in one variable satisfying the defining equations. Such a study has been made by some people who show how to completely solve the problem; those interested in this study are referred to an article by F.Beukers, R.van Luijk & R.Vidunas in Nieuw Archief voor wiskunde, June 2002. However, the question here was to find just one solution which could be found by hit and trial methods. For example, two solutions are $\{a, b, c\} = \{26, 51, 114\}$ and $\{a, b, c\} = \{-125, 99, -57\}$.

Solution 4 (Ashay Burungale)
First, we consider part (i).
For $n = 2$, an equilateral triangle with unit side length does the job. In general, we shall apply induction on n. Consider $\mathbf{R}^{n-1}$ as a hyperplane H in $\mathbf{R}^n$. Assuming the result for $n-1$, there is a set $\{a_1, \cdots, a_n\}$ of vectors in H satisfying $||a_i - a_j|| = 1$ for $1 \leq i \neq j \leq n$. Look at the 'centre of mass' $c_n = \sum_{i=1}^n a_i$ and the vectors $b_i = a_i - c_n$. Now, $\sum_{i=1}^n b_i = 0$ and $||b_i - b_j|| = ||a_i - a_j|| = 1$ for $1 \leq i \neq j \leq n$. Therefore, for every i, j we get

$$||b_i||^2 - 2 < b_i, b_j > + ||b_j||^2 = 1.$$

For a fixed i, summing over all $j \neq i$, and using $\sum_{k=1}^n b_k = 0$, we get

$$(n-1)||b_i||^2 + 2 < b_i, b_i > + \sum_{j \neq i} ||b_j||^2 = n - 1$$

which means that $n||b_i||^2 = n - 1 - \sum_{j=1}^n ||b_j||^2$. As the right side is independent of i, all the b_i's have the same length which is equal to $\sqrt{\frac{n-1}{2n}}$. If e_n is a unit vector in $\mathbf{R}^n$ orthogonal to H, then we take $a_0 = c_n + te_n$ where $t = \sqrt{\frac{n+1}{2n}}$. Note that $||a_0 - a_i|| = 1$ since

$$||a_0 - a_i||^2 = ||te_n - b_i||^2 = t^2 + ||b_i||^2 = \frac{n+1}{2n} + \frac{n-1}{2n} = 1.$$

Thus $a_0, a_1, \cdots, a_n$ form a regular n-simplex in $\mathbf{R}^n$.

Now, we look at part (ii).
Let $a_0, a_1, \cdots, a_n$ in $\mathbf{R}^n$ form a regular n-simplex. As the centre of mass c of the simplex is $\frac{\sum_{i=0}^n a_i}{n+1}$, the vectors $b_i := a_i - c$ sum to zero and as before $||b_i||$ are all equal (to β, say). Now, the dihedral angle $\delta = \pi - arccos(t)$ where $t = \frac{<b_i, b_j>}{||b_i|| ||b_j||}$, which is cosine of the angle between b_i and b_j. Taking inner product of the equality $\sum_{i=0}^n b_i = 0$ with b_1, we have $(1 + nt)\beta^2 = 0$. Thus, we get $t = -1/n$ from which $\delta = arccos(1/n)$.