Solution 1.
Apply induction on n. It is clear for $n = 1$. Assume that $n = m + 1 \geq 2$ and that the result holds for positive integers $\leq m$. If n is odd, $2^{\phi(n)} \equiv 1 \pmod{n}$ by Eulers theorem. By the induction hypothesis (since $\phi(n) < n$), the sequence is eventually constant modulo $\phi(n)$; say, $u_r \equiv c \pmod{\phi(n)}$ for large r. Consequently, $u_{r+1} = 2^{u_r} \equiv 2^c \pmod{n}$ is constant which completes this case. If $n = m + 1 = 2^k l$ for some positive integer k and odd l. Again, by induction hypothesis, the sequence is eventually constant modulo l. Now, $u_r \equiv 0 \pmod{2^k}$ for all sufficiently large r. Thus, $u_r \equiv u_{r+1} \pmod{2^k l}$ for large r onwards. The induction is complete.

Solution 2.
Note first that there are infinitely many primes with the property that they divide a number of the form $n^4 + 1$ for some n. Indeed, if $p_1, \cdots, p_r$ were all such primes, then no prime would be able to divide $(p_1 \cdots p_r)^4 + 1$! Let P denote the set of all primes with this property. For each $p \in P$, we may choose $n < p/2$ with $p|(n^4 + 1)$ because we may replace any n by its residue modulo p and further change n to $p-n$ in case $n > p/2$. Thus, for each $p \in P$, we have got hold of n with $2n < p$ and $p|(n^4 + 1)$. Of course, a particular $n^4 + 1$ has only finitely many prime divisors so that infinitely many integers n are produced from the infinite set P.

Solution 3 (Solved also by Sumitra Garai, M.Math. student from I.S.I.)
It suffices to show that for each pair $a_1, a_2 \in A$, there is a common m, n. Let $a_i^{m_i} = a_i^{n_i}$ for $i = 1, 2$ and $m_i \neq n_i$. Consider $m = m_1 m_2 + n_1 n_2, n = m_1 n_2 + m_2 n_1$. Then

$$a_1^m = (a_1^{m_1})^{m_2} (a_1^{n_1})^{n_2} = (a_1^{n_1})^{m_2} (a_1^{m_1})^{n_2} = a_1^n.$$

Similarly, $a_2^m = a_2^n$.

Solution 4.
The equality

$$\frac{1}{x_1 x_2} = \frac{1}{x_1(x_1 + x_2)} + \frac{1}{x_2(x_2 + x_1)}$$

obviously generalizes (and follows by induction on n) to :

$$\frac{1}{x_1 x_2 \cdots x_n} = \sum_{\sigma \in S_n} \frac{1}{x_{\sigma_1}(x_{\sigma_1} + x_{\sigma_2}) \cdots (x_{\sigma_1} + \cdots + x_{\sigma_n})}.$$

In fact, writing the RHS as

$$\frac{1}{x_1+\cdots+x_n}\sum_{r=1}^{n}\sum_{\sigma\in S_n,\sigma(n)=r}\frac{1}{x_{\sigma_1}(x_{\sigma_1}+x_{\sigma_2})\cdots(x_{\sigma_1}+\cdots+x_{\sigma_{n-1}})}$$

and assuming the equality for $n-1$, we get

$$\frac{1}{x_1+\cdots+x_n}\sum_{r=1}^{n}\sum_{\sigma\in S_n,\sigma(n)=r}\frac{1}{\prod_{i\neq r}x_i}$$

which is simply $\frac{1}{x_1x_2\cdots x_n}$. Now, if $(b_1,\cdots,b_n)\in B$ then b_i's are distinct and let σ be that permutation for which

$$b_{\sigma_1}<b_{\sigma_2}<\cdots<b_{\sigma_n}.$$

Then, writing $b_{\sigma_i}-b_{\sigma_{i-1}}$ as x_{σ_i}, we have

$$b_1\cdots b_n=x_{\sigma_1}(x_{\sigma_1}+x_{\sigma_2})\cdots(x_{\sigma_1}+\cdots+x_{\sigma_n}).$$

Also, $b\in B$ means $b_i\leq n$ for all n and are distinct; so $(x_1,\cdots,x_n)\in A$. Therefore,

$$\sum_{b\in B}\frac{1}{b_1\cdots b_n}=\sum_{\sigma\in S_n}\sum_{x\in A}\frac{1}{x_{\sigma_1}(x_{\sigma_1}+x_{\sigma_2})\cdots(x_{\sigma_1}+\cdots+x_{\sigma_n})}=\sum_{x\in A}\frac{1}{x_1\cdots x_n}$$

using the earlier equality.

