Final Examination

(i) Answer all questions. (ii) $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$. ***(iii)*** $\mathbb{H}=$ ***upper half plane. (iv)*** $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$. ***(v)*** $\mathbb{A}_{1,2}(0) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

1. Let $f : \mathbb{C} \to \mathbb{H}$ ***be a holomorphic function. Prove that*** f ***is a constant.***

Answer: Consider $g : \mathbb{C} \to \mathbb{C}$ ***defined by*** $g(z) = e^{if(z)}$. ***Clearly,*** g ***is holomorphic on*** $\mathbb{C}$ ***as*** f ***is so. Let*** $f = u + iv$. ***Then*** $v = Im(f) \geq 0$. ***Now for all*** $z \in \mathbb{C}$

$$|g(z)| = |e^{(-v+iu)}| = e^{-v} \leq 1$$

as $v \geq 0$. ***Therefore*** g ***is a bounded entire function. So by Liouville's theorem,*** g ***is constant and hence*** f ***is constant.***

2. Identify all the singularities of the following function and determine the nature of each singularity

$$\frac{z}{e^z - 1}.$$

Answer: Let $f(z) = \frac{z}{e^z-1}$. ***If*** $e^z - 1 = 0$, ***then*** $z = 2k\pi i$ ***for*** $k \in \mathbb{Z}$. ***Therefore the set*** $\{2k\pi i : k \in \mathbb{Z}\}$ ***is the singularity of*** f. ***Now*** $\lim_{z\to 0} zf(z) = 0$ ***and hence*** $z = 0$ ***is a removal singularity and*** $z = 2k\pi i$ ***for*** $k \neq 0$ ***are the simple poles of*** f ***as*** $\lim_{z\to 2k\pi i} f(z) = \infty$.

3. Calculate the residues of the following functions at each of the poles: $\frac{sinz}{z^2}$ ***and*** $\frac{cosz}{z^2}$.

Answer: Clearly 0 ***is a simple pole of the both functions. From the Taylor series of*** $sinz$ ***and*** $cosz$, ***we have*** $\frac{sinz}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - ...$ ***and*** $\frac{cosz}{z^2} = \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - ...$

Therefore $Res(\frac{sinz}{z^2}, 0) = 1$ ***and*** $Res(\frac{cosz}{z^2}, 0) = 0$.

4. Let $z = a$ ***be a pole of order*** n ***of a function*** f. ***Prove that*** $z = a$ ***is a pole of order*** $n + 1$ ***of*** $f^{'}$.

Answer. Since f ***has a pole of order*** n ***at*** $z = a$, ***then*** $f(z) = \frac{a_{-n}}{(z-1)^n} + \ldots + \frac{a_{-1}}{(z-1)} + G(z)$ ***where*** $a_{-n} \neq 0$ ***and*** G ***is holomorphic in some neighborhood of*** $z = a$. ***Differentiating*** f ***we have*** $f^{'}(z) = \frac{-na_{-n}}{(z-1)^{n+1}} + \ldots + \frac{-a_{-1}}{(z-1)^2} + G^{'}(z)$. ***Since*** $a_{-n} \neq 0$, ***so*** $f^{'}$ ***has a pole of order*** $n + 1$ ***at*** $z = a$.

5. Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx.$$

Answer. Let $f(z) = \frac{1}{z^4+1}$. ***Then*** $z = \frac{(1+i)}{\sqrt{2}}, \frac{(-1+i)}{\sqrt{2}}, \frac{(1-i)}{\sqrt{2}}$ ***and*** $\frac{(-1-i)}{\sqrt{2}}$ ***are the poles of*** f. ***Let*** $R > 1$ ***be any real number. Let*** γ ***be a closed curve bounded by the upper half circle with radius*** R ***and the interval*** $[-R, R]$ ***on the real axis. Then by Residue formula we have***

$$\int_\gamma f = 2\pi i[Res(f, \frac{(1+i)}{\sqrt{2}}) + Res(f, \frac{(-1+i)}{\sqrt{2}})] = 2\pi i(\frac{1}{2i}) = \frac{\pi}{\sqrt{2}}$$

where $Res(f, \frac{(1+i)}{\sqrt{2}}) = -\frac{1+i}{4\sqrt{2}}$ ***and*** $Res(f, \frac{(-1+i)}{\sqrt{2}}) = -\frac{1-i}{4\sqrt{2}}$

