Final Examination

(i) Answer all questions. (ii) $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$***. (iii)*** $\mathbb{H}$***= upper half plane. (iv)*** $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$***. (v)*** $\mathbb{A}_{1,2}(0) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

1. Let $f : \mathbb{C} \to \mathbb{H}$ ***be a holomorphic function. Prove that*** f ***is a constant.***

Answer: Consider $g : \mathbb{C} \to \mathbb{C}$ ***defined by*** $g(z) = e^{if(z)}$***. Clearly,*** g ***is holomorphic on*** $\mathbb{C}$ ***as*** f ***is so. Let*** $f = u + iv$***. Then*** $v = Im(f) \geq 0$***. Now for all*** $z \in \mathbb{C}$

$$|g(z)| = |e^{(-v+iu)}| = e^{-v} \leq 1$$

as $v \geq 0$***. Therefore*** g ***is a bounded entire function. So by Liouville's theorem,*** g ***is constant and hence*** f ***is constant.***

2. Let $f : B_1(0) \to B_1(0)$ ***be a holomorphic function. Let*** $\alpha \in B_1(0)$ ***and*** $f(\alpha) = 0$***. Prove that*** $|f(0)| \leq |\alpha|$.

Answer: For $|\alpha| < 1$***, consider*** $\phi_\alpha = \frac{\alpha - z}{1 - \bar{\alpha} z}$***. Let*** $g = f \circ \phi_\alpha$***. Then*** $g : B_1(0) \to B_1(0)$ ***is analytic. Also*** $g(0) = f(\phi_\alpha(0)) = f(\alpha) = 0$***. By Schwarz lemma, we have*** $|g(z)| \leq |z|$ ***for all*** $z \in B_1(0)$***. Put*** $z = \alpha$***, we have*** $g(\alpha) = f(\phi_\alpha) = f(0)$***. Therefore*** $|f(0)| \leq |\alpha|$***.***

3. Let $g(z) = f(z^3)$ ***where*** $f \in Hol(\mathbb{C})$ ***and*** f ***is not identically zero. Prove that***

$$Res[\frac{1}{g}; 0] = 0.$$

Answer: Since $f \in Hol(\mathbb{C})$***,*** g ***is so. Let*** $f(z) = \sum_{n=0}^{\infty} a_n z^n$***. Therefore*** $g(z) = \sum_{n=0}^{\infty} a_n z^{3n}$***. It is enough to show that the coefficient of*** $\frac{1}{z}$ ***of the Laurent series expansion of*** $\frac{1}{g}$ ***at zero is zero. If for some*** $m \geq 0$***,*** $a_m \neq 0$ ***then***

$$\frac{1}{g} = z^{-3m}(a_m + a_{m+1}z^3 + a_{m+2}z^6 + \ldots)^{-1}.$$

Clearly there does not have any $\frac{1}{z}$ ***term in the series expansion. Hence*** $Res[\frac{1}{g}; 0] = 0$.

4. Prove that $f(z) = 2 - z - e^{-z}$ ***has one root in the right half plane.***

Answer. Let $h(z) = 2 - z$ ***and*** $g(z) = -e^{-z}$ ***for*** $z \in \mathbb{C}$***. Clearly*** h, g ***are holomorphic on an open set containing circle*** $C_1(2)$ ***with centre 2 and radius 1. Also for*** $x > 0$ ***where*** $z = x + iy$

$$|h(z)| = 1 > e^{-x} = |g(z)|$$

for $z \in C_1(2)$***. Hence by Rouché's theorem,*** h ***and*** $h+g$ ***have the same number of zeros inside the circle*** $C_1(2)$***. But*** h ***has one zero inside the circle and hence*** $h+g = f$ ***has one root inside the circle. This proves that*** f ***has one root in the right half plane.***

5. Let $f \in Hol(\mathbb{C})$ ***and*** $f(0) = 0$***, and*** $f'(0) = 1$ ***and suppose that*** $|f(z)| \leq 1$ ***for all*** $z \in C_1(0)$***. Show that*** $f(z) = z$ ***for all*** $z \in \mathbb{C}$***.***