Solution 6 (Due to Professor David Savitt.)
We shall prove the (apparently) stronger statement with p replaced by any power q of p. We will prove that if $A^2=C^3$, then either $A^2=C^3=I$ or there exists C with $A=C^3, B=C^2$. Note firstly that if $A,B\in PSL_2(\mathbf{F}_q)$ satisfy $A^2=C^3$ and, if there is a solution C to $A=C^3, B=C^2$ exists in $PSL_2(\overline{\mathbf{F}_p})$, then $C=AB^{-1}$ is automatically in the subgroup G of $PSL_2(\mathbf{F}_q)$ generated by A,B. Here, of course, $\overline{\mathbf{F}_p}$ denotes an algebraic closure of $\mathbf{F}_q$. Thus, it does not matter for this problem if the matrices A,B are replaced by conjugates PAP^{-1}, PBP^{-1} for some $P\in PSL_2(\overline{\mathbf{F}_p})$. The crucial result needed for this is a knowledge of the various finite subgroups of $PSL_2(\overline{\mathbf{F}_p})$. There are many sources like Suzuki's and Dickson's texts; the latter has in sections 255 and 260 the following result :
Any finite subgroup of $PSL_2(\overline{\mathbf{F}_p})$ is either conjugate to a subgroup of $PGL_2(\mathbf{F}_p^n)$

or $PSL_2(\mathbf{F}_p^n)$ *for some* n *or the upper triangular invertible matrices or, is isomorphic to* A_4, S_4, A_5 *or* D_{2m} *where* D_{2m} *is the dihedral group of order* $2m$ *for some* $m \geq 2$ *not divisible by* p.
Using this, we may consider our $G =< A, B >$ case-by-case. If G is conjugate (which, by the observation made earlier, can be thought of as equal) to $PSL_2(\mathbf{F}_p^n)$, then it is a simple group unless $p^n = 2$ or 3 (in which cases, the result can be verified directly). As $< A^2 >=< C^3 >$ is a proper normal subgroup, this must be the identity. In the case of $PGL_2(\mathbf{F}_p^n)$ also, a similar argument works as $PSL_2(\mathbf{F}_p^n)$is its unique proper normal subgroup when $p^n > 3$.
Consider the case of upper triangular group in $PSL_2(\overline{\mathbf{F}_p})$. By lifting A, B to $X, Y \in GL_2(\overline{\mathbf{F}_p})$, we have $X^2 = tY^3$ for some constant t. Thus, $x = tX, y = tY \in GL_2(\overline{\mathbf{F}_p})$ satisfy $x^2 = y^3$. We are in the case where $G =< x, y >$ is a subgroup of the group of upper triangular invertible matrices. Then $x^2 = y^3$ gives $x_{11}^2 = y_{11}^3$ and since x_{11}, y_{11} are in a cyclic group, we have $x_{11} = a^3, y_{11} = a^2$ for some $a \in \overline{\mathbf{F}_p}^*$. Similarly, there is b for the $(2,2)$-th entries. Thus,

$$x = \begin{pmatrix} a^3 & u \\ 0 & b^3 \end{pmatrix} \ , \ y = \begin{pmatrix} a^2 & v \\ 0 & b^2 \end{pmatrix}.$$

Moreover $x^2 = y^3$ gives

$$(a^2 - ab + b^2)((a + b)u - (a^2 + ab + b^2)v) = 0.$$

From ths, it is an easy exercise to conclude that either x^2, y^3 are scalars (when at least one of $a + b, a^2 - ab + b^2, a^2 + ab + b^2$ is zero) - that is, $X^2 = Y^3 = I$ - or

$$\frac{u}{a^2 + ab + b^2} = \frac{v}{a + b}.$$

In the latter case, if this ratio is s, we have

$$x = \begin{pmatrix} a & s \\ 0 & b \end{pmatrix}^3 \ , \ y = \begin{pmatrix} a & s \\ 0 & b \end{pmatrix}^2.$$

From this, we can get C for A, B also.
The cases $A_4, S_4, A_5, D_{2m} (m > 1)$ are left as exercises as it is a routine calculation.

The problem numbered 5 will be kept open for solution one more time. Its solution will be given with the next set as that set involves a related problem.