Again

$$\int_{\gamma}\frac{1}{z^4+1}dz=\int_{-R}^{R}\frac{1}{x^4+1}dx+\int_{0}^{\pi}\frac{iRe^{i\theta}}{R^4e^{4i\theta}+1}d\theta.$$

Now $|\int_0^\pi \frac{iRe^{i\theta}}{R^4e^{4i\theta}+1}d\theta|\leq \frac{R\pi}{R^4-1}$ ***which is tending to zero as*** $R\to\infty$***. Hence from the above we have***

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx=\frac{\pi}{\sqrt{2}}.$$

6.Let Ω ***be a simply connected domain and*** $0\notin\Omega$***. Find all the branches of*** $z^{\frac{1}{2}}$ ***in*** Ω***.***

Answer. Let $z=re^{i\theta}$***,*** $-\pi<\theta\leq\pi$***. Then*** $z^{\frac{1}{2}}=r^{\frac{1}{2}}e^{\frac{i(\theta+2k\pi)}{2}}$ ***for*** k ***is any integer and*** $-\pi<\theta<\pi$***. Then for*** $k=0$***,*** $w_0=r^{\frac{1}{2}}e^{i\frac{\theta}{2}}$ ***and for*** $k=1$***,*** $w_1=r^{\frac{1}{2}}e^{i\frac{(\theta+2\pi)}{2}}=-r^{\frac{1}{2}}e^{i\frac{\theta}{2}}$ ***for*** $-\pi<\theta<\pi$***. These are the two branches of*** $z^{\frac{1}{2}}$ ***in*** Ω***.***

7. Prove that every bi-holomorphic map of $\mathbb{C}$ ***has the form*** $f(z)=az+b$***, where*** $a\neq 0$ ***and*** b ***are in*** $\mathbb{C}$***.***

Answer: Let f ***be a bi-holomorphic map on*** $\mathbb{C}$***. Then*** f ***has a pole at*** ∞ ***i.e.*** $\lim_{|z|\to\infty}|f(z)|=\infty$***. Suppose if not, then there exists a sequence of complex number*** $\{z_n\}$ ***such that*** $|z_n|\to\infty$ ***but*** $|f(z_n)|\leq M$ ***for all*** n ***for some*** $M>0$***. Since*** f ***is injective*** $\{f(z_n)\}$ ***is a non constant sequence and subsequence of*** $\{f(z_n)\}$ ***converges. Let*** $\{f(z_{n_k})\}$ ***converges to*** w_0***. Suppose*** g ***is an inverse of the holomorphic function*** f***. Then*** $g(f(z_{n_k}))\to g(w_0)$***. But*** $g(f(z_{n_k}))=z_{n_k}\to g(w_0)$ ***which is a contradiction. Hence*** f ***has a pole at*** ∞***.***
Claim: If f ***has a pole at*** ∞***, then*** f ***is a polynomial.***
Let f ***has Taylor series expansion*** $f(z)=\sum_{n=0}^{\infty}a_nz^n$***. Consider*** $g(z)=f(\frac{1}{z})=\sum_{n=0}^{\infty}a_n\frac{1}{z^n}$***. So if*** f ***has a pole of order*** m ***at*** ∞ ***then*** g ***has a pole at zero of order*** m***. So the Laurent expression of*** g ***is of the form***

$$g(z)=\frac{b_{-m}}{z^m}+\frac{b_{-(m-1)}}{z^m}+\ldots+\frac{b_{-1}}{z}+b_0+b_1z+\ldots.$$

