New Problems

1.

Show that a number of the form $\frac{[(1+\sqrt{2})^{2n+1}]+2}{4}$ can not be prime. Here $[d]$ denotes the largest integer $\leq d$.

2.

Let G be a nonempty set with an associative binary operation such that corresponding to each $g \in G$, there exists a unique element g' such that $gg'g = g$. Prove that G is a group.

3.

Find a matrix of the form $\begin{pmatrix} 0 & a & c \\ a & 0 & b \\ c & b & 0 \end{pmatrix}$ with a, b, c integers so that all eigenvalues are integral and $abc \neq 0$ and a^2, b^2, c^2 are all distinct.

4.

The regular 2-simplex is an equilateral triangle, a regular 3-simplex is a regular tetrahedron etc. In general, for any n, if $e_0, \cdots, e_n$ are $n+1$ points in $\mathbf{R}^n$ whose pairwise distances are equal, the convex hull $\{\sum_i t_i e_i : t_i \geq 0, \sum_i t_i = 1\}$ is a regular n-simplex.

(i) For each n, show that there are $n+1$ points in $\mathbf{R}^n$ whose pairwise distances are all equal to 1 (hence a regular n-simplex exists).

(ii) Compute the dihedral (acute) angle made by two adjoining faces.
There are other (solid) angles as well which can be defined as follows. Consider a small regular n-simplex centred at the origin and a large sphere about the origin. For the plane of each of the $n+1$ faces of the n-simplex (each itself an $(n-1)$-simplex), we consider the half-space of all points on the same side of that plane as the simplex. The simplex itself is the intersection of these $n+1$ half-spaces. For any $s \leq n$, the intersection of k of these half-spaces is congruent to the intersection of any other set of s half-spaces, since the group of symmetries of the regular simplex is the group of permutations of its vertices. Let angle(n, s) denote the fraction of the sphere that intersects the intersection of some particular set of s half-spaces. Thus, the dihedral angle referred to above is angle$(n, 2)$.

(iii) Show that angle$(3, 3) = 3Cos^{-1}(1/3) - \pi$.