Problems for January-March 2008
Last date for submitting solutions : 31.03.2008

1. (i) Prove that $\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} = \sum_{n=1}^{\infty} \frac{2}{n^3}$.

(ii) Obtain an expression for $\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^3}$ in terms of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

2. Given arbitrary groups G_1, G_2 get a necessary and sufficient condition for every normal subgroup of $G_1 \times G_2$ to be of the form $N_1 \times N_2$ with N_1 normal in G_1 and N_2 normal in G_2. Using such a criterion, or otherwise, show for any group G that each normal subgroup of $G \times G$ must be of the form $N_1 \times N_2$ with N_1, N_2 normal in G if, and only if, $[G, M] \leq M$ for every normal subgroup M of G.

3. Determine the number of maps $f : \mathbf{Z}_m \to \mathbf{Z}_n$ satisfying

$$f(a+b) = f(a) + f(b)\ ,\ f(ab) = f(a)f(b).$$

Here $\mathbf{Z}_m$ and $\mathbf{Z}_n$ denote the integers modulo m and the integers modulo n respectively.

4.
Let $x_1, \cdots, x_n$ be independent variables.
Define the elementary symmetric functions

$$s_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}\ (1 \leq r \leq n)$$

as usual. Denote by N_k the sum $\sum_{i=1}^{n} x_i^k$ for $k = 0, 1, \cdots$ Writing $N_l = 0$ if $l < 0$, prove the Newton identities :

$$N_k - s_1 N_{k-1} + \cdots + (-1)^n s_n N_{k-n} = 0.$$

Try to give a proof which works both for $k < n$ and for $k \geq n$.

5.
Find all real $a > 0$ satisfying $[a[(a+1)n]] = n - 1$. Here $[x]$ is the 'greatest integer' function.

6.
Given distinct real numbers a, b, c satisfying $x^2 - y = y^2 - z = z^2 - x$, compute $(a+b+1)(b+c+1)(c+a+1)$.

7.
Show that a 28×17 rectangle cannot be tiled by 4×7 rectangles. Obtain a necessary and sufficient condition for an $l \times m$ rectangle to be tile-able by $a \times b$ rectangles.