Complex Analysis Solutions *

Final Semester 2012-2013

Problem 1

Let $f(z) = \frac{1}{(z-z_1)(z-z_2)}$. Invoking the residue theorem, the integral

$$\int_\gamma f(z)dz = 2\pi i(Res_f(z_1) + Res_f(z_2)) = 2\pi i\big(\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_1}\big) = 0.$$

Problem 2

f is holomorphic in U. Therefore for any $z \in U$, we can write $f(z) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$. Notice that $\int\limits_0^{2\pi} e^{ij\theta}\overline{e^{ik\theta}}d\theta = 2\pi\delta_{j=k}$. Therefore,

$$\sup_{0\leq r<1} \int\limits_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sup_{0\leq r<1} \sum_{n=0}^{\infty} r^{2n} \left|\frac{f^{(n)}(0)}{n!}\right|^2 = \sum_{n=0}^{\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^2$$

Given that f is bounded in U, therefore the l.h.s. of (**??**) is bounded. Hence we obtain $\sum\limits_{n=0}^{\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^2 < \infty$.

The converse is not true in general. Consider the function defined by the power series $g(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$. The function g is well defined in U as the radius of convergence of the power series is 1. Now $\sum\limits_{n=0}^{\infty} \left|\frac{g^{(n)}(0)}{n!}\right|^2 = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \infty$, where as $g$is not bounded in U (choose a sequence of points in the real line approaching 1).

Problem 3

Given, $p_N(z) = \sum\limits_{k=0}^{N} c_k z^k$, $c_N \neq 0$ and $R = \max\{1, \frac{1}{|c_N|} \sum\limits_{k=0}^{N-1} |c_k|\}$.
Correction in question: Replace $B(0, R)$ to 'closed ball of radius R'. Otherwise, we can choose c_ks such that, $c_N = N$ and $c_k = -1$ for $0 \leq k \leq N-1$. Then, $R = 1$ and $z = 1$ is a zero of p. From definition $R \geq 1$.

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Consider the polynomial $q_N(z) = c_N z^N$. When $|z| = R + \epsilon > R$,

$$\begin{aligned} |p_N(z) - q_N(z)| &= |c_{N-1}z^{N-1} + \cdots + c_0| \\ &\leq |c_0| + \cdots + |c_{N-1}||z|^{N-1} \\ &\leq |c_N||z|^{N-1}R \\ &< |q_n(z)|. \end{aligned}$$

Invoking Rouche's theorem we have that the for $\epsilon > 0$, the polynomials p_N and q_N have same number of zeros in $B(R+\epsilon, 0)$. Therefore all the zeros of p_N are in the closure of the $B(R, 0)$

Problem 4

Let the f be a holomorphic function from U to U, which has a fixed point at 0. Then f satisfies the hypothesis of Schwarz's lemma for the unit disc. Let $z_0 \neq 0$ be the another fixed point of f. Then, $f(z_0) = z_0$. Therefore invoking Schwarz's lemma we get $f(z) = z$ for every $z \in U$.

Problem 5

$$\int\limits_{-\infty}^{\infty} e^{itx} e^{-\frac{x^2}{2}} = e^{-\frac{t^2}{2}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-it)^2}{2}} dx = e^{-\frac{t^2}{2}} \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \qquad (1)$$

It is enough to justify the last equality of the above equation (1). Consider the contour $\gamma_n = \gamma_{1,n} \cup \gamma_{2,n} \cup \gamma_{3,n} \cup \gamma_{4,n}$, where,
$\gamma_{1,n} = \{-n + u : 0 \leq u \leq 2n\}$,
$\gamma_{2,n} = \{-n - iu : 0 \leq u \leq t\}$,
$\gamma_{3,n} = \{-n - iu : t \geq u \geq 0\}$,
$\gamma_{4,n} = \{n + u : 0 \geq u \geq -2n\}$. Consider the analytic function $f(z) = e^{-\frac{z^2}{2}}$. For the closed loop γ_n, we have

$$\int_{\gamma_{1,n}} f(z)dz + \int_{\gamma_{2,n}} f(z)dz + \int_{\gamma_{3,n}} f(z)dz + \int_{\gamma_{4,n}} f(z)dz = \int_{\gamma_n} f(z)dz = 0.$$

Because $|f(z)| \leq e^{-n^2}$, whenever $z \in \gamma_{2,n} \cup \gamma_{3,n}$, we have

$$\lim_{n\to\infty} \left(\int_{\gamma_{2,n}} f(z)dz + \int_{\gamma_{3,n}} f(z)dz \right) = 0.$$

Therefore we have

$$\lim_{n\to\infty} \left(\int\limits_{\gamma_{1,n}} f(z)dz + \int\limits_{\gamma_{4,n}} f(z)dz \right) = 0.$$

From here it follows that

$$\int\limits_{-\infty}^{\infty} e^{-\frac{(x-it)^2}{2}} dx = \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx.$$

Problem 6

If f has a removable singularity at 0, then $f(0) = \lim_{n\to\infty} f(\frac{1}{n}) = 0$. The zero set of f has a limit point, therefore f is identically 0. If f does not have a removable singularity and has a pole at 0, then f cannot vanish in some neighborhood of 0. But $f(\frac{1}{n}) = 0$ for any $n \in \mathbb{N}$. Hence f cannot have a pole at 0. This leaves that f can have an essential singularity at 0 in which case we know that the image of any neighborhood of 0 under the mapping f is dense in $\mathbb{C}$.

Problem 7

$$\text{Sum of all residues of } f = \lim_{R\to\infty} \int\limits_{|z|=R} \frac{p(z)}{q(z)} dz \tag{2}$$

Because $deg(q) > deg(p)+1$, for R large enough we have $\left|\frac{p(z)}{q(z)}\right| \le \frac{1}{cR^2}$, whenever $|z| = R$. Choosing R large the $|\int\limits_{|z|=R} \frac{p(z)}{q(z)} dz|$ can be made as small as needed. Therefore the r.h.s. of (2) is 0.