Problems for June-August, 2009

1. (Elementary)
(i) Find all solutions in integers, of the equation $\frac{1}{k+l+m} = \frac{1}{k} + \frac{1}{l} + \frac{1}{m}$.
(ii) Find infinitely many solutions in non-zero integers of the equation

$$\frac{1}{k+l+m+n} = \frac{1}{k} + \frac{1}{l} + \frac{1}{m} + \frac{1}{n}.$$

2. (Elementary number theory)
For any natural number $n > 1$, define $E(n)$ to be the highest exponent to which a prime divides it. For instance, $E(12) = E(36) = 2$. Show that $lim_{N\to\infty} \frac{1}{N} \sum_{n=2}^{N} E(n)$ exists. Find an approximate value.

3. (Linear algebra)
This is a more difficult version of an old problem. For a 4-tuple $v := (v_1, v_2, v_3, v_4)$ of real numbers, define $T(v) = (|v_1 - v_2|, |v_2 - v_3|, |v_3 - v_4|, |v_4 - v_1|)$. Consider the transformations 'inversion' $i : v = (v_1, v_2, v_3, v_4) \mapsto (v_4, v_3, v_2, v_1)$ and 'cyclic permutation' $c : v = (v_1, v_2, v_3, v_4) \mapsto (v_2, v_3, v_4, v_1)$. Note that i is of order 2 and c is of order 4. Call $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ equivalent if w is one of the 8 elements :

$$\{v, c(v), c^2(v), c^3(v), i(v), ic(v), ic^2(v), ic^3(v)\}.$$

Here $c^2(v)$ stands for $c(c(v))$ etc. Prove that there is a special 4-tuple $v_0 = (x, y, z, w)$ such that whenever v is not equivalent to $tv_0 := (tx, ty, tz, tw)$ for any real $t > 0$, we have $T^n(v) = (0, 0, 0, 0)$ for some $n > 0$. Can you determine such a 4-tuple v_0?

4. (Groups)
For any group G, consider the set $S(G, n)$ of $n \times n$ matrices whose entries are either from G or equal to 0. Define the multiplication as matrix multiplication where we put $0.g = g.0 = 0$ for all $g \in G$. Find the structure of the group G. In particular, determine its order (if G is finite), whether it is simple etc.

5. (Rings)
Give an example of a commutative ring A for which ACCP holds (that is, any ascending chain of principal ideals stabilizes) whereas the property ACCP does not hold good for the polynomial ring $A[X]$. First, prove the easy assertion that if A is an integral domain with ACCP, then $A[X]$ continues to have ACCP.

6. (Fields)
If K is an infinite field and L is an extension field with $L \neq K$, prove that $L^\times/K^\times$ is infinite. Can it be a finitely generated group?

7. (Combinatorics)
When the determinant of a symmetric $n \times n$ matrix is evaluated, let $M(n)$ denote the number of distinct monomials which appear. For instance, $M(3) = 5$. Show that

$$1 + \sum_{n \geq 1} \frac{M(n)t^n}{n!} = \frac{exp((2t+t^2)/4}{\sqrt{1-t}}$$

for $|t| < 1$.

8. (Elementary)
Define the sequences $x(n), y(n)$ by :
$x(0) = 0, y(0) = 1, x(n) = n - y(x(n-1)), y(n) = n - x(y(n-1))$ for $n \geq 1$.
Determine closed expressions for these sequences.

9. (Topology)
Prove that the 2-torus $S^1 \times S^1$ cannot be covered by two open, contractible subsets. Can it be covered by 3 open, contractible subsets?

10. (Number theory)
Show that there is an isomorphism between the additive group of real, algebraic numbers and the multiplicative group of positive real algebraic numbers. Is there one which is order-preserving?

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