Now uniqueness of the power series of f***, we have*** $b_{-k}=a_k$ ***for*** $0\leq k\leq m$ $a_k=0$ ***for*** $k>m$***. Hence*** $f(z)=a_0+a_1z+\ldots+a_mz^m$***. But*** f ***is injective so we have*** $f(z)=az+b$ ***where*** $a\neq 0$ ***as*** f ***has a pole.***

8. Let $\epsilon>0$ ***and*** $f:B_{1+\epsilon}(0)\to\mathbb{C}$ ***be a non-constant holomorphic function. Assume that*** $|f(z)|=1$ ***if*** $|z|=1$***. (i) Prove that*** f ***has a zero in*** $\mathbb{D}$***. (ii) Prove that*** $f(\mathbb{D})$ ***contains*** $\mathbb{D}$***.***

Answer.(i) ***Suppose*** f ***has no zero in the disc*** $\mathbb{D}$***. Using maximum modulus principle we have*** $|f(z)|<1$ ***for all*** $z\in\mathbb{D}$ ***as*** $|f(z)|=1$ ***for*** $|z|=1$***. Consider*** $g(z)=\frac{1}{f(z)}$ ***for*** $z\in\mathbb{D}$***. Then*** g ***is holomorphic. Also*** $|g(z)|=\frac{1}{|f(z)|}>1$ ***for all*** $z\in\mathbb{D}$ ***and*** $|g(z)|=1$ ***for*** $|z|=1$ ***which is not possible due to maximum modulus principle. Hence*** f ***has a zero in*** $\mathbb{D}$***.***

(ii) ***For*** $a\in\mathbb{D}$ ***define*** $\phi_a:\mathbb{D}\to\mathbb{D}$ ***by***

$$\phi_a(z)=\frac{z-a}{1-\bar{a}z}.$$

Consider $g=\phi_a\circ f$***. Since*** $\phi_a(\{z\in\mathbb{C}:|z|=1\})\subset\{z\in\mathbb{C}:|z|=1\}$ ***and*** $|f(z)|=1$ ***for*** $|z|=1$***,*** $|g(z)|=1$ ***for*** $|z|=1$***. So by the first part we have*** $g(z_0)=0$ ***for some*** $z_0\in\mathbb{D}$***. That is***

$$\frac{f(z_0)-a}{1-\bar{a}f(z_0)}=0.$$

Hence $f(z_0)=a$ ***for some*** $z_0\in\mathbb{D}$***. This shows that*** $f(\mathbb{D})$ ***contains*** $\mathbb{D}$***.***

9. Let f be a holomorphic function from $\mathbb{D}$ to itself that is not the identity map z. Prove that f has at most one fixed point in $\mathbb{D}$.

Answer. Suppose if possible f has two fixed points. Let $z_1, z_2 \in \mathbb{D}$ and $z_1 \neq z_2$ such that $f(z_1) = z_1$ and $f(z_2) = z_2$. Consider $\phi_{z_1} : \mathbb{D} \to \mathbb{D}$ defined by $\phi_{z_1}(z) = \frac{z-z_1}{1-\bar{z}_1 z}$. Then ϕ_{z_1} is bi-holomorphic on $\mathbb{D}$ and in particular $\phi_{z_1}(z_1) = 0$ and $\phi_{z_1}^{-1}(0) = z_1$. Take $g = \phi_{z_1} \circ f \circ \phi_{z_1}^{-1}$. Then $g(0) = 0$. Take $w = \phi_{z_1}(z_2) \neq 0$ as ϕ_{z_1} is bi-holomorphic. Then $g(w) = w$. Hence by Schwarz lemma we have $g(z) = z$ for all $z \in \mathbb{D}$. This implies $f(z) = z$ which is a contradiction. Therefore f has at most one fixed point in $\mathbb{D}$.