Challenging problems - open to ALL
Send solutions to sury@isibang.ac.in by 31/12/2008

1. (Elementary)
Let $u_1 = 2, u_{n+1} = 2^{u_n}$ for all $n \geq 1$. For an arbitrary natural number n, show that the residues of u_k mod n eventually becomes a constant function of k.

2. (Elementary)
Show that there are arbitrarily large n for which $n^4 + 1$ has a prime divisor larger than $2n$.

3. (Elementary*)
If A is a finite ring, prove that there exist natural numbers $m \neq n$ so that $a^m = a^n$ for all $a \in A$.

4. (Elementary*)
For fixed natural numbers N and n, consider the sets
$A = \{a := (a_1, \cdots, a_n) \in \mathbf{N}^n : \sum_{i=1}^n a_i \leq N\}$ and
$B = \{b := (b_1, \cdots, b_n) \in \mathbf{N}^n : b_i \text{ are distinct, each } b_i \leq N\}$.
Prove that

$$\sum_{a \in A} \frac{1}{a_1 a_2 \cdots a_n} = \sum_{b \in B} \frac{1}{b_1 b_2 \cdots b_n}.$$

5. (Advanced level)
Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that f^2, f^3 are in $C^\infty(\mathbf{R})$. Show that $f \in C^\infty(\mathbf{R})$.

6. (Advanced level)
If p is a prime, let $\mathbf{F}_p$ denote the field of p elements and let $PSL(2, \mathbf{F}_p)$ denote the quotient $SL(2, \mathbf{F}_p)/\text{center}$. Show that any solution to $A^2 = B^3 \neq I$ in the group $PSL(2, \mathbf{F}_p)$ satisfies $A = C^3, B = C^2$ for some $C \in PSL(2, \mathbf{F}_p)$.