Answer. Since $|f(z)| \leq 1$ ***for all*** $z \in C_1(0)$***, using Maximum modulus principle we have*** $|f(z)| \leq 1$ ***for all*** $z \in \overline{B_1(0)}$***. Consider*** $g = f|_{B_1(0)}$***. Then*** $g(0) = 0$ ***and*** $g'(0) = 1$***. By Schwarz***

lemma we have $g(z) = z$ ***for all*** $z \in B_1(0)$***. Therefore*** $f(z) = z$ ***for all*** $z \in B_1(0)$***. Hence by identity theorem we have*** $f(z) = z$ ***for all*** $z \in \mathbb{C}$***.***

6. Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx.$$

Answer. Let $f(z) = \frac{1}{z^2+1}$***. Then*** $z = i, -i$ ***are the poles of*** f***. Let*** $R > 1$ ***be any real number. Let*** γ ***be a closed curve bounded by the upper half circle with radius*** R ***and the interval*** $[-R, R]$ ***on the real axis. Then by Residue formula we have***

$$\int_\gamma f = 2\pi i(Res(f,i)) = 2\pi i(\frac{1}{2i}) = \pi$$

where $Res(f,i) = \lim_{z\to i}(z-i)\frac{1}{z^2+1} = \frac{1}{2i}$***.***

Again

$$\int_\gamma \frac{1}{z^2+1} dz = \int_{-R}^{R} \frac{1}{x^2+1} dx + \int_0^\pi \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1} d\theta.$$

Now $|\int_0^\pi \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1} d\theta| \leq \frac{R\pi}{R^2-1}$ ***which is tending to zero as*** $R \to \infty$***. Hence from the above we have***

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi.$$

7. Prove there does not exist a branch of $\log(z^2-1)$ ***on*** $\mathbb{C} \setminus [-1,1]$***.***

Answer: Let $Logz$ ***be the principal branch of*** $\log z$***. Then*** $Logz$ ***is analytic on the region*** $\mathbb{C} \setminus (-\infty, 0]$ ***i.e.*** $\mathbb{C} \setminus \{z = x+iy : -\infty < x \leq 0 \ \& \ y = 0\}$***. Now*** $(z^2-1) = x^2 - y^2 - 1 + i(2xy)$***. So*** $Log(z^2-1)$ ***is not analytic on*** $\{z = x+iy : -\infty < x^2 - y^2 - 1 \leq 0 \ \& \ 2xy = 0\}$***. That is*** $Log(z^2-1)$ ***is not analytic on*** $\{z = x+iy : x = 0 \ \& \ y \in \mathbb{R}\} \bigcup [-1,1]$***. Therefore there does not exist a branch of*** $\log(z^2-1)$ ***on*** $\mathbb{C} \setminus [-1,1]$***.***

8. Prove or disprove (with justification):
(i) There exist $f \in Hol(\mathbb{C} \setminus \{0\})$ ***such that*** $f(z)^2 = z$ ***for all*** $z \in \mathbb{C} \setminus \{0\}$.
(ii) There exist $f \in Hol(\mathbb{A}_{1,2}(0))$ ***such that*** $f(z)^2 = z$ ***for all*** $z \in \mathbb{A}_{1,2}(0)$.

Answer. Both the cases there does not exist any function f ***such that*** $f(z)^2 = z$***.***

Suppose if possible $f \in Hol(\mathbb{A}_{1,2}(0))$ ***such that*** $f(z)^2 = z$ ***for all*** $z \in \mathbb{A}_{1,2}(0)$. ***Then***

$$2f(z)f^{'}(z) = 1.$$

Let $\gamma : [0,1] \to \mathbb{C}$ ***be a simple closed curve around the origin. Then*** $f(\gamma)$ ***is also closed and***

$$\int_\gamma \frac{1}{z} dz = 2\pi i.$$

Now

$$\int_{f(\gamma)} \frac{1}{z} dz = \int_0^1 \frac{f^{'}(\gamma(t))\gamma^{'}(t)}{f(\gamma)} dt = \int_\gamma \frac{f^{'}(z)}{f(z)} dz = \int_\gamma \frac{1}{2z} dz = \pi i$$

which is not possible as $f(\gamma)$ ***is closed. Hence there does not exists*** $f \in Hol(\mathbb{A}_{1,2}(0))$ ***such that*** $f(z)^2 = z$ ***for all*** $z \in \mathbb{A}_{1,2}(0)$.

Therefore there does not exist $f \in Hol(\mathbb{C} \setminus \{0\})$ ***such that*** $f(z)^2 = z$ ***for all*** $z \in \mathbb{C} \setminus \{0\}$.