Solutions due by March 31, 2009

Q 1. (Elemnetary)
Let G be a finite group of order n. For any subset S of G, put

$$S^k = \{s_1 \cdots s_k : s_i \in S\}$$

for each $k \geq 1$. Prove that S^n is always a subgroup.

Q 2. (Advanced)
There exist real functions which are continuous at each irrational and discontinuous at each rational number. For instance, $f(p/q) = 1/q$ if p/q is in its lowest terms with $q > 0$ and with $f(irrationals) = 0$ is one such. Prove that there is no function $f : \mathbf{R} \to \mathbf{R}$ which is differentiable at every irrational and discontinuous at every rational.

Q 3. (Elementary)
Prove that an equilateral triangle cannot have all its vertices to be lattice points (points whose co-ordinates are integers). More generally, show that a regular n-gon can have all vertices to be lattice points if, and only if, $n = 4$.

Q 4. (Elementary)
Prove that $\sum_{r=1}^{n} cot^2(\frac{r\pi}{n+1}) = \frac{n(2n-1)}{3}$.
More generally, for $m, n \geq 1$, show that
$\sum_{r=1}^{n} cot^{2m}(\frac{r\pi}{n+1}) = c_{2m}n^{2m} + c_{2m-1}n^{2m-1} + \cdots + c_0$ and obtain c_{2m}.

Q 5. (Elementary)
Consider the n-th cyclotomic polynomial $\Phi_n(X) = \prod_{(r,n)=1}(X - e^{2ir\pi/n})$. Determine all those natural numbers n for which all coefficients of Φ_n are non-negative.

Q 6. (Advanced)
Let B be a ring and A be a subring with the property that whenever $a \in B$ satisfies $a^n \in A$ for *every large enough* n, then $a \in A$. Show that the power series ring $A[[X]]$ satisfies the same property as a subring of $B[[X]]$.
Use this to give another proof of problem numbered 5 in the previous set (viz., if $f : \mathbf{R} \to \mathbf{R}$ is a function such that f^2, f^3 are smooth, then so is f). There is also a simpler proof of this not using the above ring-theoretic lemma.