Final Examination

(i) Answer all questions. (ii) $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$***. (iii)*** $\mathbb{H}$***= upper half plane. (iv)*** $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$***. (v)*** $\mathbb{A}_{1,2}(0) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

1. Let $f \in Hol(\mathbb{D})$ ***and assume that*** $|f(z)| \leq 1$ ***for all*** $z \in \mathbb{D}$***. If*** $f(0) = 0$***, then prove that the series***

$$\sum_{n=0}^{\infty} f(z^n)$$

converges absolutely and uniformly on $\{z \in \mathbb{C} : |z| \leq r\}, r < 1$***.***

Answer: Using Schwarz's lemma we have

$$|f(z)| \leq |z|,$$

for all $z \in \mathbb{D}$***. Therefore***

$$|f(z^n)| \leq |z^n| = |z|^n,$$

for all $z \in \mathbb{D}$ ***and*** $n \geq 0$***. Now on*** $\{z \in \mathbb{C} \leq r\}, r < 1$

$$\sum_{n=0}^{\infty} |f(z^n)| \leq \sum_{n=0}^{\infty} |z|^n \leq \sum_{n=0}^{\infty} r^n.$$

Since $r < 1$ ***the series*** $\sum_{n=0}^{\infty} r^n$ ***converges. Hence the series***

$$\sum_{n=0}^{\infty} f(z^n)$$

converges absolutely and converges uniformly as we have a uniform bound i.e. $|f(z^n)| \leq r^n$ ***for all*** $z \in \mathbb{D}$***.***

2. Let γ ***be a smooth closed curve in*** $\mathbb{C}$***. Prove that the winding number of*** γ ***is identically zero on the unbounded component of*** $\mathbb{C} \setminus \{\gamma\}$***.***

Answer. Let $W(\gamma, z)$ ***be the winding number of a closed curve*** γ ***around a point*** $z \notin \gamma$ ***and defined as***

$$W(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}.$$

Now we know that $W(\gamma, z)$ ***is constant on each component of*** $\mathbb{C} \setminus \{\gamma\}$***. Since*** $\{\gamma\}$ ***is compact, so we can find*** z ***on the unbounded component such that***

$$|\zeta - z| > M$$

for all $\zeta \in \gamma$ ***and for any given arbitrary large*** M***. Therefore***

$$|W(\gamma, z)| \leq \frac{1}{2\pi} | \int_\gamma \frac{|d\zeta|}{|\zeta - z|} \leq \frac{L(\gamma)}{2\pi M},$$

where $L(\gamma)$ ***is the length of*** γ***. Hence*** $W(\gamma, z) \to 0$ ***as*** $M \to \infty$***. But*** $W(\gamma, z)$ ***is constant on the unbounded component of*** $\mathbb{C} \setminus \{\gamma\}$***. Therefore*** $W(\gamma, z)$ ***must be zero on the unbounded component of*** $\mathbb{C} \setminus \{\gamma\}$***.***

3. Prove that there is no branch of the logarithm on $\mathbb{C}\setminus\{0\}$.

Answer: Let $G=\mathbb{C}\setminus\{0\}$ and $G^{'}=\mathbb{C}\setminus(-\infty,0]$. We will prove this by contradiction. Suppose if possible $f(z)$ is a branch of $logz$ on G. Denote $Logz$ be the principal branch of $logz$ on $G^{'}$. Then

$$Log(z)=log|z|+iarg(z),$$

where $-\pi<arg(z)<\pi$. Now $f|_{G'}$ is a branch of $logz$. Therefore it differs from the principle branch of $logz$ by $2ik\pi$ for some $k\in\mathbb{Z}$, i.e., for $z\in G^{'}$,

$$f(z)=log|z|+iarg(z)+2ik\pi,$$

where $-\pi<arg(z)<\pi$ and k is some integer. Now f is holomorphic on G in particular, f is continuous at -1. Therefore

$$\lim_{Im(z)>0,z\to-1}f(z)=-i\pi+2ik\pi$$

and

$$\lim_{Im(z)<0,z\to1}f(z)=i\pi+2ik\pi.$$

Continuity of f at -1 implies that $1=-1$ which is a contradiction. Hence there is no branch of the logarithm on $\mathbb{C}\setminus\{0\}$.

4. If $\alpha^4+\alpha^3+1=0$ for $\alpha\in\mathbb{C}$, then prove that $|\alpha|<\frac{3}{2}$.

Answer: Consider $f(z)=z^4+z^3$ and $g(z)=1$ for $z\in\mathbb{C}$. Again for $|z|=\frac{3}{2}$,

$$|f(z)|=|z^3(z+1)|=|z|^3|z+1|\leq|z|^3||z|-1|=\left(\frac{3}{2}\right)^3(\frac{1}{2})=\frac{27}{16}>1=|g(z)|.$$

We have f,g are holomorphic functions on $\mathbb{C}$ and $|f(z)|>|g(z)|$ for all $z\in C_{\frac{3}{2}}(0)$. Now f has roots at $z=0$ and $z=-1$. Hence by Rouche's theorem f and $f+g$ have the same number of zeros inside the circle $C_{\frac{3}{2}}(0)$. Therefore if α is a root of $f+g=z^4+z^3+1$, then $|\alpha|<\frac{3}{2}$.

5. Let f be a meromorphic function on $\mathbb{C}$ and let

$$|f(z)|\leq\left(\frac{|z|}{|z-1|}\right)^{\frac{3}{2}}.$$

Prove that $f=0$.

Answer: From the inequality we have $f(0)=0$ and $z=1$ is the only possible pole of f. Set $g(z)=(f(z))^2$ for $z\in\mathbb{C}$. Then g is meromorphic function on $\mathbb{C}$ and $g(0)=0$. Now rewriting the given inequality we have

$$|(z-1)^3g(z)|\leq|z|^3$$

for all $z\in\mathbb{C}$. Define $h(z)=(z-1)^3g(z)$. Then h is analytic on $\mathbb{C}$ and

$$|h(z)|\leq|z|^3.$$

Therefore h is a polynomial of degree 3 and this implies that g is constant. Hence f is constant. As $f(0)=0$, therefore f is identically zero.

6. Let $\{f_n\}$ be a sequence in $C(\bar{\mathbb{D}})\cap Hol(\mathbb{D})$. suppose that f_n converges uniformly on $\partial\mathbb{D}$ to a function f. Prove that f can be extended to a function in $C(\bar{\mathbb{D}})\cap Hol(\mathbb{D})$.

Answer. First of all $C(\bar{\mathbb{D}})$ ***is a complete metric space. Since*** $\bar{\mathbb{D}}$ ***is compact,*** $\sup|f_n - f_m|$ ***is attained in the boundary*** $\partial\mathbb{D}$ ***of*** $\mathbb{D}$***. Consider***

$$\alpha_{n,m} = \sup_{\bar{\mathbb{D}}} |f_n - f_m|$$

and

$$\beta_{n,m} = \sup_{\partial\mathbb{D}} |f_n - f_m|.$$

As $\sup_{\bar{\mathbb{D}}} |f_n - f_m| = \sup_{\partial\mathbb{D}} |f_n - f_m|$***, so***

$$\alpha_{n,m} = \beta_{n,m}.$$

Now it is given that $f_n \to f$ ***uniformly on*** $\partial\mathbb{D}$***. Therefore*** $\beta_{n,m} \to 0$ ***and hence*** $\alpha_{n,m} \to 0$ ***as*** $m, n \to \infty$***. So*** $\{f_n\}$ ***is cauchy on*** $\bar{\mathbb{D}}$***. But*** $C(\bar{\mathbb{D}})$ ***is a complete metric space so*** $\{f_n\}$ ***has a limit say*** g ***and*** $f_n \to g$ ***uniformly on*** $\bar{\mathbb{D}}$***. Therefore*** g ***is holomorphic on*** $\mathbb{D}$ ***and continuous on*** $\partial\mathbb{D}$***. Hence*** $f = g|_{\partial\mathbb{D}}$ ***i.e.,*** g ***is the extension of*** f ***to*** $\bar{\mathbb{D}}$ ***such that*** g ***is holomorphic on*** $\mathbb{D}$***.***

7. Examine the nature of the singularities of the following functions and determine the residues as the singularities $(a)\frac{1}{sin\frac{1}{z}}$ $(b)\frac{e^{-z}}{z^2}$***. Use part*** (b) ***to find***

$$\int_{|z|=3} \frac{e^{-z}}{z^2} dz.$$

Answer. Let $f(z) = sin\frac{1}{z}$***. Then*** f ***has zeros at all*** $z = \frac{1}{n\pi}$***. They are all zeros of order 1 for*** $n \neq 0$***. Therefore*** $\frac{1}{f}$ ***has simple poles at*** $z = \frac{1}{n\pi}$ ***for*** $n \in \mathbb{Z} \setminus \{0\}$***. Let*** $h(z) = \frac{e^{-z}}{z^2}$***. Then*** h ***has pole of order 2 at*** $z = 0$***. It is easy to see that***

$$Res(\frac{1}{f}, \frac{1}{n\pi}) = \frac{(-1)^{n+1}}{n^2\pi^2}$$

and

$$Res(h, 0) = -1.$$

Again

$$\int_{|z|=3} \frac{e^{-z}}{z^2} dz = 2\pi i \ \times Res(h, 0) = -2\pi i.$$

8. Let $f \in Hol(\mathbb{D})$ ***and assume that*** $|f(z)| < 1$ ***for all*** $z \in \mathbb{D}$***. Prove that***

$$\left| \frac{f(z) - f(w)}{1 - f(z)f(\bar{w})} \right| \leq \left| \frac{z - w}{1 - z\bar{w}} \right|.$$

Answer: Consider for $w \in \mathbb{D}$

$$\phi_w(z) = \frac{z - w}{1 - z\bar{w}}, \qquad z \in \mathbb{D}.$$

Define $h : \mathbb{D} \to \mathbb{D}$ ***as***

$$h(\phi_w(z)) = \phi_{f(w)}(f(z)) \quad (z \in \mathbb{D}).$$

Then h ***is holomorphic on*** $\mathbb{D}$***. Also*** $h(0) = h(\phi_w(w)) = \phi_{f(w)}(f(w)) = 0$ ***and*** $|h(\phi_w(z))| \leq 1$ ***as*** $|\phi_w(z)| < 1$ ***for*** $z \in \mathbb{D}$***. Therefore by applying Schwarz's lemma we have***

$$|h(\phi_w(z))| \leq |\phi_w(z)|$$

for all $w, z \in \mathbb{C}$***. This proves the required inequality.***